Risky Arbitrage Strategies: Optimal Portfolio Choice and Economic Implications[1]

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Abstract

We define risky arbitrages as self-financing trading strategies that have a strictly positive market price but a zero expected cumulative payoff. A continuous time cointegrated system is used to model risky arbitrages as arising from a mean-reverting mispricing component. We derive the optimal trading strategy in closed-form and show that the standard textbook arbitrage strategy is not optimal. In a calibration exercise, we show that the optimal strategy makes a sizeable difference in economic terms.

1 Introduction

Textbook arbitrages are self-financing trading strategies that have a strictly positive payoff (price) today with a zero cumulative payoff at a known future point in time. An example is simultaneously buying one asset and shorting an equal amount of another asset with the same future payoff but at a higher market price. Provided that the two asset prices converge at a known future date, investors make money up front without incurring any risk. As a consequence, there is no optimal amount to be invested in a riskless arbitrage since investors would want infinitely large positions in this strategy.

In contrast to textbook riskless arbitrage, risky arbitrages are self-financing trading strategies that have a strictly positive payoff today but a zero *expected* future cumulative payoff. One example of risky arbitrage is the well-known case where the shares of Shell and Royal-Dutch traded at different prices despite being claims on the same underlying assets. While the two stock prices could be expected to converge over time, the date where this would occur was not known *ex-ante*. Another example involves simultaneously trading A-shares in the Chinese stock market and H-shares on the Hong Kong stock exchange. Provided that the shares are held in the same firms and thus represent claims on the same assets, their prices can be expected to eventually converge. More generally, investment strategies that fall in the risky arbitrage category and are popular among institutional traders and hedge funds include relative value arbitrage, pairs trading and statistical arbitrage, see, e.g., Bondarenko (2003) and Hogan et al. (2004). Due to transaction costs, limits on capital, and capacity constraints on trading, risky arbitrage opportunities are far more common in practice than their riskless counterparts.

Following earlier studies in the literature, e.g., Gatev, Goetzmann and Rouwenhorst (2006) and Jurek and Yang (2007), in this paper we model risky arbitrages under the assumption that individual asset prices contain a random walk component, but that pairs of asset prices can be cointegrated. Our model captures the presence of a mean-reverting mispricing component that reverts to zero in the long run.[1] Importantly, the evolution of pricing errors is random and the future point in time when they revert to zero is unknown. Compared to the riskless arbitrage case, this corresponds to having a stochastic process for the elimination of pricing errors. As a consequence, and in contrast to riskless arbitrage, risk-averse investors will not hold infinite

[1] Cointegration among asset prices captures cases of statistical arbitrage involving pairs of assets that are very close substitutes (e.g. stock indexes traded through futures contracts or exchange traded mutual funds, see Hasbrouk (2003)) in which mean reversion is likely to be very fast. It also comprises cases where it is more difficult for investors to exploit mispricing and mean reversion is likely to be slow (e.g., Summers (1986)).

positions in a risky arbitrage strategy.

The question naturally arises, therefore, what the optimal trading strategy is under risky arbitrage. It is useful to first consider the standard arbitrage case where a balanced long-short position is held. This reflects that, under textbook riskless arbitrage, the objective is to offset the liability from the asset held in a short position against the payoff on the asset held in the long position. However, this is clearly very different from the objective of utility maximization which in general seeks to maximize the expected return while maintaining an optimal level of risk.

Nevertheless, the academic literature has taken the standard strategy assumed under riskless arbitrage for granted and typically only computes the optimal amount invested in this strategy, see, e.g., Liu and Longstaff (2004) and Jurek and Yang (2007). Industry practice has mirrored this (Khandani and Lo (2007)). However, under risky arbitrage this strategy is not, in general, optimal. To see why the two types of arbitrage may give rise to very different trading strategies note that, in the absence of market frictions, the standard arbitrage strategy always makes money for riskless arbitrages. In contrast, the risky arbitrage strategy may in fact lose money.

To gain intuition, consider the case where one asset is fairly priced while the other is underpriced. The riskless arbitrage strategy is long in the underpriced asset and short the same amount in the fairly priced asset. The underpriced asset has a positive alpha because its price is expected to rise more than is justified by its market exposure, while the fairly priced asset has zero alpha. The underpriced asset will be held long while the fairly priced asset will not be held if its return is independent of that of the underpriced asset after the market exposure is excluded. In this situation the fairly priced asset simply provides an additional source of pure risk without contributing alpha. Thus the optimal risky arbitrage strategy is long in the underpriced asset while holding none of the fairly priced asset. This is very different from the riskless arbitrage strategy which ceases to be optimal here because the fairly priced asset increases the risk without adding a compensating risk premium.

More generally, outside the very special and restrictive ‘symmetric’ case with completely identical assets held in the long and short positions,[2] the optimal long and short positions will not balance out against each other and the standard balanced riskless arbitrage strategy fails to be optimal since it exposes the investor to unnecessary risk. It also causes underinvestment relative to the optimal strategy for generating alpha. Returning to the earlier example with Chinese A-

[2] In particular, symmetry is required in the assets’ factor loadings, idiosyncratic risk levels and their sensitivity to the mispricing component,

shares and Hong-Kong H-shares, very different trading restrictions and market volatility on the two exchanges mean that the optimal positions are not simply one-to-one. This is economically important since the optimal trading strategy can make more efficient use of the risky assets for purposes of controlling risk and avoiding overly large leverage or short positions.

We also demonstrate that the optimal risky arbitrage strategy is not, in general, market neutral. Market neutrality puts unnecessary constraints on the trading strategy and leads to a suboptimal portfolio. The investor should optimize the portfolio of the individual assets without imposing the market neutrality constraint and the resultant market exposure can be eliminated by trading the market index. Furthermore, it is not only the relative mispricing that matters for the optimal portfolio holdings but also whether one or both of the assets are mispriced.

In summary, the contributions of our paper are as follows. First, we derive the optimal trading strategy in closed-form under the assumption that risky arbitrage opportunities arise when asset prices are cointegrated. Second, we show that the textbook arbitrage trading strategy is, in general, suboptimal and only achieves optimality under a set of highly restrictive and stringent conditions. Third, we use a calibration exercise to demonstrate that the loss incurred from following the standard arbitrage strategy can be economically significant.

Our analysis significantly generalizes existing results from the literature on arbitrage. The literature on risky arbitrages assumes that the standard strategy from riskless arbitrage should be used, see, e.g., Liu and Longstaff (2006) and Jurek and Yang (2006). We show that this need not hold in the case with two risky assets and a single risk factor and extend the results to cover an arbitrary number of risky assets and risk factors. Furthermore, our results apply to any form of arbitrage analysis, riskless or risky, which requires utility maximization. Indeed, in the presence of market frictions and limits to arbitrage, our insights also apply to riskless arbitrages since, as frictions become more important, riskless arbitrage effectively becomes risky arbitrage.

The remainder of the paper is organized as follows. Section 2 specifies our model for how asset prices evolve. Section 3 derives the optimal investment strategy in closed form, presents some special cases of particular interest and also characterizes the optimal trading strategy under constraints such as market neutrality. Section 4 considers the utility loss from pursuing suboptimal strategies that impose fixed relative weights (including market neutrality), while Section 5 generalizes our setup to incorporate multiple risk factors and many assets. Section 6 concludes.

2 Asset Prices

There are many reasons for assets to have similar payoffs and for their prices to move closely together. Examples include pairs of stocks that have the same claim to dividends and identical voting rights but are traded in different markets and two firms manufacturing products that are close substitutes. The prices of these assets can be quite different, however, because of mispricing in the markets due to, e.g., random liquidity shocks that cannot be exploited by arbitrageurs in the presence of short-selling costs. Nevertheless, over time, these differences tend to disappear.

In this section we propose a simple model that gives rise to comovements in the prices of such assets. To establish intuition, we initially focus on pairs of individual stocks and a single common risk factor, whereas Section 5 generalizes the model to allow for multiple common risk factors and an arbitrary number of assets. A special case of our model is the type of pairs trading considered by Gatev et al. (2006) whereby a winner stock is shorted and a loser stock is bought, but, as we shall see, our analysis is far more general than this example.

We assume that there is a riskless asset which pays a constant rate of return, r. Furthermore, a risky asset trading at the price P_{mt} represents the market index. We assume that this follows a geometric random walk process, i.e.

$$\frac{dP_{mt}}{P_{mt}} = (r + \mu_m)\, dt + \sigma_m dB_t, \tag{1}$$

where the market risk premium μ_m and market volatility σ_m are both constant and B_t is a standard Brownian motion. Notice that the market index is fairly priced. Papers such as Dumas, Kurshev and Uppal (2007) and Brennan and Wang (2006) assume that the market index is subject to pricing errors. We make no such assumptions here and instead concentrate on mispricing in (pairs of) individual asset prices.

In addition to the risk-free asset and the market index, we assume the presence of two risky assets whose prices P_{it}, $i = 1, 2$, evolve according to the equations

$$\frac{dP_{1t}}{P_{1t}} = (r + \beta_1 \mu_m)\, dt + \beta_1 \sigma_m dB_t + \sigma_1 dZ_t + b_1 dZ_{1t} - \lambda_1 x_t dt; \tag{2}$$

$$\frac{dP_{2t}}{P_{2t}} = (r + \beta_2 \mu_m)\, dt + \beta_2 \sigma_m dB_t + \sigma_2 dZ_t + b_2 dZ_{2t} + \lambda_2 x_t dt, \tag{3}$$

where λ_i, β_i, b_i, and σ_i are constant parameters, Z_t and Z_{it} are standard Brownian motions, and B_t, Z_t, and Z_{it} are all mutually independent for $i = 1, 2$.

In this specification, $\beta_i \sigma_m dB_t$ represents exposure to the market risk while $\sigma_i dZ_t + b_i dZ_{it}$ represents idiosyncratic risks. The presence of a common nonstationary factor is consistent with the equilibrium asset pricing model analyzed by Bossaerts and Green (1989). It is standard to assume that idiosyncratic risks in single market models are independent across different stocks with the market risk representing the only source of correlation among different assets. In our case, both assets are claims on similar fundamentals and so the presence of common idiosyncratic risk, dZ_t, is to be expected.

The final component, x_t, represents pricing errors in our model. Moreover, we assume that there exists a constant α such that the logarithms of the two asset prices $p_{it} = \ln P_{it}$ are cointegrated with cointegrating vector $(1, -\alpha)$, i.e.

$$x_t = p_{1t} - \alpha p_{2t} = \ln \left(\frac{P_{1t}}{P_{2t}^{\alpha}} \right), \tag{4}$$

is stationary.[3] Following Engle and Granger (1987), we refer to x_t as the error-correction term. In common with most of the literature, we take the price processes as exogenously given and so they are not affected by arbitrageurs' attempts at exploiting mispricing.[4]

The two asset prices in our model are correlated both because of their exposure to the same market-wide risk factor (dB_t) and a common idiosyncratic risk (dZ_t) but also due to the mean-reverting error-correction term (x_t) which will induce correlation between the two asset prices even in the absence of the two former components. As an extreme case, when there is no mispricing in either asset, $\lambda_1 = \lambda_2 = 0$ and $\sigma_1 = \sigma_2 = 0$, $\beta_1 = \beta_2$, the two prices are identical.

2.1 Cointegration Dynamics

The terms $-\lambda_1 x_t$ and $\lambda_2 x_t$ play a dual role in the above specification. First, they represent mean excess returns over the normal mean return of $r + \beta\mu_m$. They are the only source of mispricing in our model and thus the only source of abnormal returns for investors. Second, equations (2-4) constitute a continuous-time cointegrated system with $-\lambda_1 x_t$ and $\lambda_2 x_t$ as the error correction terms. Together these terms produce mean reversion that keeps mispricing stationary and pricing errors "small" compared to either of the integrated price processes p_{1t}, p_{2t}. This ensures that, in the words of Chen and Knez (1995), "closely integrated markets should assign to similar payoffs prices that are close" is valid in our model. When the terms $-\lambda_1 x_t$ and $\lambda_2 x_t$ are both absent, the risk premium is determined by $\beta_i \mu_m$ and only the market index

[3] See also Alexander (1999) and Kawasaki et al. (2003) for analyses of trading strategies when asset prices are cointegrated.

[4] See Kondor (2008) for an approach that endogenizes the price process.

and the riskless asset will be held. Neither of the individual risky assets will be held because of their extra idiosyncratic risk terms which go unrewarded.

Other statistical processes could be used to capture temporary deviations from equilibrium prices, including non-linear relations or fractional cointegration, to name a few. Our stylized model is meant to capture essential features of pricing errors while maintaining analytical tractability and allowing us to characterize the optimal trading strategy in closed form. For tractability, and to make the assumption of integrated stock prices more realistic, the cointegrated system is specified in terms of the logarithm of asset prices and not prices themselves.

Next consider the dynamics of the error correction term. It is easy to show that x_t satisfies

$$dx_t = \mu_x dt - \lambda_x x_t dt + \beta_x \sigma_m dB_t + \sigma_x dZ_t + b_x dZ_{xt}. \tag{5}$$

The mean reversion coefficient of x_t is given by

$$\lambda_x = \lambda_1 + \alpha\lambda_2, \tag{6}$$

which we assume is positive to ensure the stationarity of x_t. The mean reversion of x_t captures the temporary nature of any mispricing.

Cointegration only requires that x_t be stationary which holds provided $\lambda_x = \lambda_1 + \alpha\lambda_2 > 0.$ However, the effect of a shock to one of the prices on its own price dynamics may not be as expected. To see this, suppose $\lambda_1 = 2$, $\lambda_2 = -1$, and $\alpha = 1$, so $\lambda_x = 1 > 0$ and x_t is stationary. In this case, if $P_{2t} > P_{1t}$, the error-correction term $-(\ln P_{1t} - \ln P_{2t})$ will drive P_{2t} to be even higher and there is seemingly no mean reversion in the mispricing of P_{2t}. However, the mean reversion in the mispricing of P_{1t} is stronger because $\lambda_1 = -2\lambda_2$, and so there is mean revision in x_t and p_{1t} and p_{2t} are still cointegrated.

The long term mean of x_t is given by μ_x/λ_x, where

$$\mu_x = (1-\alpha)r + (\beta_1 - \alpha\beta_2)\mu_m - \frac{1}{2}\Big(\beta_1^2\sigma_m^2 + \sigma_1^2 + b_1^2 - \alpha(\beta_2^2\sigma_m^2 + \sigma_2^2 + b_2^2)\Big). \tag{7}$$

The mean clearly depends on the $\beta' s$ of the assets. Due to Jensen's inequality, it also depends on the differences of the variances. In many cases we would expect $\mu_x = 0$. A sufficient condition for this to hold is $\alpha = 1$ and symmetry between the other parameters, i.e. $\beta_1 = \beta_2, b_1 = b_2$ and $\sigma_1 = \sigma_2$.

The error correction term, x_t, has exposure to the market risk of

$$\beta_x = \beta_1 - \alpha\beta_2. \tag{8}$$

Finally, the volatility of the common idiosyncratic risk component in x_t is

$$\sigma_x = \sigma_1 - \alpha\sigma_2, \tag{9}$$

while the independent idiosyncratic risk component of x_t is

$$b_x dZ_{xt} = b_1 dZ_{1t} - \alpha b_2 dZ_{2t},$$

where the volatility parameter b_x is given by

$$b_x = \sqrt{b_1^2 + \alpha^2 b_2^2}. \tag{10}$$

This model is quite general. To gain intuition for price dynamics we next consider two polar cases of economic interest. These arise as special cases of our general setup.

2.2 Special Cases

A special case of particular interest arises when only one of the assets is mispriced, while the other is always correctly priced. Although intuition may suggest that only the mispriced asset should be traded, in fact we show that due to the correlation between the two assets, the mispriced asset will in fact also be held. Suppose that asset two is fairly priced while asset one is subject to mispricing; the assets are identical in all other dimensions. This can be represented by the price dynamics:

$$\begin{aligned} \frac{dP_{1t}}{P_{1t}} &= (r + \beta\mu_m)\, dt + \beta\sigma_m dB_t + \sigma dZ_t + b dZ_{1t} - \lambda_1 x_t dt; \\ \frac{dP_{2t}}{P_{2t}} &= (r + \beta\mu_m)\, dt + \beta\sigma_m dB_t + \sigma dZ_t + b dZ_{2t}. \end{aligned} \tag{11}$$

We can interpret P_{2t} as the fair price of both assets because it is not affected by the error-correction term ($\lambda_2 = 0$). However, asset 1 is mispriced due to the error-correction term $-\lambda_1 x_t = -\lambda_1(\ln P_{1t} - \ln P_{2t})$ in the dynamics of P_{1t} which, assuming that $\lambda_1 > 0$, drives mispricing of asset 1 to zero. When $P_{1t} > P_{2t}$, the price of asset 1 is overvalued so the error-correction term $-\lambda_1(\ln P_{1t} - \ln P_{2t})$ is negative and will "pull" the price P_{1t} down towards its fair value P_{2t}, presumably due to the trading of informed investors. Similarly, when $P_{1t} < P_{2t}$, the price of asset 1 is undervalued, and the error-correction term $-\lambda_1(\ln P_{1t} - \ln P_{2t})$ is positive which will "push" the price P_{1t} up towards its fair value P_{2t}.

