Realized EGARCH, CBOE VIX and Variance Risk Premium*

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Abstract

It has been documented that GARCH family models, under the locally risk-neutral valuation relationship (LRNVR), fail to explain the variance risk premium suggested by CBOE VIX index. We show that the recently proposed Realized EGARCH model, under an exponentially affine stochastic discount factor, generates reasonable levels of variance risk premium. We derive the closed-form VIX pricing formula and find empirical evidence that Realized EGARCH model provides significantly better forecasting performance for VIX index than a number of GARCH volatility models, both in-sample and out-of-sample. Realized EGARCH model benefits from its dual shock framework, more flexible dependence of return and volatility shocks and the information gain from including the realized measures of volatility.

Keywords: Realized EGARCH; High Frequency Data; Realized Variance; VIX.

JEL Classification: C10; C22; C80

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1 Introduction

In recent years, there are growing research interests in variance risk premium (VRP), which reflect the compensation for investors for taking volatility risk. It is usually measured in the difference between average variance under the physical and risk neutral measures. The former could be estimated by volatility models using returns data and the latter could be extracted from options data, known as implied volatility. The well-known VIX index launched by Chicago Board Options Exchange (CBOE) is a model-free measure of expected average variance for next 30 days under the risk neutral measure. Numerous studies, including Coval and Shumway (2001), Bakshi and Kapadia (2003) and Carr and Wu (2009), documented negative variance risk premia on average for equity and other financial assets. Recently, since Bollerslev et al. (2009), variance risk premium has been recognized as an important factor in asset pricing or a useful signal in predicting financial variables. See Zhang et al. (2009), Bollerslev et al. (2013), Londono and Zhou (2012), Bekaert et al. (2013), Andreou et al. (2013) etc.

Inspired by the great success in modeling volatility with GARCH family models, Duan (1995) pioneered in employing GARCH model for option pricing by proposing a locally risk-neutral valuation relationship (LRNVR). Under LRNVR, the one-period ahead conditional variance remains unchanged during the change of measure. Such model develops a link between expected volatility or variance in the physical measure and the risk neutral measure. Hao and Zhang (2013) investigated whether GARCH-type models under LRNVR could accommodate the variance risk premium suggested by the CBOE VIX index. They examined a number of GARCH family models and derived the model-implied VIX index under LRNVR. When the models are estimated and risk neutralized with returns data only, the model implied VIX will be significantly lower then the market CBOE VIX. Even when the models are jointly estimated with both returns and VIX, the parameters tend to be seriously distorted, for example, a very large positive price for equity risk parameter, and the model implied VIX still tend to underestimate and cannot match the CBOE VIX from various statistical aspects. They further provide the theoretical explanation by showing that the diffusion limit of the GARCH-type models under the LRNVR risk neutral measure fail to fully compensate for taking volatility risk. More precisely, GARCH-type models have only a single shock, i.e. the return shock, and thus have no room for an independent adjustment for the volatility shock.

Basically, there are two directions to alleviate this problem: adopting a more complicated way of risk neutralization or allowing independent volatility shocks in the modeling framework. Christoffersen et al. (2013) introduced a variance dependent pricing kernel to improve the option pricing performance of Heston-Nandi GARCH model, while keeping the closed-form option pricing formula. Stochastic volatility models follow the second solution by explicitly modeling volatility shocks, with the cost of more difficult model estimation due to the latent volatility process.

Realized EGARCH model, proposed by Hansen et al. (2012) and further developed Hansen and Huang (2014), includes a second shock by including the realized measure of volatility. The dual-shock nature of the model enables greater flexibility when risk neutralizing the model from the physical measure to the risk neutral measure. Contrast with stochastic volatility models, Realized EGARCH model maintains the observation-driven structure of GARCH framework and thus model parameters can be easily estimated via (quasi) maximum likelihood estimator.

In this paper, we first derive the closed-form VIX pricing formula for Realized EGARCH model, with an exponentially affine stochastic discount factor of Christoffersen et al. (2010). Then we show that Realized EGARCH model generates the best in-sample fit for the variance risk premium suggested by CBOE VIX, compared with other five models including EGARCH, GARCH, Heston-Nandi GARCH under both LRNVR and the variance dependent pricing kernel. In the rolling-window out-of-sample forecasting study, Realized EGARCH model also provides the best forecasting performance. The improvement is more significant during the recent turmoil periods.

2 Realized EGARCH model

Hansen et al. (2012)proposed the Realized GARCH model that incorporates the realized measures of volatility into the conventional GARCH framework. The realized measures of volatility are calculated from high frequency data, for example, five-minute realized variance (RV) or realized kernel (RK) of Barndorff-Nielsen et al. (2008) that uses tick trading data. Hansen et al. (2012) demonstrated that Realized GARCH outperformed GARCH family models in terms of modeling and forecasting volatility. The improved performance comes from the fact that the realized measures based on high frequency data contain more accurate information about volatility than squared daily returns used in conventional GARCH-type models. Hansen and Huang (2014) further extended it to Realized EGARCH model by using a more flexible leverage function and allowing multiple realized measures. Realized EGARCH (1,1) model under the physical measure is given by the following equations

$$r_{t+1} = r + \lambda\sqrt{h_{t+1}} - \frac{1}{2}h_{t+1} + \sqrt{h_{t+1}}z_{t+1} \tag{1}$$

$$\log h_{t+1} = \omega + \beta \log h_t + \tau(z_t) + \gamma\sigma u_t \tag{2}$$

$$\log x_t = \xi + \phi \log h_t + \delta(z_t) + \sigma u_t \tag{3}$$

where r_t is the rate of return, λ is the price for taking equity risk, h_t is the conditional variance, r is the risk-free interest rate and x_t is the realized measure of volatility. z_t and u_t are assumed bi-variate standard Gaussian. The quadratic functions$\tau(z) = \tau z + \tau_2(z^2 - 1)$ and $\delta(z) = d_1 z + d_2(z^2 - 1)$ are referred as the leverage function, accounting for dependence between return shocks and volatility shocks, which is empirically important. When $\gamma = 0$, the Realized EGARCH model shrinks to a variant of the classical EGARCH model of Nelson (1991), with a squared term replacing the absolute value term in the original EGARCH model[1].

