Consistency of 2-step estimators
of Markov Switching Dynamic Factor Models

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1 Introduction

- Knowledge about the current state of the economy is crucial for economic agents, particularly policy makers.
- Different institutions $\rightarrow$ different and sometimes not replicable methodologies.
  Turning points are identified with delays which can be important.
- Markov-Switching (MS) Models : replicable method + provides timely estimation of the probability to enter a recession (expansion)
- Initial introduction of these models was made using 1 series (Hamilton 1998,1999). ML estimation, numerical procedure.
- Introduction of MS Dynamic Factor Models (MS-DFM) : Kim (1994), Kim-Yoo (1995), Kim-Nelson (1998) : allows to use several series. ML estimation using KF (= 1-step method). Usable only with very small sets of series and parameters (4 or 5 series).

- Camacho, Perez-Quiroz, Poncela (2012) propose a 2-steps estimation method :

  i) estimation of the unobserved MS factor(s) by PCA

  ii) estimation of the MS parameters

  Comparison of 1-step and 2-step estimation methods on US data. Claim that 1-step method works far better, although the performance of both methods gets closer when the number of indicators grows.
- Doz, Petronevich (2016) : use a large set of series to analyze French business cycle. Show that the 2-steps procedure give good results.
- Aim of the present paper : prove that the 2-steps method indeed provides consistent estimators.

2 MS-DFM and two estimation methods

2.1 The model

- Factor Model :
  - x_t is a vector of N observable variables
  - x_t can be decomposed as a linear function of two unobservable latent variables f_t (common factors) and e_t (idiosyncratic components) as :

$$x_t = \Lambda f_t + e_t \tag{1}$$

    with (f_t) and (e_t) two orthogonal processes.
    In what follows, f_t has dimension 1 (i.e. one common factor).
    Results can be extended to r-dimensional factors.

- MS-DFM : (f_t) is an AR process with MS mean :

$$f_t = \beta_{S_t} + \phi_1 f_{t-1} + \ldots \phi_p f_{t-p} + \varepsilon_t \text{ where } \varepsilon_t \sim N(0, \sigma^2_{S_t}) \qquad (2)$$

In what follows $p = 1$ or $p = 2$. Results can be extended to any p.

- S_t is a Markov process which takes a finite number of values (states). In what follows S_t has two states.

$$S_t = \begin{cases} 1 & \text{if recession} \\ 0 & \text{if expansion} \end{cases}$$

S_t switches sates with transition probabilities defined as follows (Markov property) :

$$Pr(S_t = j | S_{t-1} = i,\ S_{t-2} = k, ...) = Pr(S_t = j | S_{t-1} = i) = p_{ij},\ i, j \in \{0, 1\}$$

2.2 Estimation methods

- **1-step method** :
  - Estimate parameters and factor simultaneously by ML through the Kalman filter and a numerical optimization procedure
  - The KF formulas are modified due to the MS behavior of the factor (Kim's (1994) filter).
  - The filtered probabilities $P(S_t = i|I_t)$ are also computed as an output of the filter.
  - Advantages of the procedure :
    . the MS behavior of the factor is directly taken into account
    . it can very easily accommodate individual AR process for the idiosyncratic component.
  - Drawbacks of the procedure :
    . Numerical optimization $\rightarrow$ can be done only for a very small N
    . Time consuming procedure + possible convergence problems.

- **2-steps method** :
  - 1st step : Estimate factor model (1) by PCA
  - 2nd step : If $\hat{f}_t$ is the estimated factor obtained at step 1, estimate model (2) with $\hat{f}_t$ playing the role of f_t.
    The estimation is made by ML using a numerical procedure (but here small set of parameters) and the filtered probabilities can be computed via Hamilton (1998,1999) formulas
  - Advantages of the method :
    . It can accommodate a (very) large number of series N
    . Under standard assumptions PCA gives consistent estimators and $p\lim(\hat{f}_t - f_t) = 0$
    . It can easily accommodate various specifications for the MS behaviour of f_t
    . It is very easy to implement.

– Drawbacks of the method :

. Does not take the AR behaviour of (f_t) and (e_t) into account at the 1st step (it's always the case when DFM are estimated by PCA).

