Durable-Goods Sales with Changing Values and Unobservable Arrival Dates*

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Abstract

We study the problem of a revenue-maximizing monopolist selling a durable good to a representative buyer who is forward-looking, whose value for the good changes stochastically, and for whom the value and arrival time are private information. Assuming the seller can fully commit, we consider the optimal path of posted prices and compare it with the unrestricted optimum (the “fully-optimal” mechanism). The reason for our focus on posted prices is that, presumably due to their simplicity, they are pervasive in retail markets. We show that it is often optimal for the path of prices to fluctuate or “cycle”, providing a possible explanation for the use of “sales” or periodic discounts. In contrast, the fully-optimal mechanism typically involves selling options with finite expiration dates; the associated payments can be chosen to evolve gradually and without cycles. We thereby shed light on how the seller’s ability to price discriminate across time depends on the degree of contractual sophistication.

JEL classification: D82. Keywords: sales, price cycles, durable goods, changing types, dynamic mechanism design, countercyclical mark-ups

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1 Introduction

We study the problem of a revenue-maximizing monopolist selling a durable good to a representative buyer who is forward-looking, whose value for the good changes stochastically, and for whom the value and arrival time are private information. Assuming the seller can fully commit, we consider the optimal path of posted prices and compare it with the unrestricted optimum (the “fully-optimal” mechanism). The reason for our focus on posted prices is that, presumably due to their simplicity, they are pervasive in retail markets. We show that it is often optimal for the path of prices to fluctuate or “cycle”, providing a possible explanation for the use of “sales” or periodic discounts. In contrast, the fully-optimal mechanism typically involves selling options with finite expiration dates; the associated payments can be chosen to evolve gradually and without cycles. We thereby shed light on how the seller’s ability to price discriminate across time depends on the degree of contractual sophistication.

The problem of a revenue-maximizing monopolist selling to buyers who arrive over time has been studied by a number of authors; e.g., Riley and Zeckhauser (1983), Conlisk et al. (1984), Wang (1993), Gallien (2006), Board (2008), Gershkov and Moldovanu (2009), Board and Skryzpacz (2010), Pai and Vohra (2011) and Said (2011). Motivating these papers is the observation that, in most contexts, sellers cater to an inflow of customers who purchase either as the need arises or as they learn about the existence of the product. One conclusion, which follows from a result of Stokey (1979) discussed below, is that, when the horizon is infinite, when the distribution of values is independent of time of arrival, and when there is an unlimited supply of the good (i.e., in the absence of demand-side competition for the good), intertemporal price discrimination is unprofitable. The revenue-maximizing mechanism is a constant price. To the extent sellers can find ways to make credible commitments, this is difficult to reconcile with the observation that prices for many goods fluctuate (see, e.g., Warner and Barsky (1995) for appliances and other consumer durables, and Bils and Klenow (2004) and Eichenbaum et al. (forthcoming) for more extensive data sets). In the context of airline pricing, McAfee and te Velde (2006) remark: “The
extent of price changes found in actual airline pricing is mysterious because a monopolist with commitment ability, in a standard framework, doesn’t want to engage in it at all!”

In contrast to the above literature, we consider an environment where there is uncertainty not only as to the buyer’s arrival date and initial value, but also about how that value evolves with time. We posit Markov switching of the buyer’s flow value between two levels. Changes in the flow value seem particularly natural for consumer durables, since how much a consumer values using a good at any instant depends on his current circumstances and environment.\(^1\) For example, a buyer’s value for a camera may be high if he is traveling, or if he simply has more spare time than usual, but such circumstances are not permanent. Our key finding in this environment is that the seller often finds it optimal to commit to a price path with occasional discounting, so that prices follow a “cycle”. Prices fall gradually to a periodic lowpoint (the end of the cycle) before jumping at the start of a new cycle. The buyer only ever purchases when his value is low at the end of a cycle, but always purchases immediately if his value is high. Interpreting the end of a cycle as a “sale” (as is common in the literature; see Conlisk, Gerstner and Sobel (1984)), this explains a feature of buyer behavior common in many retail markets. Buyers who do not currently have an urgent need for a product may purchase a good only if it is on sale, but buyers with an immediate need (e.g., the tourist without a camera) may purchase even at a relatively high price. Moreover, if circumstances change and a buyer develops an immediate need, he can always purchase at that point, although the corresponding price may be high.