Returning to the earlier example, there are stocks with the same dividend and voting rights that are traded on both the Hong Kong Stock Exchange and the Chinese Stock Exchange. The price of a stock traded on the Chinese exchange sometimes differs significantly from the price of the same stock traded on the Hong Kong exchange. Since the Hong Kong Stock Exchange is presumably more efficient, the stock traded in Hong Kong is more likely to be fairly priced. Thus one could model the prices of the stocks on the Hong Kong and Chinese exchanges as P_{2t} and P_{1t}, respectively.

Another special case, which we label the perfectly symmetric case, arises when all parameters of the two assets are identical and the two assets are symmetrically affected by mispricing. This case may cover stocks cross-listed on different exchanges such as ADRs or individual stocks such as Royal/Dutch and Shell where both markets are fairly similar and there is a degree of symmetry. Therefore, the following price dynamics may be appropriate

$$\begin{aligned}\frac{dP_{1t}}{P_{1t}} &= (r + \beta\mu_m)\,dt + \beta\sigma_m dB_t + \sigma dZ_t + b dZ_{1t} - \lambda x_t dt;\\ \frac{dP_{2t}}{P_{2t}} &= (r + \beta\mu_m)\,dt + \beta\sigma_m dB_t + \sigma dZ_t + b dZ_{2t} + \lambda x_t dt.\end{aligned} \tag{12}$$

We can view $(r + \beta\mu_m)\,dt + \beta\sigma_m dB_t + \sigma dZ_t$ as the return process for both assets when there is no mispricing while $-\lambda x_t dt + b dZ_{1t}$ and $\lambda x_t dt + b dZ_{2t}$ represent the pricing errors of asset 1 and 2 respectively. In this case, when $P_{1t} > P_{2t}$, the term $-\lambda x_t = -\lambda(\ln P_{1t} - \ln P_{2t})$ in the dynamics of P_{1t} will "pull" P_{1t} down towards P_{2t} while the term $\lambda x_t = -\lambda(\ln P_{2t} - \ln P_{1t})$ in the dynamics of P_{2t} will "push" P_{2t} up towards P_{1t}, thus keeping close the difference between P_{1t} and P_{2t}. A similar conclusion holds when $P_{1t} < P_{2t}$.

One could directly assume that $P_{1t} - \alpha P_{2t}$ is stationary without separate specifications for p_{1t} and p_{2t}, as in Jurek and Yang (2007). In this case, the two assets need not, strictly speaking, be cointegrated, because P_{1t} and P_{2t} are positive and therefore cannot be integrated ($I(1)$) processes, but the specification still captures mean reversion in the mispricing.

These are polar opposite cases, and the reality is likely to fall somewhere between the two cases. In the perfectly symmetric case covered by the standard arbitrage strategy, there is no need to distinguish between which asset is undervalued and which is overvalued, since only relative mispricing is needed to set up a trade. Conversely, in the more general asymmetric case, it becomes important to make this distinction. For example, if asset 1 is mispriced, while asset 2 is fairly priced, then whether asset 1 is under- or overpriced becomes important.

Note that the mispricing specified in our paper is stationary over time, whereas mispricing in Liu and Longstaff (2004) and Liu, Peleg, and Subrahmanyam (2006) are expressed in terms of a Brownian bridge and a generalized Brownian bridge. These specifications are not stationary and are useful to describe cases where mispricing will be zero for sure at some future date.

For example, on the settlement date, the difference between the spot price and future price of a futures contract has to be zero even though the underlying spot and futures prices follow non-stationary processes, cf. Brenner and Kroner (1995).

Equations (2) and (3) model a true risky arbitrage. Replacing x_t by $\frac{x_t}{T'-t}$, the above equations for $t < T'$ would represent a textbook riskless arbitrage. In this case, the price difference will become zero for sure at time T'. As pointed out by Liu and Longstaff (2004), without market frictions, the investor would hold an infinite amount of the arbitrage portfolio in this case. However, in the presence of market frictions, investors will hold finite amounts in the arbitrage strategy.

3 Optimal Investment Strategy

We next proceed to use the asset pricing model from the previous section to characterize the optimal portfolio choice for an investor with power utility. We denote the investor's portfolio weight on the market portfolio by ϕ_{mt} while the weights on the individual risky assets are given by ϕ_{it}, $i = 1, 2$.

The investor's utility is assumed to be of the constant relative risk aversion form:

$$\frac{1}{1-\gamma} E_0 \left[W_T^{1-\gamma} \right], \tag{13}$$

where W_T is the value of the investor's wealth at time T, which satisfies

$$dW_t = W_t \left(rdt + \phi_{mt} \left(\frac{dP_{mt}}{P_{mt}} - rdt \right) + \phi_{1t} \left(\frac{dP_{1t}}{P_{1t}} - rdt \right) + \phi_{2t} \left(\frac{dP_{2t}}{P_{2t}} - rdt \right) \right). \tag{14}$$

The following proposition characterizes the investor's optimal portfolio weights at time $t \leq T$:

Proposition 1 *Suppose asset prices evolve according to equations (1 - 3) and the investor has constant relative risk aversion preferences, (13). Then the optimal weight on the market portfolio is*

$$\phi_{mt}^* = \frac{\mu_m}{\gamma \sigma_m^2} + \frac{\beta_x \left(B + C \ln \left(\frac{P_{1t}}{P_{2t}^\alpha} \right) \right)}{\gamma} - (\phi_{1t}^* \beta_1 + \phi_{2t}^* \beta_2),$$

while the optimal portfolio weights of the individual assets are

$$\begin{pmatrix} \phi_{1t}^* \\ \phi_{2t}^* \end{pmatrix} = \frac{\begin{pmatrix} \sigma_2^2 + b_2^2 & -\sigma_1\sigma_2 \\ -\sigma_1\sigma_2 & \sigma_1^2 + b_1^2 \end{pmatrix}}{\gamma(\sigma_1^2 b_2^2 + \sigma_2^2 b_1^2 + b_1^2 b_2^2)} \begin{pmatrix} -\lambda_1 \ln \left(\frac{P_{1t}}{P_{2t}^\alpha} \right) + (\sigma_x \sigma_1 + b_1^2) \left(B + C \ln \left(\frac{P_{1t}}{P_{2t}^\alpha} \right) \right) \\ \lambda_2 \ln \left(\frac{P_{1t}}{P_{2t}^\alpha} \right) + (\sigma_x \sigma_2 - \alpha b_2^2) \left(B + C \ln \left(\frac{P_{1t}}{P_{2t}^\alpha} \right) \right) \end{pmatrix}.$$

B and C are functions of time t only and satisfy a system of ordinary differential equations (ODE) which is given in Appendix B.

Proposition 1 is a special case of a more general result (Proposition 5) with multiple assets and risk factors which we state in Section 5 and prove in Appendix B. The first term in the expression for the market portfolio is the standard mean-variance portfolio weight and thus depends on the market's Sharpe ratio divided by the investor's coefficient of risk aversion and market volatility. The second term is the intertemporal hedging demand for x_t which, due to its market exposure, is proportional to β_x. The third term offsets the market exposure of the individual assets which is linear in the portfolio weights ϕ_{1t}^* and ϕ_{2t}^* and proportional to the respective betas.

Turning to the expression for the holdings of the individual assets, note that parameters associated with the market index, such as β, μ_m, and σ_m, do not appear in the expression for ϕ_{1t}^* and ϕ_{2t}^* . This is because the individual assets' market exposure is hedged using the market index. In contrast, all the asset-specific parameters such as the volatility of the common and independent idiosyncratic risk components $(\sigma_1, \sigma_2, b_1, b_2)$, their sensitivity to the mispricing component (λ_1 and λ_2), the size of the mispricing ($\ln(P_{1t}/P_{2t}^{\alpha})$) in addition to the investor's attitude to risk (γ) and investment horizon (through B and C), help determine the optimal holdings of assets 1 and 2. The interaction between these parameters is quite complicated. To gain intuition for the result in Proposition 1, we next consider some special cases.

3.1 Symmetric Mispricing

As a first example, suppose that both risky assets are mispriced and the degree of mispricing is the same. To capture this we assume that $\beta_1 = \beta_2 = \beta$, $\sigma_1 = \sigma_2 = \sigma$, $b_1 = b_2 = b > 0$, $\alpha = 1$, and $\lambda_1 = \lambda_2 = \lambda > 0$. Both asset 1 and asset 2 are mispriced in this case because error-correction terms affect the dynamics of both P_{1t} and P_{2t}. Moreover, the mispricing is symmetric since both assets have the same exposure to the market and the common idiosyncratic risks, they also have the same volatility of independent idiosyncratic risks, and the same rate of mean reversion of the mispricing term.

In this case it follows from (7) - (9) that $\mu_x = \beta_x = \sigma_x = 0$. One can further show that

$B = 0$ using the equations given in Appendix B. The optimal portfolio weights simplify to

$$\begin{aligned} \begin{pmatrix} \phi_{1t}^* \\ \phi_{2t}^* \end{pmatrix} &= \frac{\begin{pmatrix} \sigma^2 + b^2 & -\sigma^2 \\ -\sigma^2 & \sigma^2 + b^2 \end{pmatrix}}{\gamma b^2 (2\sigma^2 + b^2)} \begin{pmatrix} -\lambda + b^2 C \\ \lambda - b^2 C \end{pmatrix} \ln \left(\frac{P_{1t}}{P_{2t}} \right) \\ &= \begin{pmatrix} -\left(\frac{\lambda}{\gamma b^2} - \frac{C}{\gamma} \right) \ln \left(\frac{P_{1t}}{P_{2t}} \right) \\ \left(\frac{\lambda}{\gamma b^2} - \frac{C}{\gamma} \right) \ln \left(\frac{P_{1t}}{P_{2t}} \right) \end{pmatrix}. \end{aligned} \tag{15}$$

Note that $\phi_{1t}^* = -\phi_{2t}^*$, that is, the portfolio weight of asset 1 always has the opposite sign to that of asset 2. Since both assets have the same beta, it follows that the optimal portfolio in the two assets is market neutral. Because $\phi_{1t}^* = -\phi_{2t}^*$, the following discussions will focus only on ϕ_{1t}^*.

Let us now consider the first term of ϕ_{1t}^*, i.e. $-\lambda \ln (P_{1t}/P_{2t}) / \gamma b^2$. The risk premium is $-\lambda \ln (P_{1t}/P_{2t})$, γ is the risk aversion and b^2 is the variance of the independent idiosyncratic risk. This is just the investor's myopic demand as if there is no common idiosyncratic risk. The common idiosyncratic risk, Z, does not matter here because it cancels out and thus only the independent idiosyncratic risk components remain.

The second term of ϕ_{1t}^*, $C \ln (P_{1t}/P_{2t}) / \gamma$, is the investor's intertemporal hedging demand which takes into account that the risk premium $-\lambda \ln (P_{1t}/P_{2t})$ is time varying. This term introduces a horizon dependence in the portfolio weight. For $\gamma > 1$, one can show that C is negative and its magnitude increases with the horizon. Thus, when $\gamma > 1$ the intertemporal hedging demand has the same sign as the myopic demand and the magnitude of the portfolio weight increases with the horizon. In this case, the optimal portfolio weight $\phi_{1t}^* < 0$ if and only $P_{1t} > P_{2t}$, that is, the investor will short asset 1 when it is overvalued relative to asset 2, and vice versa. This results is quite intuitive.

As a numerical example, suppose that $\lambda = 1$, so the half life of any mispricing is 1 year, $\ln (P_{1t}/P_{2t}) = 10\%$, so the price of asset 1 is about 11% higher than that of asset 2, the volatility of independent idiosyncratic risk, b, is 10% and the risk aversion coefficient $\gamma = 4$. Then, when time t is close to the end of the period ($T - t$ is near zero), the investor should short asset 1 in an amount that is 63% of his wealth; when time t is far away from the end of the period ($T - t$ is large), the investor should short asset 1 in an amount that is 114% of his wealth. The difference between 114% and 63% is due to the intertemporal hedging demand, which has been extensively discussed in the literature, see for example, Liu (2007). Since we can show that the magnitude of C decreases as the horizon decreases, as illustrated in Figure 1, the investor should monotonically reduce his asset holding to the position of the myopic demand as the investment horizon is reduced.

A broader set of scenarios is presented in Panel A of Table 1 which shows the optimal weights as a function of the magnitude of the mispricing (x) and the length of the time horizon (T). In the symmetric case, the size of the asset holdings increases as the (absolute) pricing error increases and as the investment horizon expands, but obviously the positions in assets 1 and 2 are identical in size and of opposite signs.

3.2 Asymmetric Mispricing

When the mispricing in asset 1 and 2 is symmetric, portfolio holdings in one asset must be exactly the opposite of the holdings in the other asset. While this is an interesting benchmark, asymmetries in the mispricing of two or more assets can easily arise in practice and may be due to the two assets trading on different exchanges with different trading rules and/or access to liquidity. This could give rise to differences in the volatility of the two assets' idiosyncratic risk components, i.e. $b_1 \neq b_2$, or to differences in their sensitivities to the error correction terms, i.e. $\lambda_1 \neq \lambda_2$. Other possibilities arise when the market exposures differ, $\beta_1 \neq \beta_2$, or when the volatilities of the common idiosyncratic risk component are different, $\sigma_1 \neq \sigma_2$, although these are perhaps less plausible sources of asymmetry.

Intuition for how asymmetries affect the holdings of assets 1 and 2 is fairly straightforward. For example, suppose that the volatility of the common or independent idiosyncratic components is lower for asset 1 than for asset 2 so that either $b_1 < b_2$ or $\sigma_1 < \sigma_2$. All other parameters are assumed to be the same for assets 1 and 2. In this case, when $P_{1t} > P_{2t}$ and asset 1 is overvalued, the short position in asset 1 (assuming $\gamma > 1$) exceeds the long position in asset 2 in absolute magnitude. Moreover, the effect can be quite large. Thus, assume the same parameters from the symmetric mispricing example above, but let $b_1 = 0.1$ while $b_2 = 0.2$. At short horizons when $T - t$ is close to zero, the investor should short asset 1 in the amount of -125% of his wealth and be long 93.5% in asset 2. At very long horizons, the investor should be short -220% in asset 1 and long 189% of his wealth in asset 2. Figure 2 presents the ratio of the optimal portfolio holdings, i.e. ϕ_1^*/ϕ_2^* at different horizons. These holdings are far removed from the equal-sized long-short position so commonly assumed in pairs trading.

Table 2 shows a more complete analysis of the effect on the relative asset holdings $-\phi_1/\phi_2$ of introducing various asymmetries in the asset-specific parameters, assuming a fixed value of $x = 0.1$ and varying T in the interval $[0, 1]$. Letting λ_2 vary from zero to 2 with $\lambda_1 = 1$, we see that $-\phi_1/\phi_2$ exceeds one if $\lambda_2 < 1$ while conversely $-\phi_1/\phi_2$ is less than one if $\lambda_2 > 1$. Also, the ratio of asset holdings mean reverts towards unity as the horizon gets longer.

The reverse pattern is seen when we vary b_2 between zero and 0.4, with b_1 fixed at 0.2. Now instead $-\phi_1/\phi_2$ is less than one when $b_2 < b_1$ and it is increasing in T, while for $b_2 > b_1$, we see that $-\phi_1/\phi_2$ exceeds one and is decreasing as a function of the investment horizon, T.

The effect of differences between σ_1 and σ_2 is similar to that of asymmetry in b_1 versus b_2. However, the effect is small relative to what we observed for the other parameters, at least for the range of values of σ_1, σ_2 used here.

3.3 Mispricing of One Asset

Next, we consider an example where one of the assets is mispriced ($\lambda_1 > 0$) while the other is correctly priced ($\lambda_2 = 0$). For simplicity we assume that $\beta_1 = \beta_2 = \beta$, $\sigma_1 = \sigma_2 = \sigma$, $b_1 = b_2 = b > 0$ and $\alpha = 1$. In this case, asset 1 is mispriced ($\lambda_1 > 0$) while there is no mispricing in asset 2 because there is no error-correction term ($\lambda_2 = 0$) in the dynamics of P_{2t}. Except for the impact of the error-correction terms, both assets are identical. For example, they share the same exposure to both market and common idiosyncratic risk and have the same idiosyncratic volatility.