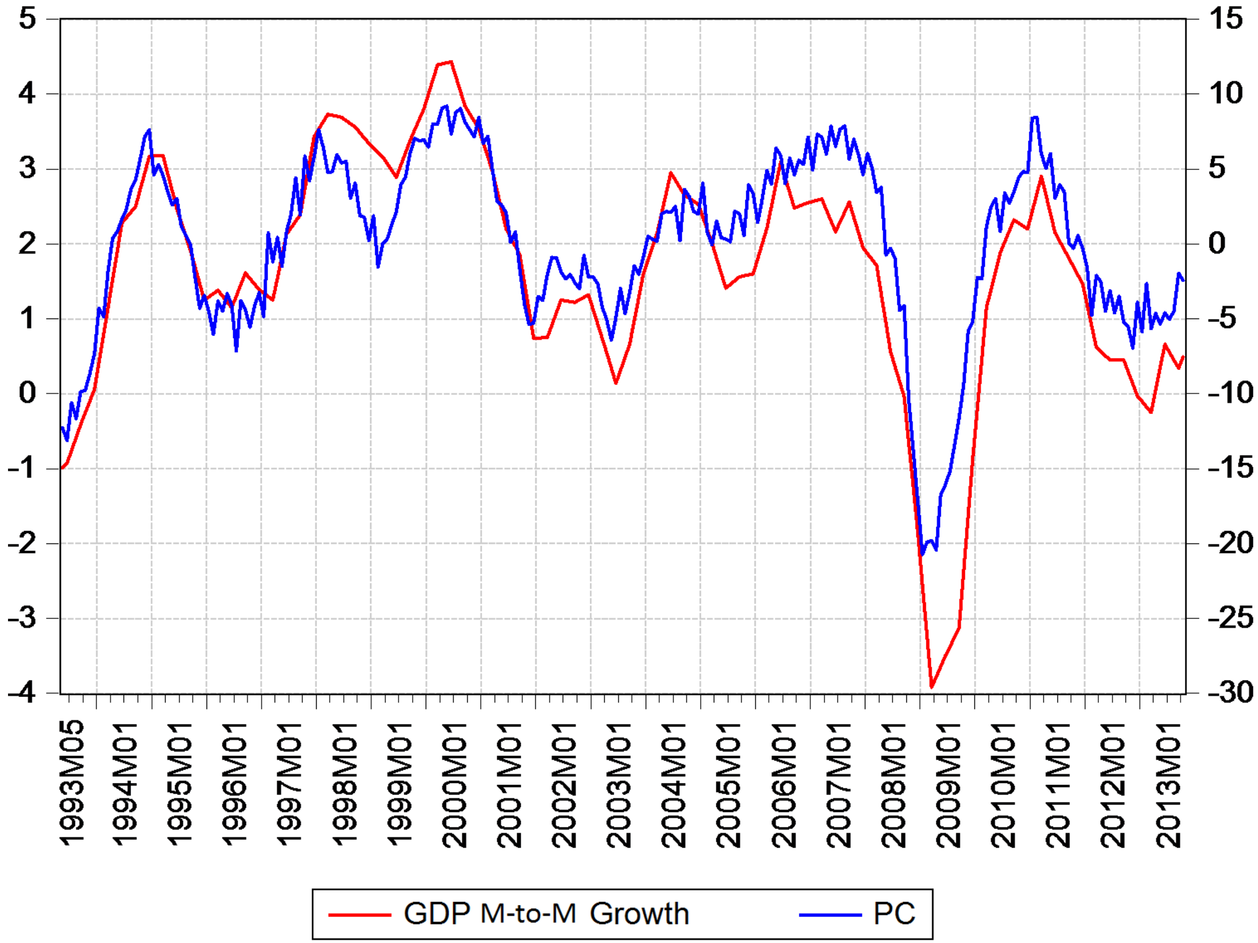
. Does not take the MS behaviour of (f_t) into account at the 1st step

- Doz-Petronevich (2016) results :

. Data = 151 monthly series for French economy concerning : production sector, financial sector, employment, households, banking system, business surveys, international trade, monetary indicators, major world economic indicators and others.
The series are stationarized and then standardized.

. 2-step method, with several specifications for the MS-AR behaviour of (f_t).