[1]See Hansen et al. (2012) for the argument in favor of the quadratic form leverage function. For comparison reason,

Equation (1) and (2) define the joint dynamics of return and volatility process, which looks very similar to that in discrete-time stochastic volatility models, such as Taylor (1986) and Kim et al. (1998). The feature is substantial for model's capacity of accommodating variance risk premium. In conventional GARCH family models, volatility shock is introduced via some function of lagged daily return. As there is no independent volatility shock term, the variance risk is only reflected through this dependence with equity risk. Therefore, in order to explain the market level of variance risk premium, the equity risk premium parameter λ will typically be inflated to an unreasonable level. Such pattern have been documented in Hao and Zhang (2013) and also confirmed in this paper. Realized EGARCH model does not suffer from this problem, due to the introduction of residual volatility shock u_t, the proportion of the volatility shock that can not been captured by z_t.

Realized EGARCH models are fundamentally different from stochastic volatility models in terms of information updating and model estimation, due to the introduction of Equation (3), the measurement equation. By substituting u_t of the variance equation with the measurement equation, the conditional volatility is actually updated by new information from both the realized standardized return and the realized measure. As h_t recursively observable, we can derive the explicit form of the joint likelihood function of $(r_t, \log x_t)$ and the maximum likelihood estimation is straightforward. In contrast, for stochastic volatility models, h_t is a latent process and in general it is not likely to derive the explicit likelihood function. Therefore, the generalized method of moment (GMM) or simulation based methods, are commonly used to estimate stochastic volatility models.

$$\log h_{t+1} = (\omega - \gamma\xi) + (\beta - \gamma\phi)\log h_t + (\tau(z_t) - \gamma\delta(z_t)) + \gamma \log x_t$$

2.1 Risk neutralization

In order to derive the model implied CBOE VIX, we need to derive the model dynamics under the risk neutral measure. Following Christoffersen et al. (2010), we choose the following exponentially affine stochastic discount factor (SDF) for risk neutralization,

$$Z_{t+1} = \frac{\exp(v_{1,t}z_{t+1} + v_{2,t}u_{t+1})}{E(\exp(v_{1,t}z_{t+1} + v_{2,t}u_{t+1}))} = \exp\left(v_{1,t}z_{t+1} + v_{2,t}u_{t+1} - \frac{v_{1,t}^2}{2} - \frac{v_{2,t}^2}{2}\right)$$

Non-arbitrage condition requires that the expected rate of return under the risk-neutral measure is just the risk-free rate.

$$\begin{aligned} E_t^Q(\exp(r_{t+1})) &= E_t(Z_{t+1}\exp(r_{t+1})) \\ &= \exp(r + \lambda\sqrt{h_{t+1}} + v_{1,t}\sqrt{h_{t+1}}) \\ &= \exp(r) \end{aligned}$$

we also estimated a Realized EGARCH model with $\tau_1 z_t + \tau_2(|z_t| - \sqrt{2/\pi})$, the exact functional form in EGARCH model. We got very similar empirical results and our main conclusions still hold.

i.e., it must be

$$v_{1,t} = -\lambda$$

To obtain the model dynamics under Q-measure, we derive the moment generating function (MGF) as below

$$\begin{aligned} E_t^Q(\exp(s_1 z_{t+1} + s_2 u_{t+1})) &= E_t(Z_{t+1}\exp(s_1 z_{t+1} + s_2 u_{t+1})) \\ &= \exp\left(-s_1\lambda + s_2 v_{2,t} + \frac{s_1^2}{2} + \frac{s_2^2}{2}\right) \end{aligned}$$

It suggests a way of changing measure from the physical world to the risk-neutral world,

$$\begin{aligned} z_{t+1}^* &= z_{t+1} + \lambda \\ u_{t+1}^* &= u_{t+1} - v_{2,t} \end{aligned}$$

Following Christoffersen et al. (2010), we just assume a time-invariate volatility risk premium $\alpha_2 = v_{2,t}$ to make sure the model remains the same dynamic structure under Q-measure. A positive value of α_2 suggests the negative variance risk premium.

Hence, the risk neutral dynamics of the model is

$$\begin{aligned} r_{t+1} &= r - \frac{1}{2}h_{t+1} + \sqrt{h_{t+1}}z_{t+1}^* && (4) \\ \log h_{t+1} &= \omega + \beta \log h_t + \tau_1(z_t^* - \lambda) + \tau_2((z_t^* - \lambda)^2 - 1) + \gamma\sigma(u_t^* + \alpha_2) && (5) \\ \log x_t &= \xi + \phi \log h_t + d_1(z_t^* - \lambda) + d_2(((z_t^* - \lambda)^2 - 1) + \sigma(u_t^* + \alpha_2) && (6) \end{aligned}$$

where $(z_t^*,\ u_t^*)$ has joint bi-variate standard Gaussian distribution. Here we can see the way of changing measure is straightforward and could be treated as a bi-variate extension of Duan (1995)'s LRNVR.