In this case, $\mu_x = 0$ and $\beta_x = 0$, and we can show that $B = 0$ using equations given in Appendix B. The optimal portfolio weights are then reduced to

$$\begin{aligned}\begin{pmatrix}\phi_{1t}^* \\ \phi_{2t}^*\end{pmatrix} &= \frac{1}{\gamma(2\sigma^2+b^2)b^2}\begin{pmatrix}\sigma^2+b^2 & -\sigma^2 \\ -\sigma^2 & \sigma^2+b^2\end{pmatrix}\begin{pmatrix}-\lambda_1+b^2C \\ -b^2C\end{pmatrix}\ln\left(\frac{P_{1t}}{P_{2t}}\right) \\ &= \frac{\ln\left(\frac{P_{1t}}{P_{2t}}\right)}{\gamma}\left(-\begin{pmatrix}\sigma^2+b^2 \\ -\sigma^2\end{pmatrix}\frac{\lambda_1}{(2\sigma^2+b^2)b^2}+\begin{pmatrix}1 \\ -1\end{pmatrix}C\right). \qquad (16)\end{aligned}$$

Since there is no mispricing in asset 2, one might expect that the investor will not hold it and its portfolio weight should be zero. This is only true if $\sigma = 0$, which implies that returns on asset 1 and 2 are completely independent. In this case, asset 2 is dominated by the market index and will not be held. Conversely, when $\sigma > 0$, the optimal portfolio weight of asset 2 is not zero because this asset can be used to hedge idiosyncratic risk that is common to assets 1 and 2. In this case, through a rotation of Brownian motions, the dynamics of asset prices can be written as

$$
\begin{aligned}
\frac{dP_{1t}}{P_{1t}} &= (r+\beta\mu_m)\,dt + \beta\sigma_m dB_t + \frac{\sigma^2 dZ'_t}{\sqrt{\sigma^2+b^2}} + \frac{\sqrt{2\sigma^2+b^2}\,b\,dZ'_{1t}}{\sqrt{\sigma^2+b^2}} - \lambda_1 x_t dt; \\
\frac{dP_{2t}}{P_{2t}} &= (r+\beta\mu_m)\,dt + \beta\sigma_m dB_t + \sqrt{\sigma^2+b^2}\,dZ'_t,
\end{aligned}
\tag{17}
$$

where

$$
\begin{aligned}
Z'_t &= \frac{\sigma Z_t + bZ_{2t}}{\sqrt{\sigma^2+b^2}}, \\
Z'_{1t} &= \frac{1}{\sqrt{2\sigma^2+b^2}}\left(\frac{\sigma(bZ_t-\sigma Z_{2t})}{\sqrt{\sigma^2+b^2}} + Z_{1t}\right).
\end{aligned}
$$

Here, Z'_t and Z'_{1t} are mutually independent. The myopic demand in the portfolio weight of asset 1, ϕ^*_{1t}, $\frac{1}{\gamma}\lambda_1 \ln\left(\frac{P_{1t}}{P_{2t}}\right)\frac{(\sigma^2+b^2)}{(2\sigma^2+b^2)b^2}$, is given by the risk premium $\lambda_1 \ln\left(\frac{P_{1t}}{P_{2t}}\right)$ divided by the risk aversion γ and the variance of the new independent idiosyncratic risk, $\frac{(2\sigma^2+b^2)b^2}{(\sigma^2+b^2)}$. This is the myopic demand as if there is no dZ'_t term in the dynamics of P_{1t}. This happens because dZ'_t risk can be completely hedged away using asset 2. In fact, the optimal portfolio weight of asset 2 is determined such that the exposure to dZ'_t risk in the optimal portfolio is zero.

As in our previous example, terms that are proportional to C are the intertemporal hedging demands that generate horizon dependence in the portfolio weights. Again, for $\gamma > 1$, C is negative and its magnitude increases with the horizon $T-t$.

Moreover, once again the simple spread strategy of taking equal-sized long-short positions in the two assets is suboptimal and the optimal strategy is not market neutral. Any market exposure for the two assets is hedged away by trading in the market index through the $-\beta(\phi^*_{1t}+\phi^*_{2t})$ term in the optimal market portfolio weight.

Under the standard market-neutral arbitrage strategy, the investor would short asset 1 and be long in asset 2 if the former asset is deemed to be overpriced ($P_{1t} > P_{2t}$), so that the combined portfolio is market neutral. However, this is not optimal here because the market-neutral portfolio has unnecessary exposure to dZ'_t which has no alpha associated with it. We will later quantify this inefficiency in terms of utility losses. To the best of our knowledge, this point has not previously been made in the literature.

As a numerical example, suppose that $\lambda_1 = 1$ so the half life of the mispricing is 1 year, $\ln(P_{1t}/P_{2t}) = 10\%$ so the price of asset 1 is about 11% higher than that of asset 2, the volatility of common idiosyncratic risk is 20%, $\sigma = 20\%$, the volatility of independent idiosyncratic risk is 20%, $b = 20\%$ and the risk aversion coefficient $\gamma = 4$. Figure 3 displays the optimal portfolio weights at different horizons, $T-t$. Close to the end of the investment horizon ($T-t$ near zero), the investor should short asset 1 in the amount of $\phi_{1t}^* = 42\%$ of his wealth while taking a long position in asset 2 in the amount of $\phi_{2t}^* = 21\%$ of his wealth. Further away from the end of the period (where $T - t$ is large), the investor should short asset 1 in the amount of $\phi_{1t}^* = 65\%$ of his wealth while taking a long position in asset 2 equal to $\phi_{2t}^* = 44\%$ of his wealth. Again, the difference in portfolio weights at different horizons is due to the investor's intertemporal hedging demand.

In the asymmetric case ($\lambda_1 = 1, \lambda_2 = 0$) shown in Panel B of Table 1, even though there is no mispricing in asset 2, this asset is held short or long depending on whether or not $x > 0$. The ratio $-\phi_1/\phi_2$ does not depend on x, but it does depend on the horizon, T, and declines monotonically from a value of two ($T - t = 0$) to a value near 1.40 for $T - t = 1$.

We have discussed two polar examples, where the assets have the same degree of mispricing in the first example and only one asset is mispriced in the second example. In reality, most cases would fall in between these two polar examples.

3.4 Constrained Arbitrage Strategies

Many popular investment strategies assume that individual assets have constant relative weights. In this section, we study investors' optimal strategy under such a constraint which takes the form $\phi_{1t} = -\kappa\phi_{2t}$. Liu and Longstaff (2004) and Liu, Peleg, and Subrahmanyam (2006), among others, directly specify the dynamics of the difference in asset prices and so one can view the strategies studied in these papers to assume that $\kappa = 1$. Under such fixed relative weight constraints, the wealth dynamics is given by

$$dW_t = W_t \left(rdt + \phi_{mt} \left(\frac{dP_{mt}}{P_{mt}} - rdt \right) + \phi_{1t} \left(\frac{dP_{1t}}{P_{1t}} - rdt \right) - \kappa\phi_{1t} \left(\frac{dP_{2t}}{P_{2t}} - rdt \right) \right), \quad (18)$$

while the optimal portfolio weights follow from the following proposition:

Proposition 2 *The optimal portfolio weights under the fixed relative weight constraint* $\phi_{1t} = -\kappa\phi_{2t}$ *are*

$$\phi_{mt}^* = \frac{\mu_m + \sigma_m^2 \beta_x \left(\check{B} + \check{C} \ln \left(\frac{P_{1t}}{P_{2t}^\alpha} \right) \right)}{\gamma \sigma_m^2} - (\beta_1 - \kappa\beta_2)\phi_1^*;$$

$$\phi_{1t}^* = \frac{-(\lambda_1 + \kappa\lambda_2) \ln \left(\frac{P_{1t}}{P_{2t}^\alpha} \right) + \left(\sigma_x(\sigma_1 - \kappa\sigma_2) + (b_1^2 + \kappa\alpha b_2^2) \right) \left(\check{B} + \check{C} \ln \left(\frac{P_{1t}}{P_{2t}^\alpha} \right) \right)}{\gamma \left((\sigma_1 - \kappa\sigma_2)^2 + b_1^2 + \kappa^2 b_2^2 \right)}.$$

Here $\check{B}$ *and* $\check{C}$ *are functions of time* t *only and satisfy a system of ordinary differential equations (ODE) which is given in Appendix C.*

Proposition 2 is a special case of a more general multivariate result (Proposition 6) which is proved in Appendix C. Our first example of a constant relative portfolio weight scheme is the market neutral strategy which sets $\kappa = \beta_1/\beta_2$, i.e. $\phi_{1t} = -\frac{\beta_2}{\beta_1}\phi_{2t}$, so $\beta_1\phi_{1t} + \beta_2\phi_{2t} = 0$. As pointed out by Gatev et al. (2006), and confirmed empirically by these authors, paired stocks are often selected to be market neutral.

Under the market neutral strategy, the optimal portfolio weights are characterized in closed form as follows:

Corollary 1 *The optimal portfolio weights under the market neutral strategy (*$\beta_1\phi_{1t} + \beta_2\phi_{2t} = 0$*) are*

$$\phi^*_{mt} = \frac{\mu_m + \sigma^2_m\beta_x\left(\check{B} + \check{C}\ln\left(\frac{P_{1t}}{P^\alpha_{2t}}\right)\right)}{\gamma\sigma^2_m};$$
$$\phi^*_{1t} = \frac{-(\lambda_1 + \frac{\beta_1}{\beta_2}\lambda_2)\ln\left(\frac{P_{1t}}{P^\alpha_{2t}}\right) + \left(\sigma_x(\sigma_1 - \frac{\beta_1}{\beta_2}\sigma_2) + (b_1^2 + \frac{\beta_1}{\beta_2}\alpha b_2^2)\right)\left(\check{B} + \check{C}\ln\left(\frac{P_{1t}}{P^\alpha_{2t}}\right)\right)}{\gamma\left((\sigma_1 - \frac{\beta_1}{\beta_2}\sigma_2)^2 + b_1^2 + (\frac{\beta_1}{\beta_2})^2 b_2^2\right)}.$$

This is a popular investment strategy but it is clearly not always optimal: If the investor really prefers a market-neutral portfolio, it can be better obtained by combining the optimal individual asset portfolio with the market index. Such a "market layover" strategy can potentially improve the performance of the combined portfolio.

Our second example of constant relative portfolio weights arises when the weights are proportional to the cointegrating vector, i.e. $\phi_{1t} = -\alpha\phi_{2t}$. Under this scheme, which we label the cointegrated strategy, the wealth process discounted by the short rate is stationary. It coincides with the market-neutral strategy if $\beta_1 = \alpha\beta_2$. This strategy is also common in the literature, see e.g., Jurek and Yang (2006). The optimal weights under this strategy are listed below:

Corollary 2 *The optimal portfolio weights under the cointegrated investment strategy are*

$$\phi^*_{mt} = \frac{\mu_m + \sigma^2_m\beta_x\left(\check{B} + \check{C}\ln\left(\frac{P_{1t}}{P^\alpha_{2t}}\right)\right)}{\gamma\sigma^2_m} - (\beta_1 - \alpha\beta_2)\phi^*_1;$$
$$\phi^*_{1t} = \frac{-(\lambda_1 + \alpha\lambda_2)\ln\left(\frac{P_{1t}}{P^\alpha_{2t}}\right) + \left(\sigma_x(\sigma_1 - \alpha\sigma_2) + (b_1^2 + \alpha^2 b_2^2)\right)\left(\check{B} + \check{C}\ln\left(\frac{P_{1t}}{P^\alpha_{2t}}\right)\right)}{\gamma\left((\sigma_1 - \alpha\sigma_2)^2 + b_1^2 + \alpha^2 b_2^2\right)}.$$

Notice that in the case with symmetric mispricing studied in the previous section, the optimal strategy is market neutral and is also a cointegrated strategy.

However, in the example where asset 1 is mispriced while asset 2 is correctly priced, we showed that the optimal portfolio strategy is neither market neutral nor cointegrated. The sub-optimal outcome stems from two sources: First, the risk premium associated with the mispricing goes under-exploited. Second, the portfolio is unnecessarily exposed to (common) idiosyncratic risk that earns no risk premium.

To gain intuition for these results, consider again the case with asymmetric risk ($\lambda_1 = 1, \lambda_2 = 0$). For this case we expect that the magnitude of the myopic demand for asset 1 in the unconstrained optimal portfolio exceeds that under the constrained optimal portfolio. To see why, notice that shocks to asset 1 have two components: one that is perfectly correlated with shocks to asset 2 (Z_t) and one that is independent of shocks to this asset (Z_{1t}). The unconstrained allocation ensures that the perfectly correlated shock is completely hedged by taking an appropriate position in asset 2. Hence the unconstrained optimal holding is determined by the risk premium and the variance of the independent shock.

The portfolio constrained to have, say, suboptimal relative weights (1,-1), earns the same risk premium as asset 1 because the risk premium of asset 2 is zero ($\lambda_2 = 0$). Furthermore, the size of the independent shock to asset 2 is the same as that for asset 1, but is not completely hedged. Thus the suboptimal portfolio earns the same risk premium but at a higher risk and so the investor will hold less of asset 1 under the constrained strategy.

By the same token, for the intertemporal hedging demand, because the unconstrained portfolio has the same risk premium as the constrained portfolio but also has lower risk, the investor would take a higher position in the unconstrained portfolio.

4 Utility loss from suboptimal strategies

Differences between the optimal and suboptimal trading strategies, while interesting in their own right, are not of economic significance unless we can demonstrate that, for sensible choices of parameter values, they lead to sizeable economic losses. In this section we address the size of the economic loss associated with the market neutral or cointegrated strategies. We first provide a simple result for computing the wealth gain of the optimal investment strategy relative to a suboptimal strategy of fixing the portfolio weights. Comparing the wealth under the two investment strategies, we have:

Proposition 3 *The wealth gain of the optimal strategy over the constant relative weight strategy* *assuming a given mispricing of x is*

$$R = e^{\frac{1}{1-\gamma}((A-\check{A})+(B-\check{B})'x+\frac{1}{2}x'(C-\check{C})x)}.$$

Using this result, it is easy to assess the investor's utility loss. Figure 4 shows the outcome of our analysis using the parameter values from the case with mispricing only in asset 1 when the benchmark is the optimal market-neutral strategy which we compare to the unconstrained optimal strategy. The utility loss from using the market-neutral strategy rises steadily from zero to around two percent at the one-year horizon. A more comprehensive analysis is shown in Table 3 which reports the wealth gain as a function of x and T. Gains grow monotonically as a function of x and T and increase from zero to 3% as we move from short horizons with little mispricing ($x = 0, T = 0$) to longer horizons with greater mispricing ($x = 0.2, T = 1$).

Another measure that is useful for evaluating how desirable the return properties of alternative investment strategies are, is to consider moments of the resulting return distribution. For example, under standard risk preferences, investors prefer returns with a higher mean and a larger (more positive) skewness, while they dislike variance and kurtosis (fat tails) of returns. The next proposition shows how to compute the moments of the returns for the optimal (unconstrained) and the optimal market neutral strategies:

Proposition 4 *The moments of the returns of the optimal trading strategy and the optimal market neutral trading strategy are*

$$\mathrm{E}_0[(W_T^*/W_0)^q] = e^{d(t)+h(t)'x+\frac{1}{2}x'g(t)x}$$

and

$$\mathrm{E}_0[(\check{W}_T/W_0)^q] = e^{\check{d}(t)+\check{h}(t)'x+\frac{1}{2}x'\check{g}(t)x},$$

respectively, where $d(t), g(t), h(t)$ and $\check{d}(t), \check{h}(t), \check{g}(t)$ are defined in Appendix (A).

This result allows us to evaluate all the moments of the terminal wealth or, equivalently, the cumulated return distribution and hence provides a complete characterization of the returns associated with a particular investment strategy. For example, for a given investment horizon, $T - t$, expected returns (the first moment) can be computed by setting $q = 1$, while risk is characterized by the standard deviation of returns which can be computed as a by-product of the moments evaluated at $q = 1$ and $q = 2$.