. OECD business cycle dating as a reference.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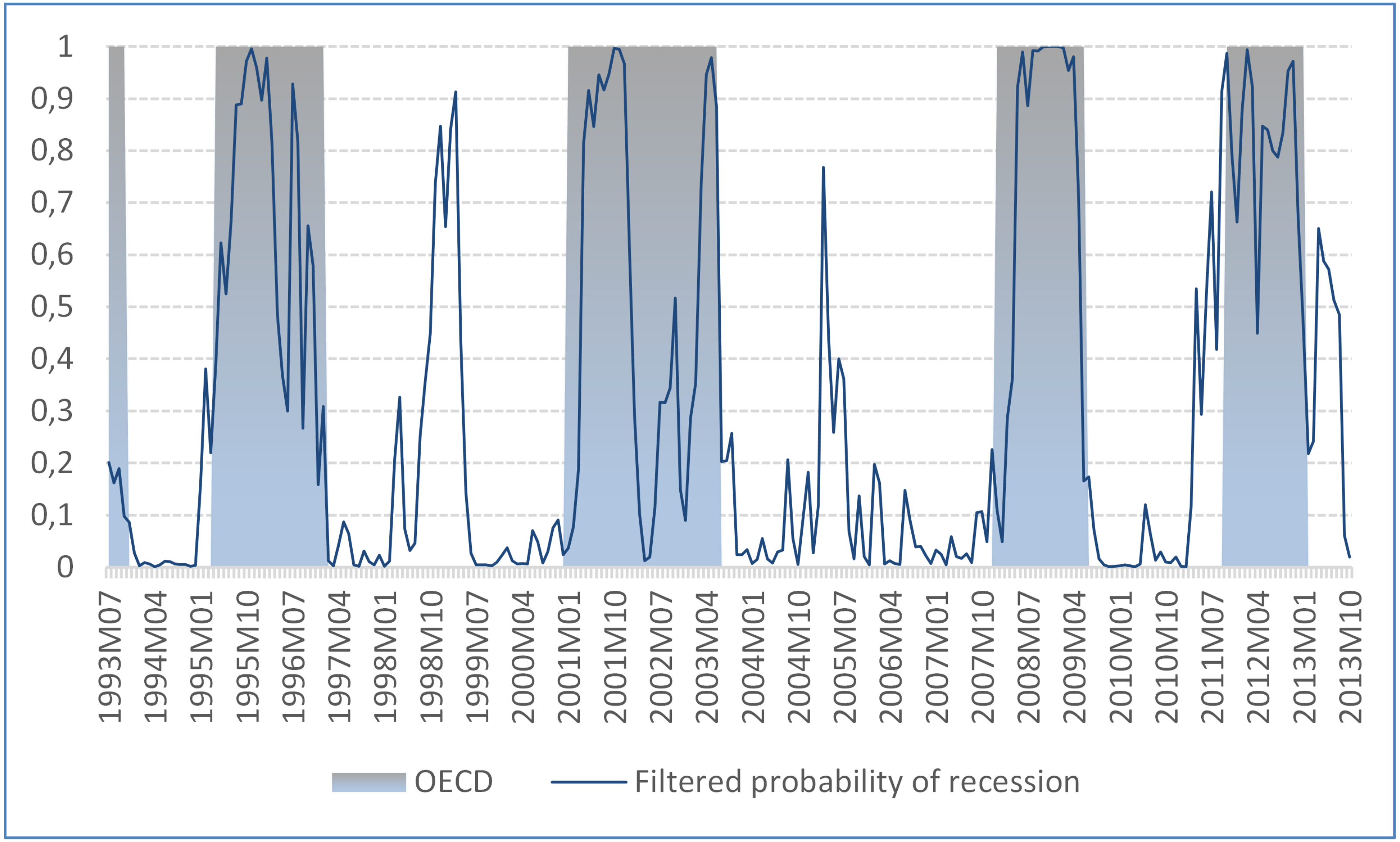