Our result about the optimality of price cycles turns out to be true even if the marginal distribution of the buyer’s value is taken to be constant over time, i.e. if his initial value upon arrival is drawn from the ergodic distribution. Considering instead a large population of buyers, this assumption implies (at least approximately) that the proportion of buyers with a high value in the

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\(^1\)Other reasons for changing values include new information coming to light about the product (see Biehl (2001) and Deb (2010)), and consumer “impulsiveness” (see Deb (2010)). Motivating a related model, Biehl (2001, p 566) suggests: “The assumption that values are invariant is common in dynamic models, but quite strong in the durable-goods context. Values are dependent upon future states of the world, and purchases are almost surely made under imperfect information about those states.”
population remains constant with time; in the words of Biehl (2001, p 568), there is “individual demand uncertainty” whilst the composition of “aggregate demand is stationary”. This is reassuring because it shows that our result remains true in a setting which seems quite natural; namely, one where the distribution of values in the population has reached its long-run level and where the time at which a buyer “arrives” or realizes he has a need for the good is independent of his value. It also clearly distinguishes the theory from the one proposed by Board (2008) where the optimality of price variation relates to changes in the elasticity of aggregate demand as new buyers arrive.

That price cycles are optimal even when the distribution of initial values is ergodic might seem at first quite surprising. It seems surprising, in particular, because when the buyer’s arrival date is known, intertemporal price discrimination is unprofitable. That is, it is either optimal to induce purchase only in case the buyer’s value is high, or to induce purchase straight away regardless of his value. This conclusion is the same in spirit as for a well-known special case of Stokey’s (1979) model, where the buyer’s arrival time is observed and the only other differences compared to our setup are the richer set of possible values and the assumption that values do not change. Stokey finds that abstaining from intertemporal price discrimination is optimal in this environment; since values do not change, the buyer then purchases, depending on his value, either straight away or not at all. One strategy for the seller that implements the optimal outcome is to set a constant price for all time. Provided the optimal price does not change with the arrival date (this is implied if the distribution of values does not depend on the arrival time), it is then immediate that this strategy remains optimal also when the buyer’s arrival date is uncertain. This explains our claim above that Stokey’s result implies the optimality of a constant price. The same conclusion does not persist in an environment with stochastically changing values, however. If arrival time is uncertain and unobserved, and if values change (both from low to high and from high to low), then, if the optimal price path ever induces the buyer to purchase after date zero when his value is low, the price path is necessarily non-monotone; the optimal pattern of posted prices exhibits cycles.

Our approach to characterizing the optimal price path involves first determining a price path that is optimal conditional on the set of dates at which the buyer is induced to purchase if his
value is low. We term this set a “sales policy”. The conditionally optimal price path turns out to induce the buyer to purchase immediately whenever his value is high. The price path is pinned down by the condition that the buyer is indifferent between purchasing and waiting at dates not in the sales policy when his value is high, together with the condition that the he is indifferent between purchasing and waiting at dates in the sales policy when his value is low. Consider an interval over which the buyer purchases only if his value is high, followed by a date at which he also purchases if his value is low. The price declines over the interval since the date at which the buyer purchases if his value is low must be associated with a relatively low price, and because the option for the buyer of waiting until this date to purchase when his value is high becomes more valuable as the date approaches. That the option becomes more valuable reflects both discounting and the reduced probability that his value switches to low whilst waiting.

Now consider the optimal choice of sales policy. In terms of the sales policy, price cycles correspond to inducing the buyer to purchase if his value is low only intermittently. A crucial consideration explaining the optimality of price cycles is the one mentioned above: if the buyer’s value is high at a given point, it may not remain high at future dates in the sales policy. Since it is opportunities to buy at dates in the sales policy when his value is high that give rise to the rent earned under the (conditionally) optimal price path, a “sparse” sales policy reduces the rents the buyer can expect to earn. Put another way, because the buyer understands his value may become low when it is high, he is effectively “impatient” to obtain the good quickly, and this impatience can be exploited most effectively if the opportunities to purchase at a low price in the future are limited. On the other hand, delayed purchase when the buyer’s value is low is inefficient. The fully-optimal policy reflects a trade-off between the concern for efficiency and the concern for rent extraction.

Whilst the composition of demand may be stationary (this is true if initial values are drawn from the ergodic distribution), the probability that the buyer has arrived (equivalently, the size of the population that has arrived) increases in the date. As a result, the trade-offs involved in determining the optimality of inducing the buyer to purchase for both values, rather than only the
high value, change with time. In particular, it becomes less desirable to drop the price to induce purchase by the buyer when his value is low at later dates. The reason is that this raises the option value of waiting whenever the buyer’s arrival is earlier, and that the probability of an earlier arrival increases with the date. The non-stationarity implies that, after a sufficiently long time, fully-optimal prices remain at a constant level such that the buyer purchases only if his value is high. A related finding, at least in our numerical examples, is that the distance between dates at which the buyer purchases if his value is low under the fully-optimal price path increases over time. The last two observations imply that, although the fully-optimal price path is non-monotone, the trend is upwards.