One implication of this result is that the optimal unconstrained portfolio will always have a higher mean and a higher variance than the optimal constrained portfolio. To see this, notice that under mean-variance utility investors will maximize expressions of the form

$$\phi'\nu - \frac{\gamma}{2}\phi'\Sigma\phi, \tag{19}$$

where γ is the coefficient of risk aversion, ν is the vector of means and Σ is the covariance matrix. Assuming a linear constraint on the portfolio weights of the form $\kappa'\phi = k$, we get the following Lagrangian:

$$\phi'\nu - \frac{\gamma}{2}\phi'\Sigma\phi + \lambda_2(\phi'\kappa - k).$$

The first order condition associated with this is $\nu - \gamma\Sigma\phi + \lambda_2\kappa = 0$, while the portfolio weights are given by

$$\phi^* = \frac{1}{\gamma}\Sigma^{-1}(\nu + \lambda_2\kappa).$$

The Lagrangian multiplier can now be solved from

$$k = \frac{1}{\gamma}\kappa'\Sigma^{-1}(\nu + \frac{\gamma k - B}{C}\kappa) \equiv \frac{1}{\gamma}(B + C\lambda_2).$$

Thus $\lambda_2 = (\gamma k - B)/C$ and the constrained weights take the form

$$\phi = \frac{1}{\gamma}\Sigma^{-1}(\nu + \frac{\gamma k - B}{C}\kappa).$$

The associated mean and variance are given by

$$\begin{aligned}
\nu'\phi &= \frac{1}{\gamma}(A + \frac{\gamma k - B}{C}B) = \frac{A}{\gamma}\left(1 - \frac{B^2}{AC}\right), \\
\phi'\Sigma\phi &= \frac{1}{\gamma^2}(A + 2\frac{\gamma k - B}{C}B + \frac{(\gamma k - B)^2}{C}) = \frac{A}{\gamma^2}\left(1 - \frac{B^2}{AC}\right).
\end{aligned}$$

Similarly, for the unconstrained optimal portfolio we have $\phi^* = \gamma^{-1}\Sigma^{-1}\nu$ and

$$\begin{aligned}
v'\phi &= \frac{A}{\gamma} > \frac{A}{\gamma}\left(1 - \frac{B^2}{AC}\right), \\
\phi'\Sigma\phi &= \frac{A}{\gamma^2} > \frac{A}{\gamma^2}\left(1 - \frac{B^2}{AC}\right).
\end{aligned}$$

It follows directly that the expected return and the variance of the optimal (unconstrained) portfolio exceeds the mean and variance of the constrained optimal portfolio.

Table 4 provides a numerical illustration of these results by presenting moments of returns as a function of x and T for both the optimal and the constrained strategy. The same parameter values as in the earlier case were assumed here with the only source of asymmetry arising from $b_1 = 0.2, b_2 = 0.1$. Higher mispricing or a longer investment horizon leads to higher expected returns for both the optimal and constrained strategies, with the expected return differential increasing as x or T rises. A parallel pattern is seen for the standard deviation of returns. Consistent with the theory, both the mean and standard deviation are higher for the optimal than for the constrained portfolio. Moreover, the Sharpe ratio rises from about 0.2 to around 2 as x or T increase and is always higher for the optimal than for the constrained portfolio strategy.

The coefficient of skew is always negative for both portfolios and, except when x and T are both very small, the constrained portfolio has a larger negative skew than the optimal portfolio, making the optimal portfolio more desirable. Conversely, both portfolios generate returns with a large positive coefficient of kurtosis. Again, except for very small values of x and T, the constrained portfolio is more fat-tailed than the optimal portfolio, making it less desirable to risk averse investors.

These results show that the optimal (unconstrained) portfolio has higher expected returns, higher standard deviation and (in most cases) a higher skew and a smaller kurtosis than the optimal market-neutral portfolio.

5 Generalization to Multiple Risk Factors and Multiple Assets

To gain intuition for the portfolio choice problem in the presence of mean-reverting mispricing and cointegrated asset prices, we have so far focused on the case with one common risk factor and two risky assets. In the interest of establishing generality of our results, we next consider the case with many risk factors, multiple cointegrating relations and multiple assets.

Suppose that there are K common risk factors and K factor assets trading at prices P^k_{mt}, $k = 1, ..., K$. In a direct generalization of (1), the dynamics of the prices of these common factor assets is given by

$$\frac{dP^k_{mt}}{P^k_{mt}} = (r + \mu^k_m)dt + (\sigma_m dB_{mt})^k, \quad k = 1, ..., K, \tag{20}$$

where μ_m is a constant $K \times 1$ vector and σ_m is a $K \times K$ constant matrix; dB_{mt} is a vector of standard Brownian motions of dimension $K \times 1$. A superscript on a vector indicates the $k'th$ element.

Moreover, as a generalization of (2) and (3) we assume that there are N individual assets with prices (for $i = 1, ..., N$)

$$\frac{dP^i_{mt}}{P^i_{mt}} = (r + (\beta\mu_m)^i)dt + (\beta\sigma_m dB_{mt})^i + (\sigma dB_t)^i + (bdZ_t)^i - (\lambda x_t)^i dt. \tag{21}$$

Here β is an $N \times K$ matrix, σ is an $N \times m$ matrix ($m < N$), b is a diagonal $N \times N$ matrix, λ is an $N \times h$ matrix. B_t is an m-dimensional standard Brownian motion; for each $i = 1, ..., N$, Z_t^i is a one-dimensional standard Brownian motion. All Brownian motions, B_{mt}, B_t, and Z_t^i, $i = 1, ..., N$, are independent of each other. Let α denote the $h \times N$ dimensional matrix of cointegrating vectors, where $h \leq N$ gives the number of cointegrating vectors. Then $x_t = \alpha \ln P_t$ is an h-dimensional process of pricing errors which satisfies

$$dx_t = (\mu_x - \lambda_x x_t)dt + \beta_x \sigma_m dB_{mt} + \sigma_x dB_t + b_x dZ_t, \tag{22}$$

where

$$\begin{aligned} \mu_x &= \alpha(r + \beta\mu_m) - \frac{1}{2}\alpha\mathcal{D}(\beta\sigma_m\sigma_m'\beta' + \sigma\sigma' + bb'); \\ \lambda_x &= \alpha\lambda; \\ \beta_x &= \alpha\beta; \\ \sigma_x &= \alpha\sigma; \\ b_x &= \alpha b. \end{aligned} \tag{23}$$

The notation $\mathcal{D}$ denotes the vector of the diagonal elements of a square matrix. The wealth process for this general case is now given by

$$dW_t = W_t\Big((r + \phi_m'\mu_m + \phi'(\beta\mu_m - \lambda x_t))dt + \phi_m'\sigma_m dB_{mt} + \phi'\beta\sigma_m dB_{mt} + \phi'\sigma dB_t + \phi' b dZ_t\Big).$$

Once again λ_x must satisfy the conditions such that x_t is stationary, i.e. the eigenvalues of $\alpha\lambda$ must be negative.

As before, the value function $J(x_t, W_t, t)$ takes the form

$$J(x_t, W_t, t) = \frac{1}{1-\gamma} \mathrm{E}_t\left[W_T^{*1-\gamma}\right],$$

where W_T^* is the wealth at time T associated with the optimal trading strategy.

The following proposition which generalizes Proposition 1 gives the optimal portfolio weights for the general case with multiple risk factors and multiple assets:

Proposition 5 *The optimal portfolio weights with multiple common factors and risky assets are*

$$\begin{aligned} \phi_{mt}^* + \beta'\phi_t^* &= \frac{1}{\gamma}(\sigma_m\sigma_m')^{-1}\mu_m + \frac{\beta_x'(B + C\alpha\ln P_t)}{\gamma}; \\ \phi_t^* &= \frac{1}{\gamma}\Big(\sigma\sigma' + bb'\Big)^{-1}\Big(-\lambda\alpha\ln P_t + (\sigma\sigma_x' + bb'\alpha)(B + C\alpha\ln P_t)\Big). \end{aligned}$$

where $B(t)$ and $C(t)$ are an $h \times 1$ dimensional vector function and an $h \times h$ dimensional symmetric matrix function of time t respectively. They satisfy a system of Riccati ODE given in Appendix B.

Proposition 5 is proved in Appendix B. The first term in the expression for ϕ^*_{mt} is again the weight from the standard mean-variance case, while the second term represents the intertemporal hedging demand. Turning to the expression for ϕ^*_t, as in the earlier case with two risky assets, the optimal demand for the risky assets can be decomposed into a myopic mean-variance portion and an intertemporal hedging portion.

Finally, as a generalization of 2 we derive the optimal portfolio weights under the fixed constant relative weights constraint, $\phi'\kappa = 0$:

Proposition 6 *Under the constraint $\kappa'\phi_t = 0$, the optimal portfolio weights with multiple common factors and risky assets are*

$$\begin{aligned}
\phi^*_{mt} + \beta'\phi^*_t &= \frac{1}{\gamma}(\sigma_m\sigma_m')^{-1}(\mu_m + \sigma_m\sigma_m'\beta_x'(\check{B} + \check{C}\alpha\ln P_t)) \\
\phi^*_t &= \frac{1}{\gamma}(\sigma\sigma' + bb')^{-1}\Big(I - \kappa(\kappa'(\sigma\sigma' + bb')^{-1}\kappa)^{-1}\kappa'(\sigma\sigma' + bb')^{-1}\Big) \\
&\quad \times\Big(-\lambda\alpha\ln P_t + (\sigma\sigma' + bb')\alpha'(\check{B} + \check{C}\alpha\ln P_t)\Big),
\end{aligned}$$

where $\check{B}(t)$ and $\check{C}(t)$ are an $h \times 1$ dimensional vector function and an $h \times h$ dimensional symmetric matrix function of time t, respectively. These functions satisfy a system of Riccati ODEs given in Appendix C.

Proposition 6, which is proved in Appendix C, provides a closed-form expression for the optimal portfolio holdings subject to the constraint that the relative weights are kept constant through time. While the intuition for the expression is more complicated than that for 2, the components of the two expressions are very similar.

This framework with multiple common factors and risky assets is very general and allows for the possibility that there may be more than just a single co-integrating vector and so there can also be multiple error-correction terms that affect the dynamics of any mispricing.

6 Conclusion

This paper studies the optimal portfolio strategy in the presence of temporary errors in the relative prices of two or more assets that are close substitutes. We assume that the relative mispricing is mean reverting and model the dynamics in the underlying asset prices through a continuous time cointegrated system. Our model is very general and can comprise multiple common risk factors, many cointegrating relations, multiple assets whose idiosyncratic risks may be correlated. We characterize how investors seek to optimally exploit predictability in pricing errors while using the factor assets to control the associated exposure to common risk factors and also account for correlated sources of idiosyncratic risk.

We show that the source of mispricing is important. Under a set of intuitive symmetry assumptions where both assets are mispriced, the standard market-neutral arbitrage strategy is optimal. However, in the presence of asymmetries such as when one asset is mispriced while the other is not, the standard long-short market-neutral strategy ceases to be optimal. Moreover, a cointegrated strategy where the portfolio weights are proportional to the cointegration vector, is not, in general, optimal.

Appendix

Propositions 1 and 2 are special cases of Propositions 5 and 6 and so we do not prove them separately here.

A Proof of Proposition 4

We are interested in deriving the moments of the portfolio return process. Given portfolio weight processes (ϕ_{mt}, ϕ_t), the wealth process is

$$dW_t = W_t\Big((r + \phi'_{mt}\mu_m + \phi'_t(\beta\mu_m - \lambda x_t))dt + (\phi'_{mt} + \phi'_t\beta)\sigma_m dB_{mt} + \phi'_t(\sigma dB_t + b dZ_t)\Big).$$

Using Ito's lemma, we have

$$\begin{aligned} d\ln W_t &= (r + \phi'_{mt}\mu_m + \phi'_t(\beta\mu_m - \lambda x_t))dt + (\phi'_{mt} + \phi'_t\beta)\sigma_m dB_{mt} + \phi'_t(\sigma dB_t + b dZ_t) \\ &\quad -\frac{1}{2}((\phi'_{mt} + \phi'_t\beta)\sigma_m\sigma'_m(\phi_{mt} + \beta'\phi_t) + \phi'_t(\sigma\sigma' + bb')\phi_t)dt. \end{aligned}$$

Thus,

$$\begin{aligned} W_T^q &= W_0^q \exp\Big(\int_0^T q(r + (\phi'_{mt} + \phi'_t\beta)\mu_m - \phi'_t\lambda x_t)dt + q(\phi'_{mt} + \phi'_t\beta)\sigma_m dB_{mt} + q\phi'_t(\sigma dB_t + b dZ_t) \\ &\quad -\frac{q}{2}((\phi'_{mt} + \phi'_t\beta)\sigma_m\sigma'_m(\phi_{mt} + \beta'\phi_t) + \phi'_t(\sigma\sigma' + bb')\phi_t)dt\Big). \end{aligned}$$

We are interested in characterizing

$$\mathrm{E}_0\left[W_T^q\right].$$

Using Girsanov's theorem, we can write

$$\begin{aligned} \mathrm{E}_0[W_T^q] &= W_0^q \mathrm{E}_0\Big[\exp\Big(\int_0^T q(r + \phi'_{mt}\mu_m + \phi'_t(\beta\mu_m - \lambda x_t))dt + q(\phi'_{mt} + \phi'_t\beta)\sigma_m dB_{mt} + q\phi'_t(\sigma dB_t + b dZ_t) \\ &\quad -\frac{q}{2}((\phi'_{mt} + \phi'_t\beta)\sigma_m\sigma'_m(\phi_{mt} + \beta'\phi_t) + \phi'_t(\sigma\sigma' + bb')\phi_t)dt\Big)\Big] \\ &= W_0^q \mathrm{E}_0^{Q_q}\Big[\exp\Big(\int_0^T q(r + (\phi'_{mt} + \phi'_t\beta)\mu_m - \phi'_t\lambda x_t)dt \\ &\quad +\frac{q^2 - q}{2}((\phi'_{mt} + \phi'_t\beta)\sigma_m\sigma'_m(\phi_{mt} + \beta'\phi_t) + \phi'_t(\sigma\sigma' + bb')\phi_t)dt\Big)\Big], \end{aligned} \tag{A-1}$$

where E^{Q_q} denotes the expectation under the equivalent martingale measure specified by the following Radon-Nykodym derivative with respect to the physical measure, P:

$$\frac{dQ_q}{dP} = \exp\Big(\int_0^T q(\phi'_{mt}+\phi'_t\beta)\sigma_m dB_{mt} + q\phi'_t(\sigma dB_t + b dZ_t) - \frac{q^2}{2}((\phi'_{mt}+\phi'_t\beta)\sigma_m\sigma'_m(\phi_{mt}+\beta'\phi_t) + \phi'_t(\sigma\sigma'+bb')\phi_t)dt\Big).$$

The standard Brownian motions under the Q_q measure are

$$\begin{aligned} dB_{mt}^{Q_q} &= dB_{mt} - q\sigma'_m(\phi_{mt} + \beta'\phi_t)dt; \\ dB_t^{Q_q} &= dB_t - q\sigma'\phi_t dt; \\ dZ_t^{Q_q} &= dZ_t - qb'\phi_t dt. \end{aligned}$$

The dynamics of x is

$$dx_t = (\mu_x - \lambda_x x_t + q\beta_x\sigma_m\sigma'_m(\phi_{mt} + \beta'\phi_t) + q\sigma_x\sigma'\phi_t + qb_x b'\phi_t)dt + \beta_x\sigma_m dB_{mt}^{Q_q} + \sigma_x dZ_t^{Q_q} + b_x dZ_{xt}^{Q_q}.$$

Suppose that ϕ_{mt} and ϕ_t are affine functions of x_t. Then the expectation in equation (A-1) is a function of x_0, $f(x,t)$. According to the Feynman-Kac formula, the function $f(x,t)$ satisfies

$$\begin{aligned} 0 \;=\; & f_t + (\mu_x - \lambda_x x + q\beta_x\sigma_m\sigma_m'(\phi_m + \beta'\phi) + q\sigma_x\sigma'\phi + qb_x b'\phi)' f_x + \frac{1}{2}\mathrm{Tr}\left[(\beta_x\sigma_m\sigma_m'\beta_x' + \sigma_x\sigma_x' + b_x b_x')f_{xx'}\right] \\ & + q(r + (\phi_m' + \phi'\beta)\mu_m - \phi'\lambda x)f + \frac{q^2 - q}{2}((\phi_m' + \phi'\beta)\sigma_m\sigma_m'(\phi_m + \beta'\phi) + \phi'(\sigma\sigma' + bb')\phi)f. \end{aligned} \tag{A-2}$$

with $f(x,T) = 1$.