2.2 The VIX pricing formula

From the risk neutral dynamics, we have

$$\log h_{t+1} = \tilde{\omega} + \beta \log h_t + v_t$$

by defining $\tilde{\omega} = \omega + \alpha_2\tilde{\sigma}$ and $\tilde{\sigma} = \gamma\sigma$

$$v_t = \tau_1(z_t^* - \lambda) + \tau_2((z_t^* - 1)^2 - 1) + \tilde{\sigma}u_t^*$$

The k step forward expected conditional variance is

$$\begin{aligned} E_t^Q[h_{t+k}] &= E_t^Q\left[\exp\left(\beta^{k-1}\log h_{t+1}\right)\exp\left(\sum_{i=0}^{k-2}\omega\beta^i\right)\exp\left(\sum_{i=0}^{k-2}\beta^{k-2-i}v_{t+1+i}\right)\right] \\ &= h^{\beta^{k-1}}\prod_{i=0}^{k-2}e^{\beta^i\tilde{\omega}}E_t^Q\left[e^{\beta^i v_{t+k-1-i}}\right] \end{aligned}$$

Let $F_i = e^{\beta^i\tilde{\omega}}E_t^Q\left[e^{\beta^i v_{t+k-1-i}}\right]$, suppress star and t on z and u. We have

$$F_i = e^{\beta^i\tilde{\omega}}E_t^Q\left[\exp\left(\beta^i\tau_2\lambda^2 - \beta^i\tau_1\lambda - \beta^i\tau_2 + \beta^i\left(\tau_2 z^2 - (2\tau_2\lambda - \tau_1)\,z\right)\right)\exp\left(\beta^i\tilde{\sigma u}\right)\right] \tag{7}$$

$$= e^{\beta^i\tilde{\omega}}\exp\left[\beta^i\tau_2\lambda^2 - \beta^i\tau_1\lambda - \beta^i\tau_2 + \frac{\beta^{2i}\tilde{\sigma}^2}{2}\right]E_t^Q\exp\left(\beta^i\left(\tau_2 z^2 - (2\tau_2\lambda - \tau_1)\,z\right)\right) \tag{8}$$

$$= e^{\beta^i\tilde{\omega}}\exp\left[\beta^i\tau_2\lambda^2 - \beta^i\tau_1\lambda - \beta^i\tau_2 + \frac{\beta^{2i}\tilde{\sigma}^2}{2} - \frac{\beta^i\left(\tau_1 - 2\tau_2\lambda\right)^2}{4\tau_2}\right]E_t^Q\left[\beta^i\tau_2\left(z + \frac{(\tau_1 - 2\tau_2\lambda)}{2\tau_2}\right)^2\right] \tag{9}$$

The last term in the third equation is essential the MGF of the non-central chi-square distribution. Therefore:

$$E_t^Q\left[\beta^i\tau_2\left(z + \frac{(\tau_1 - 2\tau_2\lambda)}{2\tau_2}\right)^2\right] = \frac{1}{\sqrt{1 - 2\beta^i\tau_2}}\exp\left[\frac{\beta^i(\tau_1 - 2\tau_2\lambda)^2}{2(1 - 2\beta^i\tau_2)\tau_2}\right] \tag{10}$$

Substitute (10) into (9), we have

$$F_i = \frac{1}{\sqrt{1 - 2\beta^i\tau_2}}\exp\left[\beta^i\tilde{\omega} + \beta^i\tau_2(\lambda^2 - 1) - \beta^i\tau_1\lambda + \frac{\beta^{2i}(\tau_1 - 2\tau_2\lambda)^2}{2(1 - 2\beta^i\tau_2)} + \frac{\beta^2\tilde{\sigma}^2}{2}\right]$$

Therefore, the model implied VIX is given by

$$VIX_t = \sqrt{\frac{252}{22}\sum_{k=1}^{22}E_t^Q(h_{t+k})} \times 100 = \sqrt{\frac{252}{22}\left[h_{t+1} + \sum_{k=2}^{22}\left(\prod_{i=0}^{k-2}F_i\right)h_{t+1}^{\beta^{k-1}}\right]} \times 100 \tag{11}$$

3 Models in comparison

To evaluate the ability of Realized EGARCH to fit and forecast the market CBOE VIX index, we include five models for comparison, i.e. Realized GARCH model with log-linear specification, GARCH, EGARCH, Heston-Nandi GARCH under LRNVR, Heston-Nandi GARCH under variance dependent pricing kernel proposed by Christoffersen et al. (2013)[2].

[2]Hao and Zhang (2013) examined GARCH, EGARCH, TGACH, AGARCH and CGARCH models. To save space, we only keeps GARCH and EGARCH models for comparison because GARCH is the benchmark and EGARCH performs the best among the five models in Hao and Zhang (2013)

3.1 Realized GARCH

The Realized GARCH model with log-linear specification is:

$$r_{t+1} = r + \lambda\sqrt{h_{t+1}} - \frac{1}{2}h_{t+1} + \sqrt{h_{t+1}}z_{t+1} \tag{12}$$

$$\log h_{t+1} = \omega + \beta \log h_t + \gamma \log x_{t-1} \tag{13}$$

$$\log x_t = \xi + \phi \log h_t + d_1 z_t + d_2(z_t^2 - 1) + \sigma u_t \tag{14}$$

Realized GARCH can be viewed as a constrained version of Realized EGARCH. By insert equation (14) to equation (13) we have

$$\log h_{t+1} = \tilde{\omega} + \tilde{\beta} \log h_t + \tilde{d}_1 z_t + \tilde{d}_2(z_t^2 - 1) + \gamma\sigma u_t \tag{15}$$

where $\tilde{\omega} = \omega + \gamma\xi$, $\tilde{\beta} = \beta + \gamma\phi$, $\tilde{d}_. = \gamma d_.$. It is clear that although Realized GARCH model can also have a leverage function in variance equation, its coefficients are, however, constrained to be proportional to those for leverage function in measurement equation. The latter leverage function is included just to model correlations between return and volatility shock.

The model generated VIX can be calculate with the help of the formula for Realized EGARCH and we will not repeat it here, to save space.