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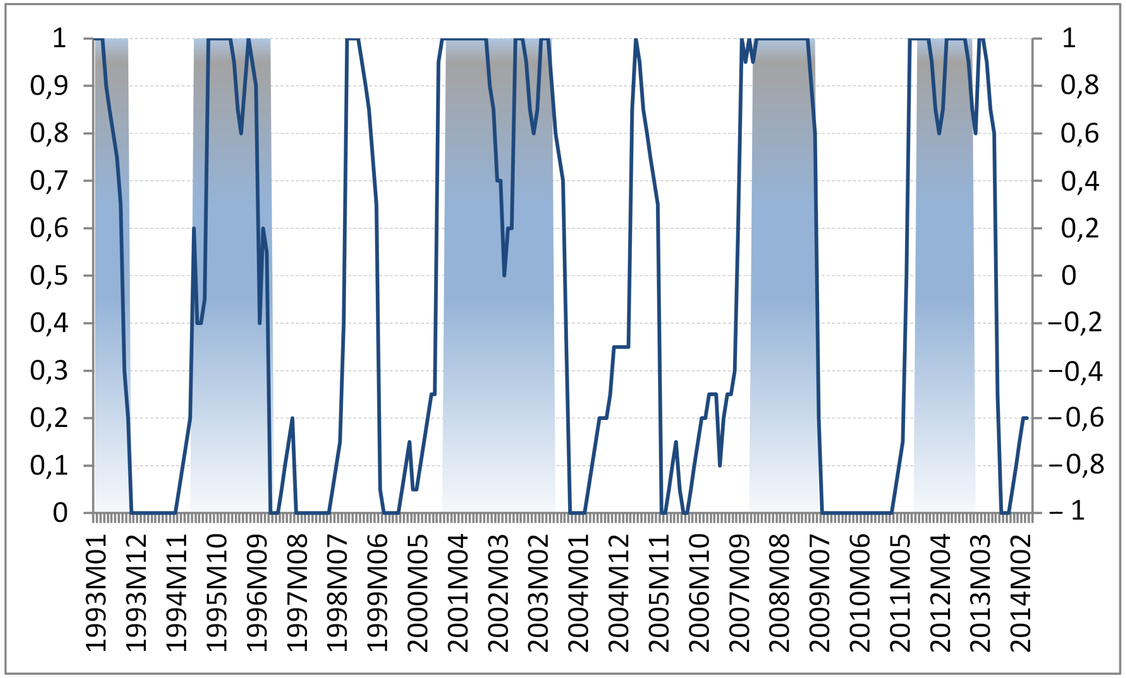
OECD
Filtered probability of recession

. Two slowdown periods (1998-1999 and 2005) are not detected by OECD due to the duration restrictions, but were indeed detected and analyzed by INSEE (see ”Note de Conjoncture”).

. Those periods were also detected by the reversal index (”indicateur de retournement”) published by INSEE.



3 Consistency of the 2-steps estimator

3.1 Consistency of ML estimators of HMM parameters when the variable is observable

- The model (AR(p) process with MS mean) is an Hidden Markov-switching Model (HMM) :

$$\begin{aligned} y_t &= \mu_{S_t} + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + \varepsilon_t \\ \varepsilon_t \quad &\text{i.i.d.} \quad \mathcal{N}(0, \sigma^2_{S_t}) \end{aligned}$$

  – y_t is observed
  – S_t is a two-state unobservable Markov variable such that :

$$P(S_t = 1 | S_{t-1} = 1) = p_1 \text{ and } P(S_t = 0 | S_{t-1} = 0) = p_0$$

- $p = 1$ below : identical proof for any p or any S_t with finite range.

- Conditional log-likelihood :

$$
\begin{aligned}
l(y_t|y_{t-1},\ldots,y_1,\theta) &= l(y_t|y_{t-1},\theta) \\
&= \frac{\pi_1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{1}{2\sigma_1^2}(y_t-\phi y_{t-1}-\mu_1)^2} + \frac{\pi_0}{\sqrt{2\pi\sigma_0^2}} e^{-\frac{1}{2\sigma_0^2}(y_t-\phi y_{t-1}-\mu_0)^2}
\end{aligned}
$$

with : $\pi_1 = P(S_t = 1) = \frac{1-p_0}{2-(p_0+p_1)}$ and $\pi_0 = P(S_t = 0) = 1 - \pi_1 = \frac{1-p_1}{2-(p_0+p_1)}$
and $\theta = (\phi, \mu_0, \mu_1, \sigma_0^2, \sigma_1^2, p_0, p_1)'$