Whilst our model does a good job at explaining the practice of periodic discounting, the upwards trend in prices is less consistent with observed practice. For example, Pashigian and Bowen (1991), in their study of apparel pricing, find it difficult to explain the use of discounts at the beginning of the season, but also note that prices are on average lower at the end of the season. Taking the beginning of the season as date zero, our baseline model seems to suggest an explanation for the first observation but not the second. In our model, discounting is attractive early in the season precisely because it only implies a rent for buyers who arrive early, but the same logic suggests prices should tend to rise with time. Our theory does become consistent with a price path that eventually declines, however, if we suppose that the value the buyer obtains from the product is smaller at later dates. This assumption makes sense in fashion markets (which, broadly defined, includes markets like consumer electronics) since consumers anticipate the value of their purchase will be eroded by the new fashion they expect to come on the market. By allowing the possibility of declining values, we are able to rationalize both the use of initial and periodic discounts as well as prices that eventually decline.

For products that have been around for a long time, the initial date the good goes on sale is less obviously focal. At the same time, it is reasonable to believe that most consumers cannot wait indefinitely to obtain a good. We extend our model to allow for stochastic departure of the buyer from the market, and show that, provided the departure rate is sufficiently large, the pattern of
prices eventually reaches a steady state. One possibility is that steady state prices involve repeated cycles, with a constant time between periodic low points.

That the trade-off between rent extraction and efficiency is a dynamic one suggests a plausible explanation for the well-documented phenomenon of counter-cyclical mark-ups. We explain why periods during which the probability of arrival is high may be more likely to be associated with lower prices. During these times, the concern for limiting the rents of buyers who arrive earlier is less important relative to the objective of obtaining revenue from buyers arriving at the time in question, so the price is more likely to be set to induce purchase even if the buyer’s value is low. We discuss the relative merits of this theory compared to several existing theories (e.g., Rotemberg and Saloner (1986), Warner and Barsky (1995) and Chevalier et al. (2003)).

It is worth pointing out that the key trade-offs discussed above are particular to a model where the seller has commitment ability, particularly because a seller without the power to commit does not internalize the effect of the prices she sets on buyers’ past purchase decisions. The case of commitment is clearly an important benchmark: To the extent sellers can profit by making credible commitments, we might expect them to do so — if not explicitly, then by establishing a reputation. The case of no commitment is of interest as well, however, and has been given considerable attention in the literature. Of particular relevance to our paper, Conlisk, Gerstner and Sobel (1984) and Sobel (1991) (which we discuss in more detail below) find that equilibrium prices may cycle when they are set by a seller without commitment power facing buyers who arrive over time and have constant values.\footnote{It would also be of interest to consider these models assuming that buyers’ values change, but the problem seems best posed maintaining the assumption of price-setting at discrete intervals, an assumption which allows the seller a degree of commitment power. The continuous-time framework, which we adopt for tractability in this paper, may therefore be less helpful. In any case, we leave the possibility that the seller cannot commit to future research.}

Having considered optimal posted prices, we characterize the fully-optimal mechanism, again assuming that the seller commits. The fully-optimal mechanism can be implemented by offering an option contract with an up-front fee whose strike price depends on the date the contract is requested by the buyer and the length of time it has been held. The option expires after finite
time, after which the buyer can never obtain the good. The buyer executes the option whenever his
value is high, or, if his value is still low, on the expiration date. The advantage of such contracts,
compared with posted prices, is that the length of time a buyer waits when his value is low can
be tailored to his arrival time, allowing more consistent intertemporal price discrimination across
arrival times. We discuss the importance of this observation in light of related mechanisms that
are used in practice. We also explain how the optimal option contracts depend on the date that
the buyer arrives.

The rest of the paper is set out as follows. Next we expand our discussion of the relevant
literature. Section 3 examines the optimal path of posted prices, and Section 4 examines the fully-
optimal mechanism and compares it to the optimal posted price mechanism. Section 5 concludes.
Two appendices collect proofs for the various results: Appendix A for posted prices, and Appendix
B for the fully-optimal mechanism.

1.1 Related literature

Other explanations for sales/price cycles.

Several alternative theories have been proposed to explain sales phenomena. Some of these
theories view sales as arising as a result of oligopolistic competition. Salop and Stiglitz (1977),
Shilony (1977), Rosenthal (1980) and Varian (1980), for example, provide static models of price
dispersion in which oligopolistic firms randomize their choice of prices. By considering repeated
play, these models suggest an explanation for random price fluctuations over time. Rosenthal
(1982), for example, addresses the possibility formally. Maskin and Tirole (1988) study how
“Edgeworth cycles”, or cycles of price wars, can arise in a duopoly model with partial commitment
to future prices. Competition is central to the pricing patterns suggested by these theories, whilst
we obtain price fluctuations in the absence of competition.