3.2 GARCH and EGARCH model

GARCH and EGARCH model are commonly used as benchmarks of GARCH family models. Unlike the linear GARCH model, EGARCH model is a log-linear model and the exponential function makes it easier to adjust to drastic volatile changes. With the same return equation as defined above, GARCH and EGARCH specify the following volatility equations respectively,

- GARCH

$$h_{t+1} = \omega + \beta h_t + \alpha h_t z_t^2$$

- EGARCH

$$\log h_{t+1} = \omega + \beta \log h_t + \tau_1 z_t + \tau_2(|z_t| - \sqrt{2/\pi})$$

Using the results in Hao and Zhang (2013) (Prop.1-4 in their paper), the model implied VIX pricing formula is given by

GARCH

$$VIX_t = \sqrt{\frac{252}{22}\sum_{k=1}^{22} E_t^Q(h_{t+k})} \times 100 = \sqrt{252\sigma_h^2 + \frac{252}{22}\frac{1-\beta^{22}}{1-\beta}(h_{t+1} - \sigma_h^2)} \times 100$$

where $\sigma_h^2 = \omega/(1-\beta)$.

- EGARCH

$$VIX_t = \sqrt{\frac{252}{22}\left[h_{t+1} + \sum_{k=2}^{22}\left(\prod_{i=0}^{k-2} F_i\right) h_{t+1}^{\beta^{k-1}}\right]} \times 100$$

where

$$\begin{aligned} F_i &= \exp\left[\beta\left(\omega - \tau_2\sqrt{\frac{2}{\pi}}\right)\right]\{\exp\left[-\beta^i(\tau_1-\tau_2)\lambda + \frac{\beta^{2i}(\tau_2-\tau_1)^2}{2}\right]\Phi[\lambda - \beta^i(\tau_1-\tau_2)] \\ &\quad + \exp\left[-\beta^i(\tau_1+\tau_2)\lambda + \frac{\beta^{2i}(\tau_2+\tau_1)^2}{2}\right]\Phi[\beta^i(\tau_1+\tau_2)-\lambda]\} \end{aligned}$$

3.3 Heston-Nandi GARCH model under LRNVR

The Heston-Nandi GARCH model (Heston and Nandi (2000)) is one of the most popular discrete-time option pricing model. It specifies different return and volatility equations to ensure closed-form option pricing formula.

$$\begin{aligned} r_{t+1} &= r + \lambda h_{t+1} - \frac{1}{2}h_{t+1} + \sqrt{h_{t+1}}z_{t+1} \\ h_{t+1} &= \omega + \beta h_t + \alpha(z_t - \delta\sqrt{h_t})^2 \end{aligned}$$

The LRNVR risk neutralization follows yields a risk neutral dynamics by letting $z_t^* = z_t + \lambda\sqrt{h_t}$,

$$\begin{aligned} r_{t+1} &= r - \frac{1}{2}h_{t+1} + \sqrt{h_{t+1}}z_{t+1}^* \\ h_{t+1} &= \omega + \beta h_t + \alpha(z_t^* - (\delta+\lambda)\sqrt{h_t})^2 \end{aligned}$$

It is easy to see that the unconditional mean of h_t is $\sigma_h^2 = \tilde{\omega}/(1-\tilde{\beta})$, where $\tilde{\omega} = \omega+\alpha$, $\tilde{\beta} = \beta+\alpha(\delta+\lambda)^2$. Therefore, the k step forward expected conditional variance is

$$E_t^Q(h_{t+k}) = \sigma_h^2 + \tilde{\beta}^{k-1}(h_{t+1} - \sigma_h^2)$$

Therefore, the model implied VIX can be expressed as

$$VIX_t = \sqrt{\frac{252}{22}\sum_{k=1}^{22} E_t^Q(h_{t+k})} \times 100 = \sqrt{252\sigma_h^2 + \frac{252}{22}\frac{1-\tilde{\beta}^{22}}{1-\tilde{\beta}}(h_{t+1} - \sigma_h^2)} \times 100$$

3.4 Heston-Nandi GARCH under variance dependent pricing kernel

In line with stochastic volatility models, Christoffersen et al. (2013) introduced a variance dependent pricing kernel to improve the option pricing performance of Heston-Nandi GARCH model. Unlike

LRNVR, now the one-step conditional volatility does no remains the same after the measure change. Citing the results in Christoffersen et al. (2013), we can write the risk neutral dynamics as below

$$\begin{aligned} r_{t+1} &= r - \frac{1}{2}h^*_{t+1} + \sqrt{h^*_{t+1}}z^*_{t+1} \\ h^*_{t+1} &= \omega^* + \beta^* h^*_t + \alpha^*(z^*_t - \delta^*\sqrt{h^*_t})^2 \end{aligned}$$

where z^*_t has a standard normal distribution and

$$\begin{aligned} h^*_t &= h_t/(1-2\alpha\alpha_2) \\ \omega^* &= \omega/(1-2\alpha\alpha_2) \\ \alpha^* &= \alpha/(1-2\alpha\alpha_2)^2 \\ \delta^* &= (\lambda+\delta-1/2)(1-2\alpha\alpha_2)+1/2 \end{aligned}$$

where α_2 is the free parameter included in the variance dependent pricing kernel[3]. Following the similar procedures for GARCH model, the model implied VIX can be expressed as

$$VIX_t = \sqrt{252\sigma_h^{*2} + \frac{252}{22}\frac{1-\tilde{\beta^*}^{22}}{1-\tilde{\beta^*}}(h^*_{t+1} - \sigma_h^{*2})}$$

where $\sigma_h^{*2} = \tilde{\omega^*}/(1-\tilde{\beta^*})$ and $\tilde{\omega^*} = \omega^* + \alpha^*$, $\tilde{\beta^*} = \beta^* + \alpha^*\delta^{*2}$. As suggested in Christoffersen et al. (2013), α_2 is estimated under the transformation $1/(1-\alpha\alpha_2)$, Unlike LRNVR, the variance dependent kernel will induce a transformation of one-step-ahead conditional variance after the change of measure.