A.1 Optimal (unconstrained) Strategy

Substituting the optimal portfolio weights ϕ^*, ϕ_m^* into (A-2), we get

$$\begin{aligned}
0 \;=\; & f_t + \left(\mu_x - \alpha\lambda x + \frac{q}{\gamma}\beta_x\sigma_m\sigma_m' \left[(\sigma_m\sigma_m')^{-1}\mu_m + \beta_x'(B+Cx)\right]\right)' f_x \\
& + \left(\frac{q}{\gamma}\left[\sigma_x\sigma' + b_x b'\right]\left(\sigma\sigma' + bb'\right)^{-1}\Big(-\lambda x + (\sigma\sigma_x' + bb'\alpha')(B+Cx)\Big)\right)' f_x \\
& + \frac{1}{2}\mathrm{Tr}\left[\alpha(\beta\sigma_m\sigma_m'\beta + \sigma\sigma' + bb')\alpha' f_{xx'}\right] + qrf \\
& + \frac{q}{\gamma}\Big((\sigma_m\sigma_m')^{-1}\mu_m + \beta_x'(B+Cx)\Big)'\mu_m f \\
& - \frac{q}{\gamma}\Big(-\lambda x + (\sigma\sigma_x' + bb'\alpha')(B+Cx)\Big)'\Big(\sigma\sigma' + bb'\Big)^{-1}\lambda x f \\
& + \frac{q^2-q}{2\gamma^2}(\mu_m'(\sigma_m\sigma_m')^{-1} + (B+Cx)'\beta_x)\sigma_m\sigma_m'\left[(\sigma_m\sigma_m')^{-1}\mu_m + \beta_x'(B+Cx)\right] f \\
& + \frac{q^2-q}{2\gamma^2}\Big(-\lambda x + (\sigma\sigma_x' + bb'\alpha')(B+Cx)\Big)'\Big(\sigma\sigma' + bb'\Big)^{-1} \\
& \times\Big(-\lambda x + (\sigma\sigma_x' + bb'\alpha')(B+Cx)\Big) f.
\end{aligned}$$

The coefficients are quadratic in x and we can conjecture that $f(x;q) = e^{d(t)+h(t)'x+\frac{1}{2}x'g(t)x}$ and derive an ODE for d, h, and g. Substituting the functional form of f into the above equation, we get

$$\begin{aligned}
0 \;=\; & d_t + h_t'x + \frac{1}{2}x'g_t x + \left(\mu_x - \alpha\lambda x + \frac{q}{\gamma}\beta_x\sigma_m\sigma_m'\left[(\sigma_m\sigma_m')^{-1}\mu_m + \beta_x'(B+Cx)\right]\right)'(h+gx) \\
& + \left(\frac{q}{\gamma}\left[\sigma_x\sigma' + b_x b'\right]\left(\sigma\sigma' + bb'\right)^{-1}\Big(-\lambda x + (\sigma\sigma_x' + bb'\alpha')(B+Cx)\Big)\right)'(h+gx) \\
& + \frac{1}{2}\mathrm{Tr}\left[\alpha(\beta\sigma_m\sigma_m'\beta + \sigma\sigma' + bb')\alpha'\Big((h+gx)(h+gx)' + g\Big)\right] + qr \\
& + \frac{q}{\gamma}\Big((\sigma_m\sigma_m')^{-1}\mu_m + \beta_x'(B+Cx)\Big)'\mu_m - \frac{q}{\gamma}\Big(-\lambda x + (\sigma\sigma_x' + bb'\alpha')(B+Cx)\Big)'\Big(\sigma\sigma' + bb'\Big)^{-1}\lambda x \\
& + \frac{q^2-q}{2\gamma^2}\Big(\mu_m + \sigma_m\sigma_m'\beta_x'(B+Cx)\Big)'(\sigma_m\sigma_m')^{-1}\Big(\mu_m + \sigma_m\sigma_m'\beta_x'(B+Cx)\Big) \\
& + \frac{q^2-q}{2\gamma^2}\Big(-\lambda x + (\sigma\sigma_x' + bb'\alpha')(B+Cx)\Big)'\Big(\sigma\sigma' + bb'\Big)^{-1}\Big(-\lambda x + (\sigma\sigma_x' + bb'\alpha')(B+Cx)\Big).
\end{aligned}$$

Setting the coefficients of the various powers of x to zero, we get the following system of ODEs

$$\begin{aligned}
0 \;=\; & d_t + \left(\mu_x + \frac{q}{\gamma}\beta_x\left[\mu_m + \sigma_m\sigma_m'\beta_x'B\right]\right)' h \\
& + \left(\frac{q}{\gamma}\alpha(\sigma\sigma' + bb')\alpha' B\right)' h \\
& + \frac{1}{2}\mathrm{Tr}\left[\alpha(\beta\sigma_m\sigma_m'\beta' + \sigma\sigma' + bb')\alpha'\Big(hh' + g\Big)\right] + qr \\
& + \frac{q}{\gamma}\mu_m'(\sigma_m\sigma_m')^{-1}\Big(\mu_m + \sigma_m\sigma_m'\beta_x'B\Big) \\
& + \frac{q^2-q}{2\gamma^2}\Big(\mu_m + \sigma_m\sigma_m'\beta_x'B\Big)'(\sigma_m\sigma_m')^{-1}\Big(\mu_m + \sigma_m\sigma_m'\beta_x'B\Big) \\
& + \frac{q^2-q}{2\gamma^2}B'(\sigma\sigma_x' + bb'\alpha')'\Big(\sigma\sigma' + bb'\Big)^{-1}(\sigma\sigma_x' + bb'\alpha')B.
\end{aligned}$$

$$
\begin{aligned}
0 \;=\; & h_t + \left(-\alpha\lambda + \frac{q}{\gamma}\beta_x\sigma_m\sigma_m'\beta_x'C\right)' h + g'\left(\mu_x + \frac{q}{\gamma}\beta_x\sigma_m\sigma_m'[(\sigma_m\sigma_m')^{-1}\mu_m + \beta_x'B]\right) \\
& + \left(\frac{q}{\gamma}\alpha\Big(-\lambda + (\sigma\sigma_x' + bb'\alpha')C\Big)\right)' h \\
& + g'\frac{q}{\gamma}\alpha(\sigma\sigma_x' + bb'\alpha')B \\
& + g'\alpha(\beta\sigma_m\sigma_m'\beta' + \sigma\sigma' + bb')\alpha' h \\
& + \frac{q}{\gamma}C\alpha\beta\mu_m - \frac{q}{\gamma}\lambda'\Big(\sigma\sigma' + bb'\Big)^{-1}\Big((\sigma\sigma_x' + bb'\alpha')B\Big) \\
& + \frac{q^2 - q}{\gamma^2}C\beta_x(\mu_m + \sigma_m\sigma_m'\beta_x'B) \\
& + \frac{q^2 - q}{\gamma^2}\Big(-\lambda + (\sigma\sigma_x' + bb'\alpha')C\Big)'\Big(\sigma\sigma' + bb'\Big)^{-1}(\sigma\sigma_x' + bb'\alpha')B.
\end{aligned}
$$

$$
\begin{aligned}
0 \;=\; & g_t + \left(-\alpha\lambda + \frac{q}{\gamma}\beta_x\sigma_m\sigma_m'\beta_x'C\right)' g + g'\left(-\alpha\lambda + \frac{q}{\gamma}\beta_x\sigma_m\sigma_m'\beta_x'C\right) \\
& + \frac{q}{\gamma}\Big(-\lambda + (\sigma\sigma_x' + bb'\alpha')C\Big)'\alpha' g + \frac{q}{\gamma}g'\alpha\Big(-\lambda + (\sigma\sigma_x' + bb'\alpha')C\Big) \\
& + g'\alpha\Big(\beta\sigma_m\sigma_m'\beta' + \sigma\sigma' + bb'\Big)\alpha' g + \frac{2q}{\gamma}\lambda'\Big(\sigma\sigma' + bb'\Big)^{-1}\lambda \\
& - \frac{q}{\gamma}\Big(C\alpha\lambda + \lambda'\alpha'C\Big) + \frac{q^2 - q}{\gamma^2}C'\beta_x(\sigma_m\sigma_m')\beta_x'C \\
& + \frac{q^2 - q}{\gamma^2}\Big(-\lambda + (\sigma\sigma_x' + bb'\alpha')C\Big)'\Big(\sigma\sigma' + bb'\Big)^{-1}\Big(-\lambda + (\sigma\sigma_x' + bb'\alpha')C\Big).
\end{aligned}
$$

The expected portfolio return is given by $f(x;1)$ and the variance of the portfolio return is given by $f(x;2) - f^2(x;1)$. Higher order moments can be derived for other values of q in a similar manner.

A.2 Constrained Strategy

The constrained strategy satisfies $\kappa'\phi_t = 0$. Equation (A-2) then becomes

$$
\begin{aligned}
0 \;=\; & f_t + (\mu_x - \lambda_x x + q\beta_x\sigma_m\sigma_m'(\phi_m + \beta'\phi) + q\sigma_x\sigma'\phi + qb_x b'\phi)' f_x + \frac{1}{2}\mathrm{Tr}\left[(\beta_x\sigma_m\sigma_m'\beta_x' + \sigma_x\sigma_x' + b_x b_x')f_{xx'}\right] \\
& + q(r + (\phi_m + \beta'\phi)'\mu_m - \phi_t'\lambda x)f + \frac{q^2 - q}{2}((\phi_m + \beta'\phi)'\sigma_m\sigma_m'(\phi_m + \beta'\phi) + \phi'(\sigma\sigma' + bb')\phi)f.
\end{aligned}
$$

As we have show in the proof of Proposition 6, the optimal value of $\phi_m + \beta'\phi$ is given by

$$
(\phi_m + \beta'\phi) = \frac{1}{\gamma}(\sigma_m\sigma_m')^{-1}\Big(\mu_m + \sigma_m\sigma_m'\beta_x'(\check{B} + \check{C}x)\Big).
$$

and the optimal value of ϕ is given by

$$
\begin{aligned}
\phi \;&=\; \frac{1}{\gamma}(\sigma\sigma' + bb')^{-1}\Big(I - \kappa(\kappa'(\sigma\sigma' + bb')^{-1}\kappa)^{-1}\kappa'(\sigma\sigma' + bb')^{-1}\Big)\Big(-\lambda x + (\sigma\sigma' + bb')\alpha'(\check{B} + \check{C}x_t)\Big) \\
&=\; \frac{1}{\gamma}(\sigma\sigma' + bb')^{-1}(B_1 + C_1 x_t).
\end{aligned}
$$

Using the equations for $\phi_m + \beta'\phi$, ϕ and λ_x, β_x, σ_x, and b_x, we get

$$\begin{aligned} 0 \;=\;& f_t + (\mu_x - \lambda_x x + q\beta_x \sigma_m \sigma_m' (\phi_m + \beta'\phi) + q\alpha(\sigma\sigma' + bb'))' f_x + \frac{1}{2}\mathrm{Tr}\left[\alpha(\beta\sigma_m\sigma_m'\beta' + \sigma\sigma' + bb')\alpha' f_{xx'}\right] \\ & + q(r + (\phi_m + \beta'\phi)\mu_m - \phi'\lambda x) f + \frac{q^2 - q}{2}\Big(\sigma_m\sigma_m'(\phi_m + \beta'\phi)^2 + \phi'(\sigma\sigma' + bb')\phi\Big) f. \end{aligned}$$

Substituting ϕ_m and ϕ into the above equation, we get

$$\begin{aligned} 0 \;=\;& f_t + \left(\mu_x - \alpha\lambda x + \frac{q\beta_x}{\gamma}\Big(\mu_m + \sigma_m\sigma_m'\beta_x'(\check{B} + \check{C}x)\Big) + \frac{q}{\gamma}\alpha(B_1 + C_1 x)\right)' f_x + \frac{1}{2}\mathrm{Tr}\left[\alpha(\beta\sigma_m\sigma_m'\beta' + \sigma\sigma' + bb')\alpha' f_{xx'}\right] \\ & + \left(qr + \frac{q}{\gamma}\mu_m'(\sigma_m\sigma_m')^{-1}\Big(\mu_m + \sigma_m\sigma_m'\beta_x'(\check{B} + \check{C}x)\Big) - \frac{q}{\gamma}(B_1 + C_1 x)'(\sigma\sigma' + bb')^{-1}\lambda x\right) f \\ & + \frac{q^2 - q}{2\gamma^2}\Big((\mu_m + \sigma_m\sigma_m'\beta_x'(\check{B} + \check{C}x))'(\sigma_m\sigma_m')^{-1}(\mu_m + \sigma_m\sigma_m'\beta_x'(\check{B} + \check{C}x)) + (B_1 + C_1 x_t)'(\sigma\sigma' + bb')^{-1}(B_1 + C_1 x)\Big) f. \end{aligned}$$

If we conjecture that $\check{f} = e^{\check{d}(t) + \check{h}(t)'x + \frac{1}{2}x'\check{g}(t)x}$, we can write the ODE for $\check{d}$, $\check{h}$, and $\check{g}$. We get the following PDE

$$\begin{aligned} 0 \;=\;& \check{d}_t + \check{h}_t' x + \frac{1}{2}x'\check{g}_t x + \left(\mu_x - \alpha\lambda x + \frac{q\beta_x}{\gamma}\Big(\mu_m + \sigma_m\sigma_m'\beta_x'(\check{B} + \check{C}x)\Big) + \frac{q}{\gamma}\alpha(B_1 + C_1 x)\right)'(\check{h} + \check{g}x) \\ & + \frac{1}{2}\mathrm{Tr}\left[\alpha(\beta\sigma_m\sigma_m'\beta' + \sigma\sigma' + bb')\alpha'\Big((\check{h} + \check{g}x)(\check{h} + \check{g}x)' + \check{g}\Big)\right] \\ & + qr + \frac{q}{\gamma}\mu_m'(\sigma_m\sigma_m')^{-1}\Big(\mu_m + \sigma_m\sigma_m'\beta_x'(\check{B} + \check{C}x)\Big) - \frac{q}{\gamma}(B_1 + C_1 x)'(\sigma\sigma' + bb')^{-1}\lambda x \\ & + \frac{q^2 - q}{2\gamma^2}\left(\Big(\mu_m + \sigma_m\sigma_m'\beta_x'(\check{B} + \check{C}x)\Big)'(\sigma_m\sigma_m')^{-1}\Big(\mu_m + \sigma_m\sigma_m'\beta_x'(\check{B} + \check{C}x)\Big) + (B_1 + C_1 x)'(\sigma\sigma' + bb')^{-1}(B_1 + C_1 x)\right). \end{aligned}$$

Furthermore, we get the following system of ODEs

$$\begin{aligned} 0 \;=\;& \check{d}_t + \left(\mu_x + \frac{q\alpha\beta}{\gamma}(\mu_m + \sigma_m\sigma_m'\beta'\alpha'\check{B}) + \frac{q}{\gamma}\alpha B_1\right)'\check{h} + \frac{1}{2}\mathrm{Tr}\left[\alpha(\beta\sigma_m\sigma_m'\beta' + \sigma\sigma' + bb')\alpha'\Big(\check{h}\check{h}' + g\Big)\right] \\ & + qr + \frac{q}{\gamma}\mu_m'(\sigma_m\sigma_m')^{-1}\Big(\mu_m + \sigma_m\sigma_m'\beta'\alpha'\check{B}\Big) \\ & + \frac{q^2 - q}{2\gamma^2}\left(\Big(\mu_m + \sigma_m\sigma_m'\beta'\alpha'\check{B}\Big)'(\sigma_m\sigma_m')^{-1}\Big(\mu_m + \sigma_m\sigma_m'\beta'\alpha'\check{B}\Big) + B_1'(\sigma\sigma' + bb')^{-1}B_1\right); \\ 0 \;=\;& \check{h}_t + \left(-\lambda'\alpha' + \frac{q}{\gamma}\sigma_m\sigma_m'\check{C}'\alpha\beta\beta'\alpha' + \frac{q}{\gamma}C_1'\alpha'\right)\check{h} + \check{g}'\left(\mu_x + \frac{q}{\gamma}\beta_x(\mu_m + \sigma_m\sigma_m'\beta_x'\check{B}) + \frac{q}{\gamma}\alpha B_1\right) \\ & + \check{g}'\alpha(\beta\sigma_m\sigma_m'\beta' + \sigma\sigma' + bb')\alpha'\check{h} \\ & + \frac{q}{\gamma}\check{C}'\alpha\beta\mu_m - \frac{q}{\gamma}\lambda'(\sigma\sigma' + bb')^{-1}B_1 + \frac{q^2 - q}{\gamma^2}\Big(\check{C}\alpha\beta(\mu_m + \sigma_m\sigma_m'\beta'\alpha'\check{B}) + C_1'(\sigma\sigma' + bb')^{-1}B_1\Big); \\ 0 \;=\;& \check{g}_t + \left(-\alpha\lambda + \frac{q}{\gamma}\beta_x\sigma_m\sigma_m'\beta_x'\check{C} + \frac{q}{\gamma}\alpha C_1\right)'\check{g} + \check{g}'\left(-\alpha\lambda + \frac{q}{\gamma}\beta_x\sigma_m\sigma_m'\beta_x'\check{C} + \frac{q}{\gamma}\alpha C_1\right) \\ & + \check{g}'\alpha(\beta\sigma_m\sigma_m'\beta' + \sigma\sigma' + bb')\alpha'\check{g} \\ & - \frac{q}{\gamma}(C_1'(\sigma\sigma' + bb')^{-1}\lambda + \lambda'(\sigma\sigma' + bb')^{-1}C_1) + \frac{q^2 - q}{\gamma^2}\Big(\check{C}\alpha\beta(\sigma_m\sigma_m')\beta'\alpha'\check{C} + C_1'(\sigma\sigma' + bb')^{-1}C_1\Big). \end{aligned}$$

The expected portfolio return is given by $\check{f}(x;1)$ and the variance of the portfolio return is given by $\check{f}(x;2) - \check{f}^2(x;1)$. Again generalizations to higher order moments are straightforward.