- Maximisation of the log-likelihood :

$$\hat{\theta} = \text{Argmax} Q_T(\theta) = \text{Argmax} \frac{1}{T} \sum_{t=1}^{T} \log l(y_t|y_{t-1},\ldots,y_1,\theta)$$

- No analytical solution : numerical optimization (e.g. EM-algorithm)
- General (complicated) proofs of MLE consistency for HMMs have been given in a very general framework (Leroux (1992), Cappe, Moulines, Rydden (2005) and others). Here : simpler proof for the particular case under study $\rightarrow$ extension to the 2-step estimator.

- In those mentioned papers, it is shown that $\hat{\theta}$ converges to θ^* or to a $\tilde{\theta}$ which is equivalent to θ^*, using the following definition :

$$\tilde{\theta} \sim \theta^* \Leftrightarrow P_{\tilde{\theta}} = P_{\theta^*}$$

It will be also the case here.
In practise, two parameters will be equivalent if they are equal up to an index permutation.

- Newey-Mac Fadden (1984) consistency conditions :
if $Q^*(\theta) = \mathrm{p}\lim Q_T(\theta)$ and if θ^* is the true value of θ, then the following conditions are sufficient to get : $\mathrm{p}\lim \hat{\theta} = \theta^*$
i) θ^* is the unique maximum of $Q^*(\theta)$
ii) $\theta \in \Theta$, with Θ a compact set containing θ^*
iii) Q^* is a continuous function
iv) $Q_T(\theta)$ converges in probability to $Q^*(\theta)$ uniformly on Θ

- In the present framework, (S_t) and (y_t) are stationary and

$$Q^*(\theta) = E_{\theta^*}\left(\log l(y_t|y_{t-1},\ldots,y_1,\theta)\right) = E_{\theta^*}\left(\log l(y_t|y_{t-1},\theta)\right)$$

- Sketch of the proof :

  ii) It can be easily shown that the maximization of $Q_T(\theta)$ can be done on a compact subset of the initial parameter space

  i) From the information inequality, it can be seen that θ^* is a maximum of Q^*.
  From the same inequality, it can be proved that any $\tilde{\theta}$ such that $Q^*(\tilde{\theta}) = Q^*(\theta^*)$ is equivalent to θ^*.
  Thus, any $\tilde{\theta}$ which is equivalent to θ is a potential limit for $\hat{\theta}$.

  iii) Can be obtained from Lebesgue continuity theorem.

  iv) It is sufficient to prove that (Q_T) is stochastically equicontinuous on Θ. This can be obtained by showing that :

$$\operatorname*{Sup}_{\theta\in\Theta} \|\nabla_\theta Q_T(\theta)\| = O_P(1)$$

3.2 Consistency of the 2-step estimator for an MS-DFM

- The model is now :

$$\begin{array}{rcl} x_t & = & \Lambda f_t + e_t \\ f_t & = & \mu_{S_t} + \phi_1 f_{t-1} + \cdots + \phi_p f_{t-p} + \varepsilon_t \\ \varepsilon_t & \text{i.i.d.} & \mathcal{N}(0, \sigma^2_{S_t}) \end{array}$$

  where
  - x_t is a $(n \times 1)$ vector of observed variables
  - f_t is a $(r \times 1)$ vector of unobserved factors
  - (f_t) and (e_t) are uncorrelated at all leads and lags
  - usual assumptions about Dynamic Factor Models (DFM) are satisfied

– S_t is a two-state unobservable Markov variable such that :

$$P(S_t = 1|S_{t-1} = 1) = p_1 \text{ and } P(S_t = 0|S_{t-1} = 0) = p_0$$

- Two-step estimator :

  1. Estimate the factor model (e.g. by PCA, but can be also done using QML) without taking the MS structure into account, and obtain an approximation $\hat{f}_t$ of f_t.
  2. Estimate an HMM for $\hat{f}_t$.