4 Empirical Results

4.1 Model Estimation

We use maximum likelihood (ML) method to estimate model parameters by maximizing the joint log-likelihood functions of observed variables in the physical dynamics and the risk neutral dynamics (CBOE VIX index)[4]. The first part involves the joint log-likelihoods of both returns and realized measures for Realized EGARCH/Realized GARCH, and only return likelihood for other models.

$$l_P = \sum_{t=1}^{T} \log f(r_t, \log x_t) \;\; or \;\; = \sum_{t=1}^{T} \log f(r_t)$$

Since we have obtained the model implied VIX formula, we can construct a likelihood function related to model's goodness-of-fit to the market CBOE VIX index. By assume that the error between

[3]In Christoffersen et al. (2013), this free parameter is denote as ξ, we rename it as α_2 because it takes the same job as the free parameter we used in our Realized GARCH model.

[4]Although it is not a must to include VIX data in the estimation process for GARCH, EGARCH and original Heston-Nandi GARCH model, VIX information will significantly improves those model's fitting of VIX index. It is worth to point out that this improvement does not necessarily mean that those model's ability of describing the return dynamics is also improved. In fact, such procedure will distort those models from the underling dynamics

market VIX (VIX^{Market}) and model implied VIX (VIX^{Model}) to be a normal distribution $N(0, \sigma_e^2)$, the log-likelihood for VIX is given by

$$l_{VIX} = -\frac{T}{2}\log(2\pi) - T\log\sigma_e - \sum_{t=1}^{T}\frac{(VIX_t^{Market} - VIX_t^{Model})^2}{2\sigma_e^2}$$

The full log-likelihood function is therefore the sum of the two parts, reflecting the balance between the model's goodness-of-fit for both physical and risk neutral dynamics. Such joint-likelihood method is widely used in recent option pricing literature such as Christoffersen et al. (2012), Christoffersen et al. (2013), Hao and Zhang (2013) etc.

4.2 Data

Our empirical analysis is based on daily data for S&P 500 stock index. The full-sample starts from July 2003 to June 2013, spanning a thirteen-year period. We obtain the daily VIX index from the CBOE website, daily return and realized kernel (RK) from the Realized Library at Oxford-Man Institute. For out-of-sample forecast, we use a 750-day rolling window method on daily basis. The three years of observations from July 2000 to June 2003 are used as pre-sample to get the first parameter for the out-of-sample analysis. We also divide the full-sample into two sub-sample periods, using the beginning of 2008 as the break-point, to check if the model performances are different before and after the financial crisis.

4.3 Empirical Results

Throughout this section, we use the following abbreviations: REG for Realized EGARCH, RG for Realized GARCH, EG for EGARCH, G for GARCH, HNG for Heston-Nandi GARCH and HNG-V for Heston-Nandi GARCH with variance dependent pricing kernel.

4.3.1 Full-sample parameter estimation

In this section, we present the full-sample estimate of parameters for different models based on joint estimation. Results are listed in Table 4.3.1.

The most notable finding in Table 4.3.1 is the different estimates of λ, the market price for equity risk. They are all positive, which is consistent with the positive equity risk premium, but the magnitudes are much smaller in Realized EGARCH and Realized GARCH models. Our estimates for EGARCH and GARCH models are very close to that in Hao and Zhang (2013) and the values are much larger than the parameters when the models are estimated with only returns data. The underlying reason is that the positiveλ has to be inflated so much to account for the negative volatility risk premium. The estimates for λ are even larger for the two Heston-Nandi GARCH models, although using a variance dependent pricing kernel gives a smaller estimate. We do not see that happening in the Realized EGARCH and

Realized GARCH models, when the volatility shocks are explicitly introduced via the measurement equation.

The persistence parameters are close to one for all of the models, indicating a high persistent in physical and risk neutral process of volatility. The estimates for τ_1 and d_1 are negative for both Realized EGARCH, Realized GARCH and EGARCH models, suggesting a strong negative correlation between return and volatility shocks. This finding is consistent with the so-called leverage effect, that is, a large negative return is usually followed by an increase in volatility.

The values of the optimized log-likelihood functions measure the model's goodness-of-fit for the observed data, in terms of the distribution. We present the log-likelihoods for r_t, VIX_t, (r_t, VIX_t), and (r_t, VIX_t, RK_t). In all cases, Realized EGARCH gives the best fit, followed by Realized GARCH and then EGARCH. The gain of Realized EGARCH over Realized GARCH models are statistically significant, when the conventional likelihood ratio tests are applied.

4.3.2 Conmparison of actual and model implied VIX

Table 2 presents the full-sample fit of the CBOE VIX index across different models. Comparisons are made based the goodness-of-fit between the model implied VIX and market VIX and their other summary statistics.

Realized EGARCH model consistently provides the smallest bias, standard deviation, mean squared error and mean absolute error, followed by Realized GARCH, with a little underestimation. EGARCH, GARCH and HNG, with LRNVR, tend to underestimate VIX, whiling HNG-V tend to overestimate to some extent. The correlation between model generated VIX and market VIX are highest for REG/RG and lowest for two Heston-Nandi GARCH models. Comparing other statistical measures, Realized EGARCH model also give close results to that of the market VIX.

Among the models without realized measures, EGARCH seems the best model and GARCH gives the worst performance. Although HNG model is popular in option pricing practice due to the quasi-analytical formula, it seems that there is not a large improvement in handling VIX fit.

To evaluate the model's performance during different sample periods, we also estimate the models for the two sub-samples divided by the start of 2008 and present their goodness-of-fit measures in Table 3. For both sub-samples, Realized EGARCH and Realized GARCH are still the best models, which slightly estimate VIX during the first relatively calm period and slightly overestimate during the second relatively turmoil sub-sample. On average, the goodness-of fit for all models are better during the first calm sub-sample. HNG-V outperforms HNG in turmoil period. more substantially, indicating the volatility dependent pricing kernel works better in a more volatile situation.