B Proof of Proposition 5

We use the dynamic programming principle to solve the optimization problem in Proposition 5. To this end we define the value function $J(x_t, W_t, t)$

$$J(x_t, W_t, t) = \frac{1}{1-\gamma}\,\mathrm{E}_t\left[W_T^{*1-\gamma}\right],$$

where W_T^* is the wealth at time T associated with the optimal investment strategy. The dynamic programming principle implies that the value function J satisfies the multivariate HJB equation

$$\begin{aligned}
0 = \max\ & J_t + (\mu_x - \lambda_x x)' J_x + \frac{1}{2}\mathrm{Tr}\Big((\beta_x\sigma_m\sigma_m'\beta_x' + \sigma_x\sigma_x' + b_x b_x')J_{xx'}\Big)\\
&+\Big(r + \phi_m'\mu_m + \phi'(\beta\mu_m - \lambda x)\Big)WJ_W\\
&+\Big(\beta_x\sigma_m\sigma_m'(\phi_m + \beta'\phi) + \sigma_x\sigma'\phi + \alpha bb'\phi\Big)' WJ_{xW}\\
&+\frac{1}{2}\Big((\phi_m' + \phi'\beta)\sigma_m\sigma_m'(\phi_m + \beta'\phi) + \phi'\sigma\sigma'\phi + \phi' bb'\phi\Big)W^2 J_{WW}.
\end{aligned}$$

We conjecture that

$$J(x, W, t) = \frac{1}{1-\gamma}W^{1-\gamma}e^{A(t)+B(t)'x+\frac{1}{2}x'C(t)x},$$

where $A(t)$, $B(t)$, and $C(t)$ are a scalar function, an $h\times 1$ vector function, and an $h\times h$ symmetric matrix function of time t, respectively.

Substituting this into the HJB equation, we obtain the following expression

$$\begin{aligned}
0 = \max\ & A_t + B_t'x + \frac{1}{2}x'C_t x + (\mu_x - \lambda_x x)'(B + Cx)\\
&+\frac{1}{2}\mathrm{Tr}\Big(\beta_x\sigma_m\sigma_m'\beta_x' + \sigma_x\sigma_x' + b_x b_x')((B+Cx)(B+Cx)' + C)\Big)\\
&+\Big(r + \phi_m'\mu_m + \phi'(\beta\mu_m - \lambda x))\Big)(1-\gamma)\\
&+\Big(\beta_x\sigma_m\sigma_m'(\phi_m + \beta'\phi) + \sigma_x\sigma'\phi + \alpha bb'\phi\Big)'(B + Cx)(1-\gamma)\\
&+\frac{1}{2}\Big((\phi_m' + \phi'\beta)\sigma_m\sigma_m'(\phi_m + \beta'\phi) + \phi'\sigma\sigma'\phi + \phi' bb'\phi\Big)(-\gamma)(1-\gamma).
\end{aligned} \tag{B-3}$$

The first order conditions for the optimal values of $\phi_m + \beta'\phi$ and ϕ are

$$\begin{aligned}
&\mu_m + \sigma_m\sigma_m'\beta_x'(B + Cx) - \sigma_m\sigma_m'\gamma(\phi_m + \beta'\phi) = 0;\\
&-\lambda x + \Big(\sigma\sigma_x' + bb'\alpha'\Big)(B + Cx) - \Big(\sigma\sigma' + bb'\Big)\phi\gamma = 0.
\end{aligned}$$

These form a system of linear equations in ϕ_m and ϕ that can be solved to get

$$\begin{aligned}
\phi_m^* &= \frac{1}{\gamma}(\sigma_m\sigma_m')^{-1}\mu_m + \frac{\beta_x'(B + Cx)}{\gamma} - \beta'\phi^*,\\
\phi^* &= \frac{1}{\gamma}\Big(\sigma\sigma' + bb'\Big)^{-1}\Big(-\lambda x + \Big(\sigma\sigma_x' + bb'\alpha'\Big)(B + Cx)\Big).
\end{aligned}$$

The optimal portfolio weights given in the proposition are obtained from this using that $x_t = \alpha\ln P_t$. Substituting the optimal portfolio weights into equation (B-3), we have

$$\begin{aligned}
0 = {} & A_t + B_t'x + \frac{1}{2}x'C_t x + (\mu_x - \lambda_x x)'(B + Cx)\\
&+\frac{1}{2}\mathrm{Tr}\Big(\beta_x\sigma_m\sigma_m'\beta_x' + \sigma_x\sigma_x' + b_x b_x')((B+Cx)(B+Cx)' + C)\Big)\\
&+(1-\gamma)r + \frac{(1-\gamma)}{2\gamma}\left(\mu_m + \sigma_m\sigma_m'\beta_x'(B + Cx)\right)'(\sigma_m\sigma_m')^{-1}\left(\mu_m + \sigma_m\sigma_m'\beta_x'(B + Cx)\right)\\
&+\frac{(1-\gamma)}{2\gamma}\Big(-\lambda x + (\sigma\sigma_x' + bb'\alpha')(B + Cx)\Big)'(\sigma\sigma' + bb')^{-1}\Big(-\lambda x + (\sigma\sigma_x' + bb'\alpha')(B + Cx)\Big).
\end{aligned}$$

This is a quadratic equation in x which must hold for all values of x. Thus, the coefficients of all powers of x should be zero, which leads to the following system of ODEs

$$
\begin{aligned}
&0 = A_t + \mu_x' B + \frac{1}{2}\mathrm{Tr}\Big((\beta_x\sigma_m\sigma_m'\beta_x' + \sigma_x\sigma_x' + b_x b_x')(BB' + C)\Big) \\
&+(1-\gamma)r + \frac{(1-\gamma)}{2\gamma}\left(\mu_m + \sigma_m\sigma_m'\beta_x' B\right)'(\sigma_m\sigma_m')^{-1}\left(\mu_m + \sigma_m\sigma_m'\beta_x' B\right) \\
&+\frac{(1-\gamma)}{2\gamma}B'(\sigma\sigma_x' + bb'\alpha')'(\sigma\sigma' + bb')^{-1}(\sigma\sigma_x' + bb'\alpha')B; \\
&0 = B_t + C\mu_x - \lambda_x' B + C(\beta_x\sigma_m\sigma_m'\beta_x' + \sigma_x\sigma_x' + b_x b_x')B \\
&+\frac{(1-\gamma)}{\gamma}C\beta_x\left(\mu_m + \sigma_m\sigma_m'\beta_x' B\right) \\
&+\frac{(1-\gamma)}{\gamma}\Big(-\lambda + (\sigma\sigma_x' + bb'\alpha')C\Big)'\Big(\sigma\sigma' + bb'\Big)^{-1}(\sigma\sigma_x' + bb'\alpha')B; \\
&0 = C_t - (\lambda_x' C + C\lambda_x) + C\Big(\beta_x\sigma_m\sigma_m'\beta_x' + \sigma_x\sigma_x' + b_x b_x'\Big)C \\
&+\frac{(1-\gamma)}{\gamma}C\beta_x\sigma_m\sigma_m'\beta_x' C \\
&+\frac{(1-\gamma)}{\gamma}\Big(-\lambda + (\sigma\sigma_x' + bb'\alpha')C\Big)'\Big(\sigma\sigma' + bb'\Big)^{-1}\Big(-\lambda + (\sigma\sigma_x' + bb'\alpha')C\Big).
\end{aligned}
$$

The boundary conditions in this case are $A(T) = B(T) = C(T) = 0$.

C Proof of Proposition 6

For the constrained strategy we have $\kappa'\phi_t = 0$. Market neutral strategies arise as a special case when $\kappa = \beta$. The HJB equation then reduces to

$$
\begin{aligned}
&0 = \max\ J_t + (\mu_x - \lambda_x x)' J_x + \frac{1}{2}\mathrm{Tr}\Big((\beta_x\sigma_m\sigma_m'\beta_x' + \sigma_x\sigma_x' + b_x b_x')J_{xx'}\Big) \\
&+\Big(r + (\phi_m + \beta'\phi)'\mu_m - \phi'\lambda x\Big)WJ_W \\
&+\Big(\beta_x\sigma_m\sigma_m'(\phi_m + \beta'\phi) + \sigma_x\sigma'\phi + \alpha bb'\phi\Big)'WJ_{xW} \\
&+\frac{1}{2}\Big((\phi_m + \beta'\phi)'\sigma_m\sigma_m'(\phi_m + \beta'\phi) + \phi'\sigma\sigma'\phi + \phi' bb'\phi\Big)W^2 J_{WW} - \phi'\kappa\mathcal{L},
\end{aligned}
$$

where $\mathcal{L}$ is the Lagrangian multiplier for the constraint. We again conjecture that

$$
J(x, W, t) = \frac{1}{1-\gamma}W^{1-\gamma}e^{\check{A}(t) + \check{B}(t)'x + \frac{1}{2}x'\check{C}(t)x},
$$

where $\check{A}(t)$, $\check{B}(t)$, and $\check{C}(t)$ are a scalar function, an $h \times 1$ dimensional vector function, and an $h \times h$ dimensional symmetric matrix function of time t, respectively. Substituting this conjecture into the HJB equation, we get

$$
\begin{aligned}
&0 = \max \check{A}_t + \check{B}_t' x + \frac{1}{2}x'\check{C}_t x + (\mu_x - \lambda_x x)'(\check{B} + \check{C}x) \\
&+\frac{1}{2}\mathrm{Tr}\Big((\beta_x\sigma_m\sigma_m'\beta_x' + \sigma_x\sigma_x' + b_x b_x')(\check{C} + (\check{B} + \check{C}x)(\check{B} + \check{C}x)')\Big) \\
&+\Big(r + (\phi_m + \beta'\phi)'\mu_m - \phi'\lambda x\Big)(1-\gamma) \\
&+\Big(\beta_x\sigma_m\sigma_m'(\phi_m + \beta'\phi) + \sigma_x\sigma'\phi + \alpha bb'\phi\Big)'(1-\gamma)(\check{B} + \check{C}x) \\
&+\frac{1}{2}\Big((\phi_m + \beta'\phi)'\sigma_m\sigma_m'(\phi_m + \beta'\phi) + \phi'\sigma\sigma'\phi + \phi' bb'\phi\Big)(-\gamma)(1-\gamma) - \phi'\kappa\mathcal{L}. \qquad \text{(C-4)}
\end{aligned}
$$

The first order conditions for the optimal $\phi_m + \beta'\phi$ and ϕ are

$$\mu_m + \sigma_m\sigma_m'\beta_x'(\check{B} + \check{C}x) - \gamma\sigma_m\sigma_m'(\phi_m + \beta'\phi) = 0;$$
$$-\lambda x + \Big(\sigma\sigma_x'\phi + bb'\alpha'\Big)(\check{B} + \check{C}x) - \gamma(\sigma\sigma' + bb')\phi - \kappa\mathcal{L} = 0.$$

Thus, thee optimal $\phi_m + \beta'\phi$ is given by

$$(\phi_m + \beta'\phi) = \frac{1}{\gamma}(\sigma_m\sigma_m')^{-1}(\mu_m + \sigma_m\sigma_m'\beta_x'(\check{B} + \check{C}x))$$

while the optimal ϕ is

$$\phi = \frac{1}{\gamma}(\sigma\sigma' + bb')^{-1}\Big(- \lambda x + (\sigma\sigma' + bb')\alpha'(\check{B} + \check{C}x) - \kappa\mathcal{L}\Big),$$

where $\mathcal{L}$ is the Lagrangian multiplier for the constraint $\kappa'\phi_t = 0$. Using the constraint

$$0 = \kappa'(\sigma\sigma' + bb')^{-1}\Big(- \lambda x + (\sigma_x\sigma' + \alpha bb')'(\check{B} + \check{C}x) - \kappa\mathcal{L}\Big),$$

the Lagrangian multiplier can be obtained as

$$\mathcal{L} = (\kappa'(\sigma\sigma' + bb')^{-1}\kappa)^{-1}\kappa'(\sigma\sigma' + bb')^{-1}\Big(- \lambda x + (\sigma_x\sigma' + \alpha bb')'(\check{B} + \check{C}x)\Big).$$

The optimal constrained portfolio weight ϕ is

$$\begin{aligned}
\phi &= \frac{1}{\gamma}(\sigma\sigma' + bb')^{-1}\Big(I - \kappa(\kappa'(\sigma\sigma' + bb')^{-1}\kappa)^{-1}\kappa'(\sigma\sigma' + bb')^{-1}\Big)\Big(- \lambda x + (\sigma\sigma' + bb')\alpha'(\check{B} + \check{C}x)\Big) \\
&= \frac{1}{\gamma}(\sigma\sigma' + bb')^{-1}(B_1 + C_1 x),
\end{aligned}$$

where the last equality defines B_1 and C_1:

$$\begin{aligned}
B_1 &= \Big(I - \kappa(\kappa'(\sigma\sigma' + bb')^{-1}\kappa)^{-1}\kappa'(\sigma\sigma' + bb')^{-1}\Big)(\sigma\sigma' + bb')\alpha'\check{B}; \\
C_1 &= \Big(I - \kappa(\kappa'(\sigma\sigma' + bb')^{-1}\kappa)^{-1}\kappa'(\sigma\sigma' + bb')^{-1}\Big)\Big(- \lambda + (\sigma\sigma' + bb')\alpha'\check{C}\Big).
\end{aligned}$$

Substituting the optimal portfolio weight back in equation (C-4), we get

$$\begin{aligned}
&0 = \check{A}_t + \check{B}_t'x + \frac{1}{2}x'\check{C}_t x + (\mu_x - \lambda_x x)'(\check{B} + \check{C}x) \\
&+ \frac{1}{2}\mathrm{Tr}\Big((\beta_x\sigma_m\sigma_m'\beta_x' + \sigma_x\sigma_x' + b_x b_x')(\check{C} + (\check{B} + \check{C}x)(\check{B} + \check{C}x)')\Big) + r(1-\gamma) \\
&+ \frac{1-\gamma}{2\gamma}\Big((\mu_m + \sigma_m\sigma_m'\beta_x'(\check{B} + \check{C}x))'(\sigma_m\sigma_m')^{-1}(\mu_m + \sigma_m\sigma_m'\beta_x'(\check{B} + \check{C}x)) + (B_1 + C_1 x)'(\sigma\sigma' + bb')^{-1}(B_1 + C_1 x)\Big).
\end{aligned}$$

Comparing the coefficient of various powers of x, we get the following system of ODEs:

$$\begin{aligned}
&0 = \check{A}_t + \mu_x'\check{B} + \frac{1}{2}\mathrm{Tr}\Big((\beta_x\sigma_m\sigma_m'\beta_x' + \sigma_x\sigma_x' + b_x b_x')(\check{C} + \check{B}\check{B}')\Big) + r(1-\gamma) \\
&+ \frac{1-\gamma}{2\gamma}\Big((\mu_m + \sigma_m\sigma_m'\beta_x'\check{B})'(\sigma_m\sigma_m')^{-1}(\mu_m + \sigma_m\sigma_m'\beta_x'\check{B}) + B_1'(\sigma\sigma' + bb')^{-1}B_1\Big); \\
&0 = \check{B}_t + \check{C}\mu_x - \lambda_x'\check{B} + \check{C}(\beta_x\sigma_m\sigma_m'\beta_x' + \sigma_x\sigma_x' + b_x b_x')\check{B} \\
&+ \frac{1-\gamma}{\gamma}\Big(\check{C}\beta_x(\mu_m + \sigma_m\sigma_m'\beta_x'\check{B}) + C_1(\sigma\sigma' + bb')^{-1}B_1\Big); \\
&0 = \check{C}_t - (\lambda_x'\check{C} + \check{C}\lambda_x) + \check{C}(\beta_x\sigma_m\sigma_m'\beta_x' + \sigma_x\sigma_x' + b_x b_x')\check{C} \\
&+ \frac{1-\gamma}{\gamma}\Big(\check{C}\beta_x(\sigma_m\sigma_m')^{-1}\beta_x'\check{C} + C_1(\sigma\sigma' + bb')^{-1}C_1\Big).
\end{aligned}$$