- Newey-Mac Fadden (1984) extension :

  using the same notations and conditions as before, if $\hat{\hat{\theta}}$ satisfies :

$$Q_T(\hat{\hat{\theta}}) \geq \underset{\theta \in \Theta}{\text{Max}} Q_T(\theta) + o_P(1)$$

  then :

$$\text{p}\lim \hat{\hat{\theta}} = \theta^*$$

- Application in our framework :
  - f_t plays the role which was played by y_t in the previous section
  - Q_T cannot be computed since f_t is not observable :

$$\begin{aligned} Q_T(\theta) &= \frac{1}{T}\sum_{t=1}^{T}\log l(f_t|f_{t-1},\ldots,f_1,\theta) \\ &= \frac{1}{T}\sum_{t=1}^{T}\log\left(\frac{\pi_1}{\sqrt{2\pi\sigma_1^2}}e^{-\frac{1}{2\sigma_1^2}(f_t-\phi f_{t-1}-\mu_1)^2}+\frac{\pi_0}{\sqrt{2\pi\sigma_0^2}}e^{-\frac{1}{2\sigma_0^2}(f_t-\phi f_{t-1}-\mu_0)^2}\right) \end{aligned}$$

  - Q_T is replaced by $\hat{Q}_T$:

$$\hat{Q}_T(\theta) = \frac{1}{T}\sum_{t=1}^{T}\log\left(\frac{\pi_1}{\sqrt{2\pi\sigma_1^2}}e^{-\frac{1}{2\sigma_1^2}(\hat{f}_t-\phi \hat{f}_{t-1}-\mu_1)^2}+\frac{\pi_0}{\sqrt{2\pi\sigma_0^2}}e^{-\frac{1}{2\sigma_0^2}(\hat{f}_t-\phi \hat{f}_{t-1}-\mu_0)^2}\right)$$

  - $\hat{\hat{\theta}}$ is defined by $\hat{\hat{\theta}} = \text{Argmax}\hat{Q}_T(\theta)$.

– To get the consistency result, it is sufficient to prove that : $Q_T(\hat{\hat{\theta}}) \geq \underset{\theta\in\Theta}{\text{Max}} Q_T(\theta) + o_P(1)$.

– Remark : Here the consistency result will be analogous to the result obtained in section 1 :

$$\hat{\hat{\theta}} \text{ converges to a } \tilde{\theta} \text{ which is equivalent to } \theta^*.$$

- Sketch of the proof
  – $\forall\theta \in \Theta$, define

$$g(f_t, f_{t-1}, \theta) = \log\left(\sum_{i=0}^{1} \frac{\pi_i}{\sqrt{2\pi\sigma_i^2}} e^{-\frac{1}{2\sigma_i^2}(f_t-\phi f_{t-1}-\mu_i)^2}\right)$$

so that

$$|Q_T(\theta) - \hat{Q}_T(\theta)| = |\frac{1}{T}\sum_{t=1}^{T}\Big(g(f_t, f_{t-1}, \theta) - g(\hat{f}_t, \hat{f}_{t-1}, \theta)\Big)|$$

– It can be shown that $\forall \theta \in \Theta$:

$$|\frac{\partial g}{\partial f_t}(f_t, f_{t-1}, \theta)| \leq M \text{ and } |\frac{\partial g}{\partial f_{t-1}}(f_t, f_{t-1}, \theta)| \leq M$$

– Thus :

$$\begin{aligned} |Q_T(\theta) - \hat{Q}_T(\theta)| &\leq \frac{1}{T}\sum_{t=1}^{T} |g(f_t, f_{t-1}, \theta) - g(\hat{f}_t, \hat{f}_{t-1}, \theta)| \\ &\leq M\frac{1}{T}\sum_{t=1}^{T} |f_t - \hat{f}_t|^2 = o_p(1) \text{ uniformly in } \theta \end{aligned}$$

– As $\hat{Q}_T(\hat{\hat{\theta}}) = \underset{\theta\in\Theta}{\text{Max}}\hat{Q}_T(\theta)$, we have :

$$\hat{Q}_T(\hat{\hat{\theta}}) \geq \hat{Q}_T(\hat{\theta})$$

where $\hat{\theta} = \text{Argmax } Q_T(\theta)$.

– This can be written as :

$$Q_T(\hat{\hat{\theta}}) + o_P(1) \geq Q_T(\hat{\theta}) + o_P(1)$$

which completes the proof.