4.3.3 Conmparison of actual and model implied variance risk premium

In this section we compare the actual and model implied and variance risk premium series. The model implied variance risk premium at time t is defined as the difference between the model implied VIX at time t and the annualized 22 days averaged fitted volatility. i.e.

$$VRP_t^{Model} = \left(\sqrt{\frac{252}{22} \sum_{i=1}^{22} E_t^Q(h_{t+i})} - \sqrt{\frac{252}{22} \sum_{i=1}^{22} E_{t+i-1}^P(h_{t+i})} \right) \times 100$$

Notice that the model implied variance risk premium is not a concept of forecast since it bears the looking backward bias. The market counterpart, the actual variance risk premium is calculated with their market counterparts. We want to investigate whether the models are able to match the observed levels of variance risk premium throught the sample period.

$$VRP_t^{Actual} = \left(VIX_t - \sqrt{\frac{252}{22} \times IF \times \sum_{i=1}^{22} RK_{t+i}} \right) \times 100$$

where the inflated parameter (IF) is defined as $IF = Var(ret)/Mean(RK)$ which is designed to transform open-to-close volatility to close-to-close volatility.

Table 1 plot the time seris of the actual and model implied variance risk premium. It is obvious that models realized measures provide the best match. Garch and HNG give the poorest performance. Using the variance dependent kernel seems to improve the performance of Heston-Nandi GARCH model.

4.3.4 Out-of-sample forecasting comparison

We also conduct out-of-sample rolling window forecasting comparison, to examine if the fit improvements in the previous section are due to in-sample over-fit. We present the bias, MAE, RMSE for the three sample periods in Table 4, and DM statistics are also reported to evaluate if the improvements are statistically significant. In all of the three sample periods, Realized EGARCH provides the best out-of-sample fit, whether MAE or RMSE is used as the comparison standard. During the turmoil period, the improvement is more substantial and likely to be statistically significant. Due to possible outliers of HNG-V, the difference is not significant at 5% level (it is significant at 10% level). Unlike the in-sample results, Realized EGARCH tend to overestimate the VIX index during calm period and underestimate during the more volatile sub-sample. Its average out-of-sample forecasting bias is positive.

HNG and HNG-V are not good under both sub-samples. The volatility dependent pricing kernel does make HNG model better but not good enough. One possible reason is that the information is linearly rather than exponentially depends on standardized shock for HNG than log-linear models REG/RG/EG. Those results indicate that if a linear model is used, it is better to use non-standardized shocks like the original GARCH model.