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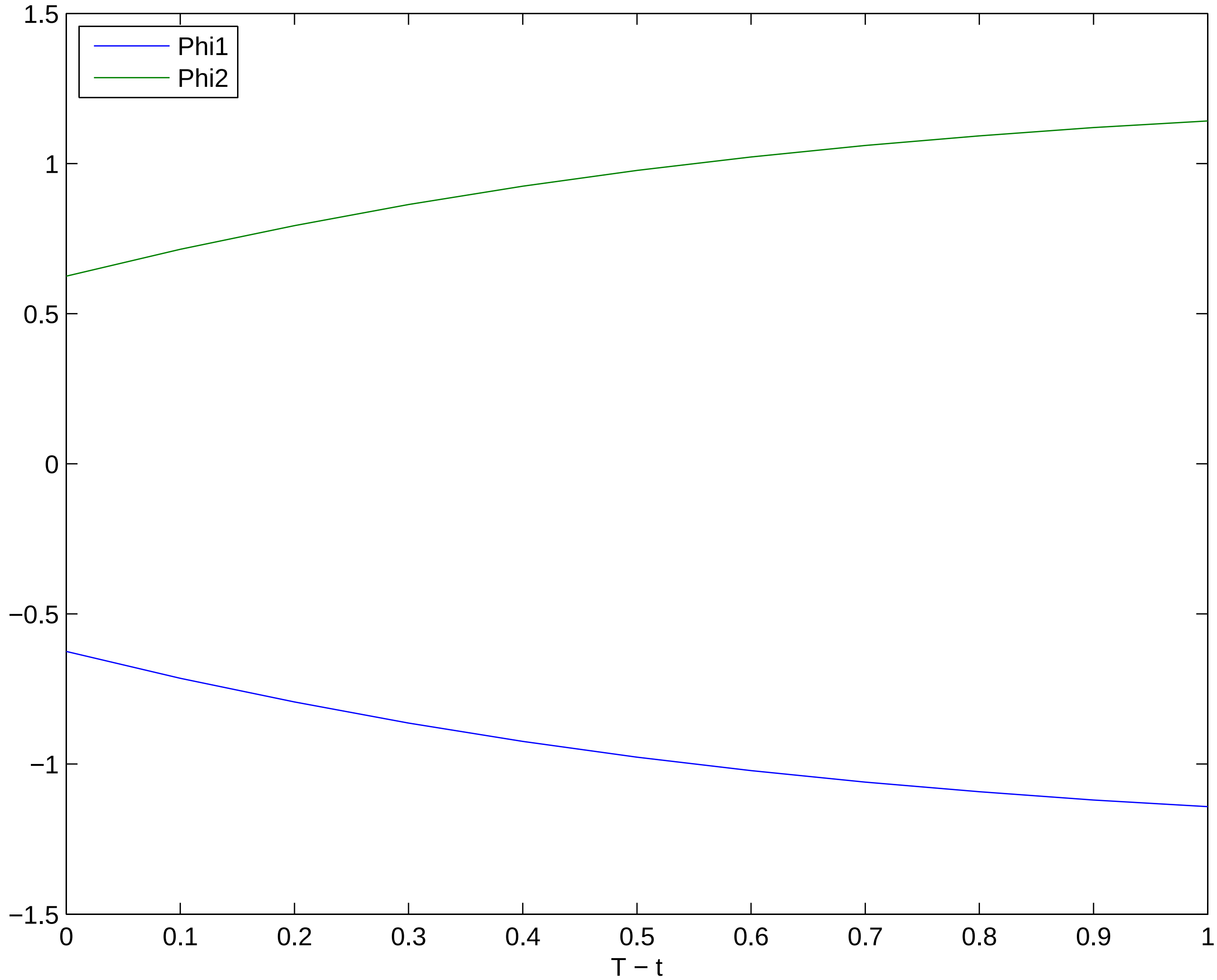

Figure 1: Optimal portfolio weights under symmetric mispricing. The computations assume the following parameter values: Volatility of common idiosyncratic risk: $\sigma_1 = \sigma_2 = 20\%$. Volatility of independent idiosyncratic risk: $b_1 = b_2 = 20\%$; Sensitivity to error correction term: $\lambda_1 = \lambda_2 = 1$. Size of mispricing: $x_t = 10\%$. Coefficient of risk aversion, $\gamma = 4$.

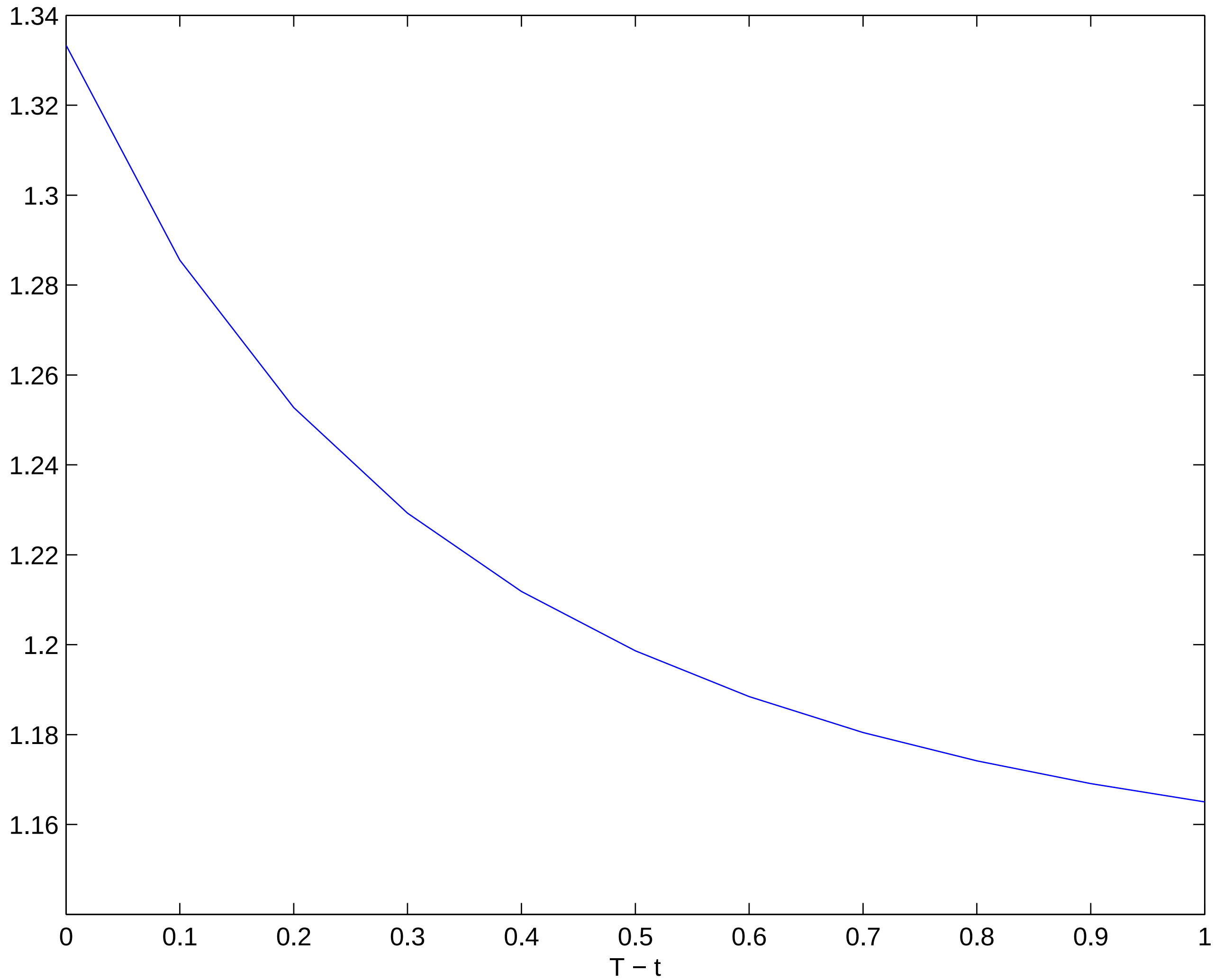

Figure 2: Ratio of optimal portfolio weights under asymmetric mispricing. The computations assume the following parameter values: Volatility of common idiosyncratic risk: $\sigma_1 = \sigma_2 = 20\%$. Volatility of independent idiosyncratic risk: $b_1 = 10\%$, $b_2 = 20\%$; Sensitivity to error correction term: $\lambda_1 = \lambda_2 = 1$. Size of mispricing: $x_t = 20\%$. Coefficient of risk aversion, $\gamma = 4$.

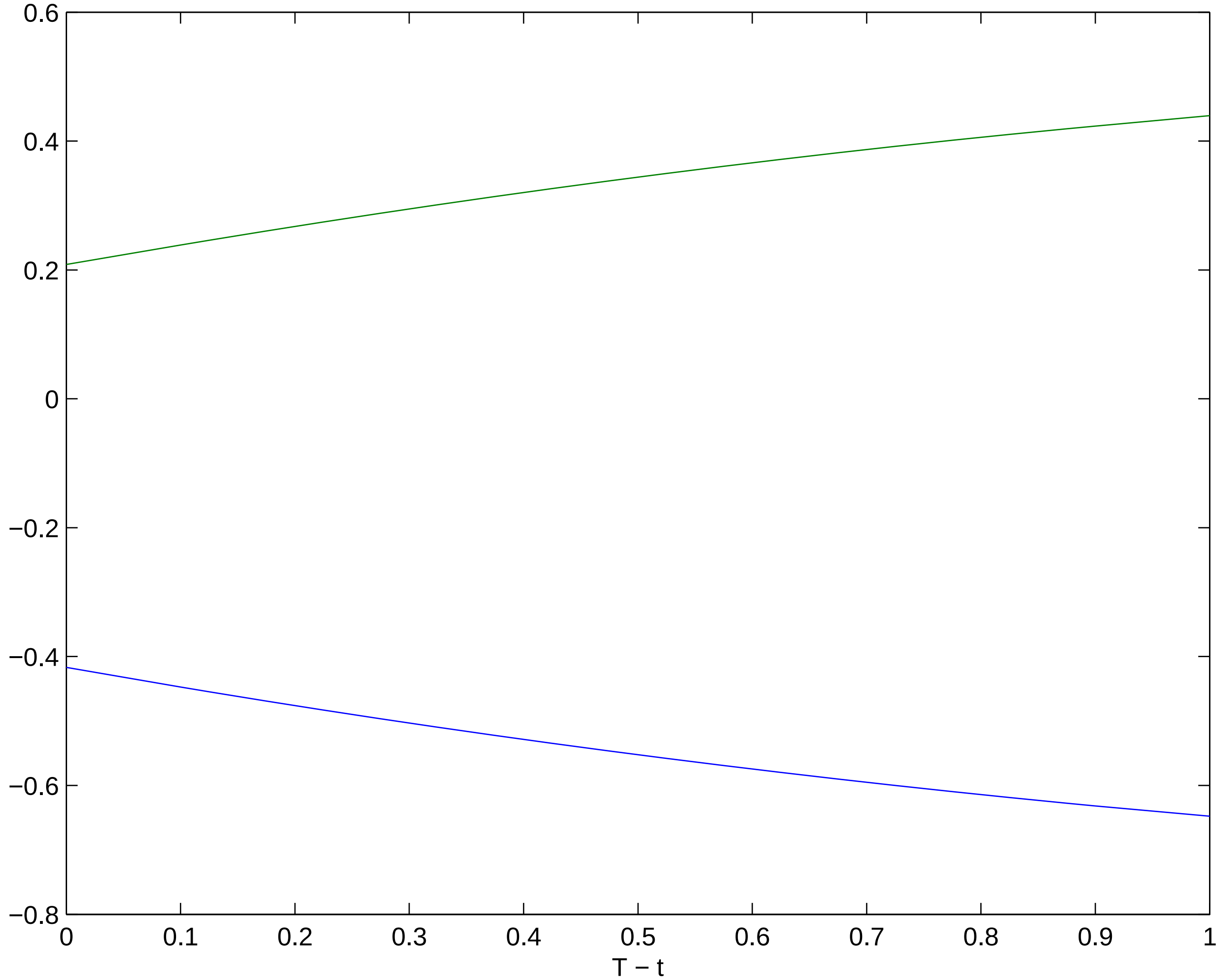

Figure 3: Optimal portfolio weights under mispricing only in asset 1. The lower line tracks the (short) investment in asset 1, while the upper line tracks holdings in asset 2. The computations assume the following parameter values: Volatility of common idiosyncratic risk: $\sigma_1 = \sigma_2 = 20\%$. Volatility of independent idiosyncratic risk: $b_1 = b_2 = 20\%$; Sensitivity to error correction term: $\lambda_1 = 1$, $\lambda_2 = 0$. Size of mispricing: $x_t = 20\%$. Coefficient of risk aversion, $\gamma = 4$.

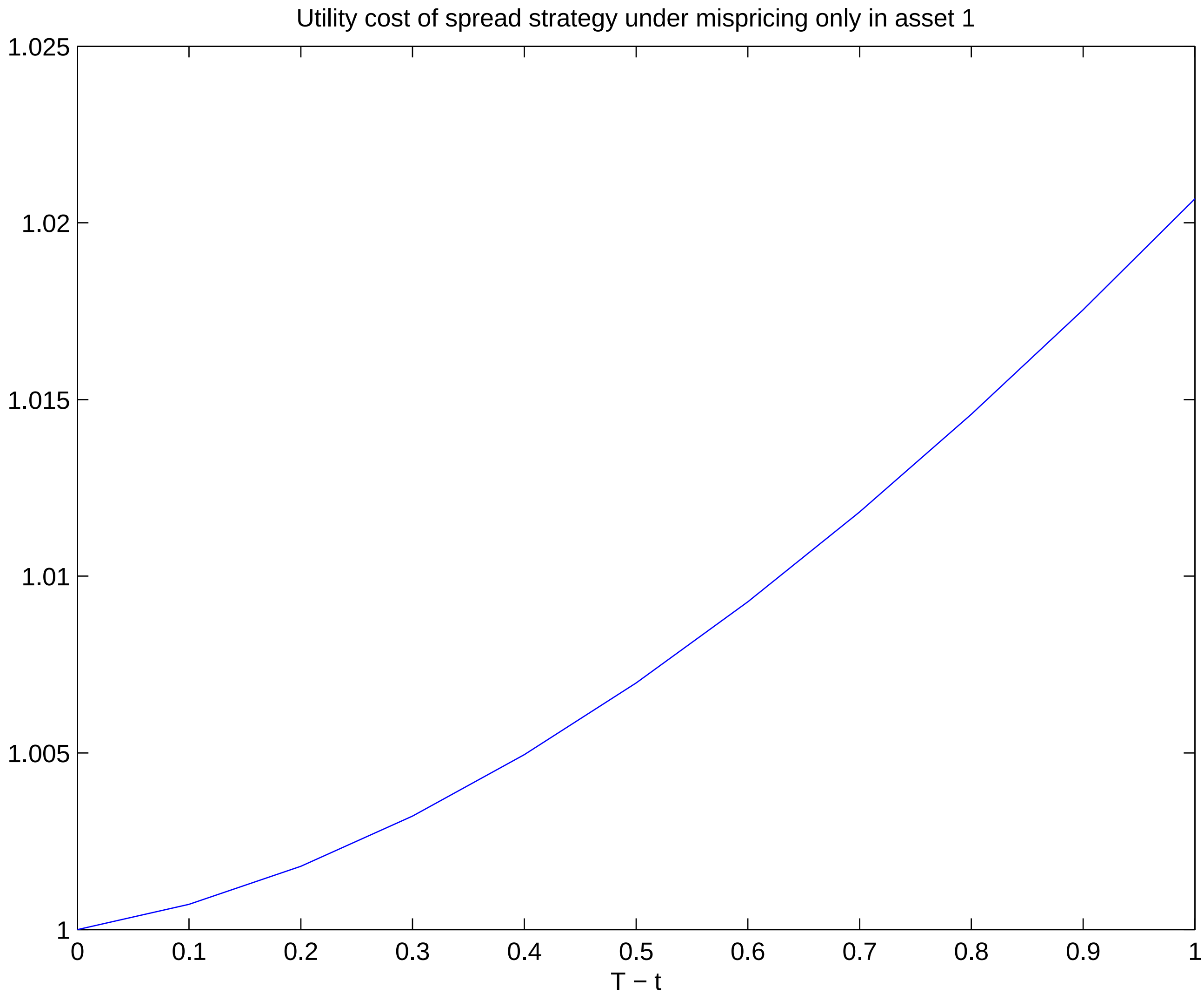

Figure 4: Utility cost of using market neutral spread strategy under mispricing in asset 1. The computations assume the following parameter values: Volatility of common idiosyncratic risk: $\sigma_1 = \sigma_2 = 20\%$. Volatility of independent idiosyncratic risk: $b_1 = b_2 = 20\%$; Sensitivity to error correction term: $\lambda_1 = 1$, $\lambda_2 = 0$. Size of mispricing: $x_t = 20\%$. Coefficient of risk aversion, $\gamma = 4$.