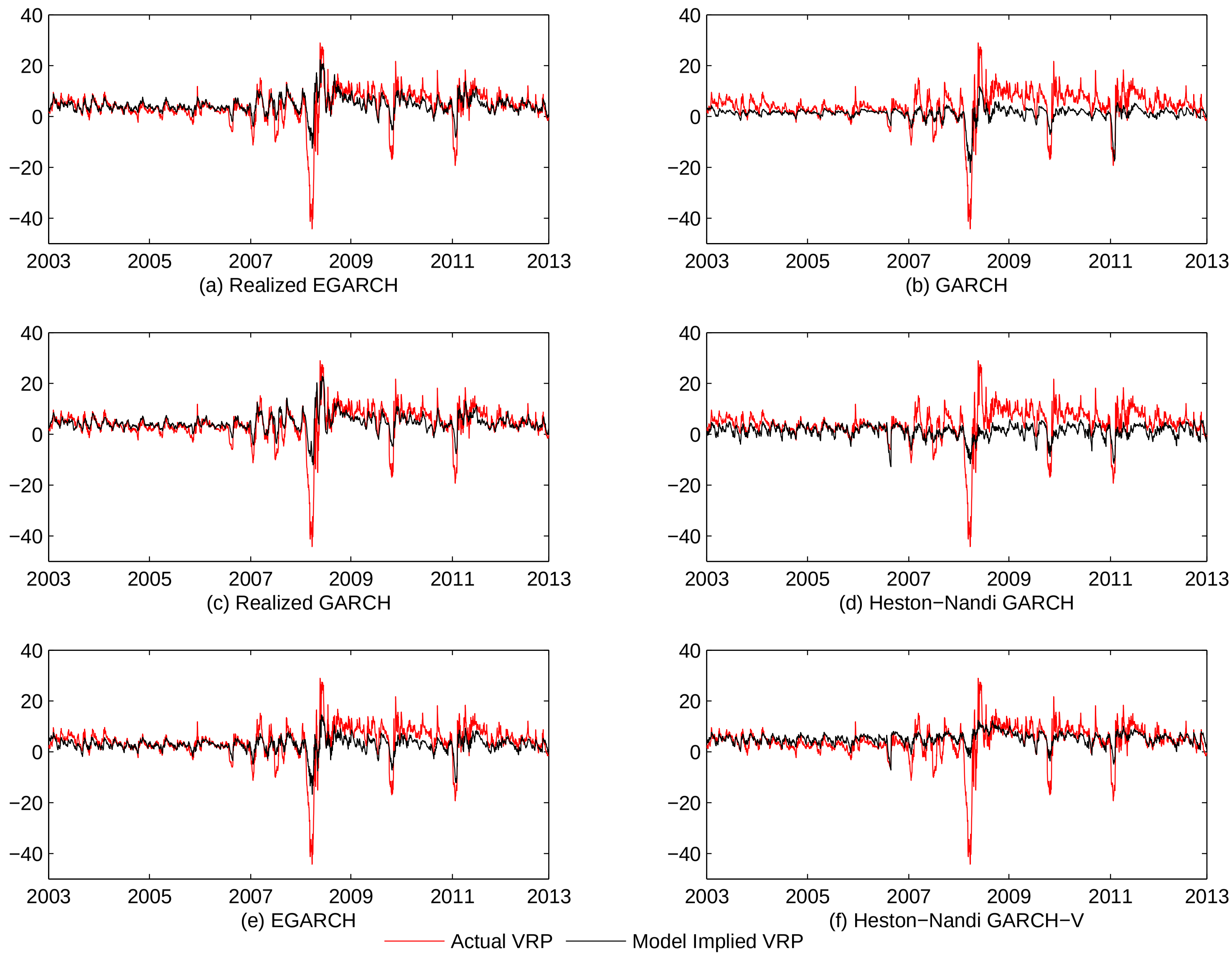

Figure 1: Actual and Model Implied Variance Risk Premium

5 Conclusion

We show that the Realized EGARCH model proposed in Hansen et al. (2012) and Hansen and Huang (2014), under an exponentially affine stochastic discount factor, generates reasonable levels of variance risk premium, which is not possible for other conventional GARCH-family models. We derive the closed-form VIX pricing formula and find empirical evidence that Realized EGARCH model provides significantly better forecasting performance for VIX index than a number of GARCH volatility models, both in-sample and out-of-sample. To sum, Realized EGARCH shares all of the benefits below: 1) it exploits information contained in realized measures of volatility; 2) a flexible leverage function that accounts for return-volatility dependence; 3)whiling remaining its GARCH-like modeling framework and estimation convenience, the model allows independent return and volatility shock and this dual shock nature leaves a room for variance risk premium. 4) combined with a popular pricing kernel, the model generates analytical pricing form of CBOE VIX index, that is, the expected average integrated variance under the risk neutral measure.

References

Andreou, E., E. Ghysels, and A. Kourtellos (2013): "Should macroeconomic forecasters use daily financial data and how?" *Journal of Business and Economic Statistics*, 31, 240–251.

Bakshi, G. and N. Kapadia (2003): "Delta-Hedged Gains and the Negative Market Volatility Risk Premium," *Review of Financial Studies*, 16, 527–566.

Barndorff-Nielsen, O. E., P. R. Hansen, A. Lunde, and N. Shephard (2008): "Designing Realized Kernels to Measure the ex post Variation of Equity Prices in the Presence of Noise," *Econometrica*, 76, 1481–1536.

Bekaert, G., M. Hoerova, and M. Lo Duca (2013): "Risk, uncertainty and monetary policy," *Journal of Monetary Economics*, 60, 771–788.

Bollerslev, T., J. Marrone, L. Xu, and H. Zhou (2013): "Stock return predictability and variance risk premia: statistical inference and international evidence," *Journal of Financial and Quantitative Analysis*, Forthcoming.

Bollerslev, T., G. Tauchen, and H. Zhou (2009): "Expected Stock Returns and Variance Risk Premia," *Review of Financial Studies*, 22, 4463–4492.

Carr, P. and L. Wu (2009): "Variance Risk Premiums," *Review of Financial Studies*, 22, 1311–1341.

Christoffersen, P., R. Elkamhi, B. Feunou, and K. Jacobs (2010): “Option Valuation with Conditional Heteroskedasticity and Non-Normality,” *Review of Financial Studies*, 23(5), 2139–2183.

Christoffersen, P., B. Feunou, K. Jacobs, and N. Meddahi (2012): “The Economic Value of Realized Volatility: Using High-Frequency Returns for Option Valuation,” *Journal of Financial and Quantitative Analysis, forthcoming.*

Christoffersen, P., S. L. Heston, and K. Jacobs (2013): “Capturing Option Anomalies with a Variance-Dependent Pricing Kernel,” *Review of Financial Studies*, 26(8), 1963–2006.

Coval, J. and T. Shumway (2001): “Expected option returns,” *Journal of Finance*, 56, 983–1009.

Duan, J.-C. (1995): “The GARCH option pricing model,” *Mathematical Finance*, 5(1), 13–32.

Hansen, P. R. and Z. Huang (2014): “Exponential GARCH Modeling with Realized Measures of Volatility,” *Working Paper*.

Hansen, P. R., Z. Huang, and H. H. Shek (2012): “Realized GARCH: A Complete Model of Returns and Realized Measures of Volatility,” *Journal of Applied Econometrics*, 27(6), 877–906.

Hao, J. and J. E. Zhang (2013): “GARCH Option Pricing Models, the CBOE VIX, and Variance Risk Premium,” *Journal of Financial Econometrics*, 11, 556–580.

Heston, S. L. and S. Nandi (2000): “A Closed-Form GARCH Option Valuation Model,” *Review of Financial Studies*, 13, 585–625.

Kim, S., N. Shephard, and S. Chib (1998): “Stochastic Volatility: Likelihood Inference and Comparison with ARCH Models,” *Review of Economic Studies*, 65, 361–93.

Londono, J. M. and H. Zhou (2012): “Variance Risk Premiums and the Forward Premium Puzzle,” *International Finance Discussion Papers 1068*, Board of Governors of the Federal Reserve System (U.S.).

Nelson, D. B. (1991): “Conditional Heteroskedasticity in Asset Retruns: A New Approach,” *Econometrica*, 59, 347–370.

Taylor, S. (1986): *Modelling Financial Time Series.*, Chichester: John Wiley.

Zhang, B. Y., H. Zhou, and H. Zhu (2009): “Explaining Credit Default Swap Spreads with the Equity Volatility and Jump Risks of Individual Firms,” *Review of Financial Studies*, 22, 5099–5131.

Table 1: Parameter estimation with full sample

Model	REG	RG	EG	G	HNG	HNGV
λ	0.029	0.031	0.232	0.334	9.125	2.356
	(0.011)	(0.021)	(0.024)	(0.019)	(0.834)	(1.469)
ω	-0.067	-0.098	-0.001	1.68×10^{-6}	-2.09×10^{-6}	-1.39×10^{-6}
	(0.011)	(0.048)	(0.004)	(1.25×10^{-7})	(2.70×10^{-7})	(5.92×10^{-8})
β	0.993	0.871	0.990	0.944	0.906	0.943
	(0.001)	(0.058)	(4.52×10^{-4})	(0.004)	(0.006)	(0.002)
α				0.049	4.37×10^{-6}	1.96×10^{-6}
				(0.004)	(3.94×10^{-7})	(1.67×10^{-8})
δ					134.154	159.143
					(9.065)	(4.220)
τ_1	-0.066		-0.071			
	(0.004)		(0.003)			
τ_2	0.011		0.088			
	(0.002)		(0.005)			
γ	0.064	0.114				
	(0.006)	(0.053)				
ξ	0.388	0.193				
	(0.272)	(0.354)				
ϕ	1.080	1.060				
	(0.029)	(0.039)				
d_1	-0.085	-0.075				
	(0.010)	(0.014)				
d_2	0.120	0.125				
	(0.011)	(0.013)				
σ_u^2	0.263	0.279				
	(0.010)	(0.025)				
α_2	1.405	0.430				1.221
	(0.163)	(0.194)				(0.036)
π^P	0.993	0.992	0.990	0.994	0.984	0.993
$l(r_t)$	8232.57	8218.32	8126.90	7994.82	8074.33	8127.13
$l(VIX_t)$	-5905.39	-6168.79	-6335.45	-6480.41	-6894.29	-6930.90
$l(r_t, VIX_t)$	2327.18	2049.53	1791.45	1514.41	1180.05	1196.23
$l(r_t, VIX_t, \log x_t)$	2327.18	2049.53				

Note: Robust standard errors are in parenthesis. The α_2 reported for HNG-V is the value of $(1-2\alpha\alpha_2)^{-1}$ and the implied value of α_2 is 92305.598. The persistence parameter π^P is measured by $\beta + \phi\gamma$ for Realized GARCH and β for other models.

Table 2: Summary statistic comparison between model generated VIX and market VIX

Model	Bias	STD	MAE	RMSE	λ	π^P	Corr
REG	0.004	2.557	2.016	2.556	0.029	0.993	0.966
RG	0.015	2.840	2.123	2.840	0.031	0.992	0.957
EG	-0.195	3.029	2.340	3.035	0.232	0.990	0.951
G	-0.346	3.198	2.473	3.216	0.334	0.994	0.946
HNG	-0.069	3.794	2.562	3.793	9.125	0.984	0.926
HNG-V	0.192	3.845	2.580	3.849	2.356	0.993	0.930
Model	**AR1**	**AR10**	**AR30**	**Mean**	**Var**	**Skew**	**Kurt**
REG	0.996	0.926	0.735	20.385	90.030	2.516	10.935
RG	0.997	0.918	0.731	20.397	88.051	2.573	11.372
EG	0.994	0.941	0.779	20.186	87.889	2.377	9.948
G	0.997	0.953	0.789	20.035	89.695	3.041	13.708
HNG	0.994	0.943	0.818	20.312	69.375	1.635	5.595
HNG-V	0.995	0.951	0.831	20.573	61.112	1.674	5.717
VIX	0.982	0.918	0.775	20.381	96.705	2.996	10.074

Note: π^P is the persistence parameter under P-measure: REG/EG(β), G($\beta+\alpha$), HNG($\beta+\alpha\delta^2$), HNG-V($\beta+\alpha\delta$). For VIX index, it is the slope coefficient when an $AR(1)$ model is applied to the time series.

Table 3: Full-sample fit of different models

	REG	RG	EG	G	HNG	HNG-V
Full-sample: 2003/7-2013/6						
Bias	0.004	0.015	-0.195	-0.332	-0.219	0.327
RMSE	2.556	2.840	3.035	3.218	3.801	3.838
MAE	2.016	2.123	2.340	2.477	2.552	2.590
Sub-sample: 2003/7-2007/12						
Bias	0.004	0.005	-0.065	-0.115	-0.301	0.101
RMSE	1.563	1.739	1.766	1.833	1.933	1.877
MAE	1.224	1.344	1.390	1.435	1.297	1.319
Sub-sample: 2008/1-2013/6						
Bias	-0.052	-0.043	-0.255	-0.269	-3.327	0.295
RMSE	2.737	3.230	3.438	3.509	6.271	4.003
MAE	2.140	2.341	2.741	2.681	4.704	2.804

Note: Bias is defined as model implied VIX minus Market VIX.

Table 4: Bias, MAE, RMSE and Diebold-Mariano statistic for VIX forecast

Model	Bias	RMSE	DM Stat.	p-value	MAE	DM stat.	p-value
Full-sample: 2003/7-2013/6							
REG	0.159	2.872	-	-	2.148	-	-
RG	0.521	3.159	2.548	0.005	2.361	2.663	0.004
EG	-0.348	3.449	4.736	0.000	2.590	4.677	0.000
G	0.226	3.360	4.312	0.000	2.497	3.906	0.000
HNG	-0.278	4.433	2.117	0.017	2.916	3.501	0.000
HNG-V	0.338	4.075	1.563	0.059	2.703	2.587	0.005
Sub-sample: 2003/7-2007/12							
REG	0.482	1.941	-	-	1.521	-	-
RG	1.082	2.380	4.340	0.000	1.955	4.408	0.000
EG	-0.428	2.164	1.827	0.034	1.637	1.131	0.129
G	0.637	2.072	1.647	0.050	1.647	1.490	0.068
HNG	0.167	2.428	2.427	0.008	1.809	2.494	0.006
HNG-V	0.349	2.390	2.094	0.018	1.809	2.443	0.007
Sub-sample: 2008/1-2013/6							
REG	-0.106	3.455	-	-	2.662	-	-
RG	0.060	3.678	1.422	0.078	2.695	0.301	0.382
EG	-0.282	4.221	5.113	0.000	3.373	5.713	0.000
G	-0.112	4.128	4.418	0.000	3.196	3.924	0.000
HNG	-0.642	5.563	2.095	0.018	3.826	3.288	0.001
HNG-V	0.329	5.056	1.495	0.068	3.437	2.187	0.014

Note: Bias is defined as model implied VIX minus Market VIX; DM statistic is calculated with Newey-West standard errors. The p-value is for H0: REG is as good as the model in the corresponding row against Ha: REG is better than the model in the corresponding row.