Table 1: Effect of changes in the state variable, x, on optimal portfolio holdings

A: Symmetric case ($\lambda_1 = \lambda_2 = 1$)

	Asset 1				Asset 2			
Horizon, x	-0.2	-0.15	-0.1	-0.05	-0.2	-0.15	-0.1	-0.05
0	1.25	0.94	0.63	0.31	-1.25	-0.94	-0.63	-0.31
0.1	1.43	1.07	0.71	0.36	-1.43	-1.07	-0.71	-0.36
0.2	1.59	1.19	0.79	0.40	-1.59	-1.19	-0.79	-0.40
0.3	1.73	1.30	0.86	0.43	-1.73	-1.30	-0.86	-0.43
0.4	1.85	1.39	0.92	0.46	-1.85	-1.39	-0.92	-0.46
0.5	1.95	1.47	0.98	0.49	-1.95	-1.47	-0.98	-0.49
0.6	2.04	1.53	1.02	0.51	-2.04	-1.53	-1.02	-0.51
0.7	2.12	1.59	1.06	0.53	-2.12	-1.59	-1.06	-0.53
0.8	2.18	1.64	1.09	0.55	-2.18	-1.64	-1.09	-0.55
0.9	2.24	1.68	1.12	0.56	-2.24	-1.68	-1.12	-0.56
1	2.28	1.71	1.14	0.57	-2.28	-1.71	-1.14	-0.57

B: Asymmetric case: mispricing in asset 1 only ($\lambda_1 = 1, \lambda_2 = 0$)

	Asset 1				Asset 2			
Horizon, x	-0.2	-0.15	-0.1	-0.05	-0.2	-0.15	-0.1	-0.05
0	0.83	0.63	0.42	0.21	-0.42	-0.31	-0.21	-0.10
0.1	0.89	0.67	0.45	0.22	-0.48	-0.36	-0.24	-0.12
0.2	0.95	0.71	0.48	0.24	-0.54	-0.40	-0.27	-0.13
0.3	1.01	0.75	0.50	0.25	-0.59	-0.44	-0.29	-0.15
0.4	1.06	0.79	0.53	0.26	-0.64	-0.48	-0.32	-0.16
0.5	1.10	0.83	0.55	0.28	-0.69	-0.52	-0.34	-0.17
0.6	1.15	0.86	0.57	0.29	-0.73	-0.55	-0.37	-0.18
0.7	1.19	0.89	0.60	0.30	-0.77	-0.58	-0.39	-0.19
0.8	1.23	0.92	0.61	0.31	-0.81	-0.61	-0.41	-0.20
0.9	1.26	0.95	0.63	0.32	-0.85	-0.64	-0.42	-0.21
1	1.30	0.97	0.65	0.32	-0.88	-0.66	-0.44	-0.22

Note: This table shows the optimal holdings in two risky assets for different time horizons (T-t) and different values of the state variable, x, that captures mispricing in the model:

$\frac{dP_{mt}}{P_{mt}} = (r + \mu_m)dt + \sigma_m dB_t$

$\frac{dP_{1t}}{P_{1t}} = (r + \beta_1 \mu_m)dt + \beta_1 \sigma_m dB_t + \sigma_1 dZ_t + b_1 dZ_{1t} - \lambda_1 x_t dt$

$\frac{dP_{2t}}{P_{2t}} = (r + \beta_2 \mu_m)dt + \beta_2 \sigma_m dB_t + \sigma_2 dZ_t + b_2 dZ_{2t} + \lambda_2 x_t dt,$

where we assume the following parameter values: $\beta_1 = \beta_2 = 1, \sigma_1 = \sigma_2 = 0.2, \alpha = 1,$ $b_1 = b_2 = 0.2, \mu_m = 0.06, r = 0.03, \sigma_m = 0.15.$

The coefficient of relative risk aversion is set at $\gamma = 4$.

Table 2: Effect of differences in risk parameters on the ratio of optimal portfolio holdings $(-\phi_1/\phi_2)$

	λ_2 ($\lambda_1 = 1$):				b_2 ($b_1 = 0.2$):				σ_2 ($\sigma_1 = 0.2$):			
Horizon	0	0.5	1.5	2	0	0.1	0.3	0.4	0.05	0.1	0.15	0.25
0	2.00	1.25	0.88	0.80	0.67	0.75	1.42	2.00	0.58	0.70	0.84	1.17
0.1	1.87	1.22	0.89	0.83	0.70	0.78	1.35	1.82	0.63	0.73	0.86	1.15
0.2	1.78	1.20	0.90	0.85	0.73	0.80	1.31	1.70	0.66	0.76	0.87	1.13
0.3	1.71	1.19	0.91	0.87	0.75	0.81	1.28	1.62	0.68	0.77	0.88	1.12
0.4	1.65	1.17	0.92	0.87	0.76	0.82	1.26	1.56	0.69	0.78	0.89	1.11
0.5	1.61	1.16	0.92	0.88	0.77	0.83	1.24	1.51	0.71	0.79	0.89	1.11
0.6	1.57	1.16	0.93	0.88	0.78	0.84	1.23	1.48	0.72	0.80	0.90	1.10
0.7	1.54	1.15	0.93	0.89	0.78	0.84	1.22	1.45	0.72	0.81	0.90	1.10
0.8	1.51	1.14	0.93	0.89	0.79	0.84	1.21	1.43	0.73	0.81	0.90	1.09
0.9	1.49	1.14	0.93	0.89	0.79	0.85	1.20	1.41	0.73	0.82	0.91	1.09
1	1.47	1.14	0.93	0.89	0.79	0.85	1.20	1.39	0.74	0.82	0.91	1.09

Note: This table shows the effect that differences in the parameters of two risky asset prices has on the ratio of the optimal portfolio holdings, $-\phi_1/\phi_2$. The following model is assumed:

$\frac{dP_{mt}}{P_{mt}} = (r + \mu_m)dt + \sigma_m dB_t$

$\frac{dP_{1t}}{P_{1t}} = (r + \beta_1\mu_m)dt + \beta_1\sigma_m dB_t + \sigma_1 dZ_t + b_1 dZ_{1t} - \lambda_1 x_t dt$

$\frac{dP_{2t}}{P_{2t}} = (r + \beta_2\mu_m)dt + \beta_2\sigma_m dB_t + \sigma_2 dZ_t + b_2 dZ_{2t} + \lambda_2 x_t dt$

where the parameter values are set at $x = 0.1, \beta_1 = \beta_2 = 1, \sigma_1 = \sigma_2 = 0.2, \alpha = 1,$
$b_1 = b_2 = 0.2, \lambda_1 = \lambda_2 = 1, \mu_m = 0.06, r = 0.03, \sigma_m = 0.15.$
The coefficient of relative risk aversion is set at $\gamma = 4$.

Table 3. Wealth gains for the optimal versus the constrained strategy

Horizon, x	0	0.05	0.1	0.15	0.2
0.1	0.02	0.04	0.11	0.22	0.38
0.2	0.09	0.13	0.25	0.46	0.75
0.3	0.19	0.24	0.41	0.70	1.11
0.4	0.32	0.38	0.59	0.95	1.45
0.5	0.48	0.55	0.79	1.20	1.78
0.6	0.65	0.73	1.00	1.45	2.09
0.7	0.85	0.93	1.21	1.70	2.39
0.8	1.06	1.14	1.44	1.95	2.68
0.9	1.28	1.36	1.67	2.20	2.95
1	1.50	1.59	1.90	2.44	3.22

Note: This table shows how the wealth gains (measured in percentage terms) depend on the size of the mispricing component (x) and the length of the investment horizon (T).
The following model is assumed:

$$\frac{dP_{mt}}{P_{mt}} = (r + \mu_m)dt + \sigma_m dB_t$$

$$\frac{dP_{1t}}{P_{1t}} = (r + \beta_1\mu_m)dt + \beta_1\sigma_m dB_t + \sigma_1 dZ_t + b_1 dZ_{1t} - \lambda_1 x_t dt$$

$$\frac{dP_{2t}}{P_{2t}} = (r + \beta_2\mu_m)dt + \beta_2\sigma_m dB_t + \sigma_2 dZ_t + b_2 dZ_{2t} + \lambda_2 x_t dt$$

where $\beta_1 = \beta_2 = 1, \sigma_1 = \sigma_2 = 0.2, \alpha = 1, b_1 = b_2 = 0.2,$
$\lambda_1 = 2, \lambda_2 = 0, \mu_m = 0.06, r = 0.03, \sigma_m = 0.15.$
The coefficient of relative risk aversion is set at $\gamma = 4$.

Table 4: Moments of returns for the optimal unconstrained and market neutral strategies

A. Mean returns

horizon, x	optimal weights 0	0.05	0.1	0.15	0.2	constrained weights 0	0.05	0.1	0.15	0.2
0.1	0.012	0.0165	0.0307	0.0547	0.0895	0.0115	0.0157	0.0286	0.0506	0.0821
0.2	0.0323	0.0401	0.065	0.1083	0.172	0.0307	0.0379	0.0606	0.0999	0.1575
0.3	0.0593	0.0696	0.1032	0.1623	0.2509	0.056	0.0653	0.0959	0.1495	0.2293
0.4	0.0919	0.1042	0.1451	0.2178	0.3283	0.0863	0.0975	0.1346	0.2003	0.2995
0.5	0.1296	0.1434	0.1906	0.2755	0.4057	0.1212	0.1337	0.1766	0.253	0.3696
0.6	0.1718	0.1869	0.2399	0.3357	0.484	0.1602	0.1739	0.2218	0.308	0.4404
0.7	0.2183	0.2346	0.2928	0.3989	0.5641	0.203	0.2177	0.2702	0.3654	0.5126
0.8	0.2689	0.2862	0.3494	0.4651	0.6465	0.2495	0.2651	0.3219	0.4256	0.5867
0.9	0.3236	0.3418	0.4097	0.5348	0.7317	0.2995	0.316	0.3769	0.4887	0.6631
1	0.3822	0.4013	0.4738	0.608	0.82	0.3531	0.3702	0.4352	0.5548	0.7422

B. Standard deviation

horizon, x	optimal weights					constrained weights				
0.1	0.045	0.0535	0.0745	0.1019	0.134	0.044	0.052	0.0716	0.0974	0.1274
0.2	0.0724	0.08	0.1013	0.1326	0.1729	0.0703	0.0774	0.0971	0.126	0.1631
0.3	0.0953	0.102	0.1221	0.1543	0.1986	0.0921	0.0982	0.1166	0.1458	0.1859
0.4	0.1157	0.1218	0.1411	0.1734	0.2204	0.1113	0.1168	0.1341	0.1631	0.2047
0.5	0.1349	0.1406	0.1595	0.1922	0.2414	0.1291	0.1342	0.1509	0.1798	0.2227
0.6	0.1535	0.1591	0.178	0.2116	0.2631	0.1463	0.1512	0.1678	0.197	0.2414
0.7	0.1721	0.1777	0.197	0.2318	0.2861	0.1632	0.1682	0.1849	0.2149	0.2612
0.8	0.191	0.1968	0.2168	0.2532	0.3107	0.1804	0.1854	0.2027	0.2338	0.2824
0.9	0.2105	0.2165	0.2375	0.2759	0.337	0.1979	0.2032	0.2211	0.2538	0.3051
1	0.2309	0.2371	0.2592	0.2999	0.365	0.2161	0.2216	0.2404	0.2748	0.3292

C. Sharpe ratio

horizon, x	optimal weights					constrained weights				
0.1	0.1993	0.2529	0.3715	0.5079	0.6454	0.1942	0.2447	0.3577	0.4885	0.6211
0.2	0.3628	0.4264	0.5829	0.7713	0.9599	0.3512	0.4118	0.5621	0.7445	0.9289
0.3	0.5273	0.5939	0.7709	0.9933	1.2174	0.5095	0.5736	0.7451	0.9629	1.185
0.4	0.6901	0.7563	0.9429	1.1863	1.435	0.6672	0.7313	0.9137	1.1546	1.4046
0.5	0.8487	0.9123	1.1008	1.3544	1.618	0.8217	0.8839	1.0697	1.3233	1.5917
0.6	1.001	1.0607	1.2457	1.501	1.7707	0.9711	1.0299	1.2138	1.4715	1.7493
0.7	1.1454	1.2004	1.3783	1.6289	1.8973	1.1138	1.1684	1.3466	1.6018	1.8812
0.8	1.2809	1.331	1.4995	1.7408	2.0024	1.2487	1.2987	1.4688	1.7165	1.9914
0.9	1.4071	1.4521	1.6099	1.8391	2.0899	1.3752	1.4204	1.5809	1.818	2.0838
1	1.5237	1.5638	1.7103	1.9257	2.1633	1.493	1.5334	1.6836	1.9081	2.1619

D. Skew

horizon, x	optimal weights					constrained weights				
0.1	-0.7518	-0.9975	-0.9776	-0.7432	-0.5037	-0.7448	-1.0165	-1.032	-0.8083	-0.5694
0.2	-0.8819	-0.9967	-1.035	-0.8828	-0.668	-0.9046	-1.0401	-1.1148	-0.9819	-0.7721
0.3	-0.8624	-0.9228	-0.9711	-0.8901	-0.7286	-0.9074	-0.9832	-1.0661	-1.0115	-0.8623
0.4	-0.8053	-0.8372	-0.8753	-0.8336	-0.7191	-0.8661	-0.9094	-0.9782	-0.9664	-0.8711
0.5	-0.7356	-0.7515	-0.7756	-0.751	-0.6688	-0.8076	-0.8316	-0.8818	-0.8872	-0.8284
0.6	-0.6628	-0.6692	-0.681	-0.6616	-0.5987	-0.7426	-0.7547	-0.788	-0.7962	-0.7581
0.7	-0.5911	-0.5919	-0.5942	-0.5748	-0.5223	-0.676	-0.6807	-0.7005	-0.7054	-0.6771
0.8	-0.5223	-0.5197	-0.5155	-0.4943	-0.4473	-0.6105	-0.6106	-0.6205	-0.6204	-0.5958
0.9	-0.4573	-0.4528	-0.4442	-0.4212	-0.3771	-0.5474	-0.5447	-0.5477	-0.5429	-0.5191
1	-0.3962	-0.3907	-0.3795	-0.3553	-0.3132	-0.4874	-0.4832	-0.4816	-0.4731	-0.4492

E. Kurtosis

horizon, x	optimal weights					constrained weights				
0.1	5.6766	5.5865	4.4976	3.5684	3.0598	5.5538	5.623	4.6241	3.6657	3.1146
0.2	5.4941	5.4816	4.8592	3.9674	3.299	5.5555	5.6167	5.0627	4.1428	3.4127
0.3	5.0325	5.0765	4.8178	4.191	3.5641	5.1653	5.2574	5.0664	4.4338	3.7482
0.4	4.7456	4.7566	4.6122	4.1909	3.6822	4.9145	4.9588	4.8758	4.4682	3.9174
0.5	4.4983	4.4865	4.3818	4.0905	3.6989	4.6844	4.6954	4.6419	4.3745	3.9588
0.6	4.2833	4.2588	4.168	3.9528	3.6525	4.475	4.4653	4.414	4.2243	3.9125
0.7	4.0982	4.0674	3.9823	3.8117	3.5761	4.2875	4.2656	4.209	4.06	3.8191
0.8	3.9399	3.9068	3.8257	3.6828	3.492	4.1215	4.0929	4.0311	3.9037	3.709
0.9	3.8054	3.7723	3.6957	3.5717	3.4126	3.9759	3.9442	3.8794	3.7648	3.6006
1	3.6915	3.6599	3.5886	3.4793	3.3438	3.8489	3.8166	3.7515	3.6461	3.5032

Note: This table shows the mean, standard deviation, Sharpe ratio, skew and kurtosis of portfolio returns for the optimal and the constrained (fixed relative weights) portfolios as a function of the mispricing component (x) and the investment horizon (T). The model assumed in the calculations is given in equations (1)-(3).

The table assumes the following parameter values: $\beta_1 = \beta_2 = 1$, $\sigma_1 = \sigma_2 = 0.2$, $\alpha = 1$, $b_1 = b_2 = 0.2$, $\lambda_1 = 2$, $\lambda_2 = 0$, $\mu_m = 0.06$, $r = 0.03$, $\sigma_m = 0.15$, $\gamma = 4$.