There are, however, other theories of sales which do not require competition between firms. An
early example is Salop (1977), who proposes a model in which the monopolist chooses a distribution
of prices across stores as a way to price discriminate against those with high costs of searching for a
low price realization. As with the early models of price dispersion in competitive settings, low prices may be interpreted as sales if play is repeated. Although search seems a plausible explanation for price fluctuations, a richer dynamic story is missing which seems important to explain how consumers form correct beliefs about the distribution of prices as well as patterns of price setting across time.

Conlisk, Gerstner and Sobel (1984) and Sobel (1991) show how, when a monopolist seller of a durable good cannot make long-term commitments, the Coasian dynamic together with the assumption that buyers arrive over time leads to fluctuating prices. After a sufficient length of time charging high prices, the seller cannot resist the urge to drop the price to induce purchase by the buyers that have arrived with low values. In these models, where buyer values do not change, long-term commitments not to drop the price would increase total revenue. Indeed, as discussed above, price fluctuations are no longer optimal in case the seller can find a way to credibly commit to future prices. This might be interpreted as lending support to a view sometimes expressed in the business press, that firms would do better by committing not to hold sales (see, for instance, Kadet (2004)). Some firms, such as Apple, are well known for their commitment to seldom offering discounts (see, e.g., Sutter (2010, November 29)). Our paper suggests to the contrary that periodic discounts may be an optimal response by firms to the fluctuating values of their customers rather than the result of an inherent inability to commit. At the same time, the paper shows that in environments where firms do seem to make long-term commitments, such as by creating a reputation for certain pricing patterns or offering price schedules that pertain to long horizons, price fluctuations can be consistent with profit-maximizing behavior.

3 Other notable contributions are Sobel (1984) and Pesendorfer (2002), who consider competition among sellers, and Chevalier and Kashyap (2010). These papers posit heterogeneous degrees of patience among buyers – in particular, high types are impatient and purchase immediately if at all. The reason for the assumption, which seems a strong requirement, is to reduce the equilibrium set, which is often large in the model of the aforementioned papers (see, in particular, Sobel, 1991). Another related contribution is Villas-Boas (2004). The Villas-Boas model provides an explanation of price cycles for non-durables based on (i) the monopolist seller’s inability to commit, (ii) its ability to recognize customers who have purchased previously, and (iii) the fact that a new generation of customers arrives in each period (in particular, he assumes overlapping generations).

4 Whilst the extent to which firms can make long-term commitments is an ongoing debate, Waldman (2003) suggests that it is crucial to recognize commitment ability in analyzing durable-good pricing, arguing (p 135) that
Our paper also relates to a broader literature which finds the optimality of rising or falling prices over time. In these papers, prices typically rise or fall monotonically, and there is no prediction of periodic discounting. The finding of falling prices is much more common. Several studies focus on how the option value of delaying sale may fall over time. For example, in different environments, Gallego and Van Ryzin (1994), Gershkov and Moldovanu (2009), Board and Skrzypacz (2010) and Horner and Samuelson (2011) examine optimal pricing when there is a known expiry date for the good. Prices either tend to fall gradually over time, or abruptly at the end, depending on such matters as whether consumers are forward-looking/strategic as well as on the seller’s ability to commit.\footnote{Nocke and Peitz (2007) show that commitments to falling prices and rationing can be an optimal response to uncertainty about aggregate demand. Lazear (1986) and Courty and Li (1999) present models where prices fall over time due to learning about the distribution of demand and the seller’s inability to commit.}

Some models, however, predict that prices rise over time. One set of models, which includes the one in our paper, is based on buyers’ private uncertainty about future values. Dana (2001) shows how advanced purchase discounts can be used to target buyers who are certain of their value for the good, and Deb (2010) argues that an increasing price path may be optimal in case the buyer’s value is subject to occasional shocks. Among other reasons, Su (2007) shows that correlation between waiting costs and values can lead the optimal price path to feature prices that either increase or decrease over time. Several authors, such as Das Varma and Vettas (2001), Gallien (2006) and Moldovanu (2009) document how prices tend to increase as inventory is depleted.

\textbf{Dynamic mechanism design}

The part of our paper on the fully-optimal mechanism is related more closely to the literature on dynamic mechanism design with stochastically evolving values, which includes works such as

\"analyses that assume no commitment at all is possible miss many of the real world implications\".

\footnote{It is unclear whether the use of declining price paths in these situations should be described as “price discrimination”. Stokey (1979) discusses this question and considers the case of a monopolist lowering his price in response to falling costs. She asks “Is this strategy “discriminatory”? Since consumers would pay different prices at different dates in the corresponding competitive market, the mere presence of price variation seems an inadequate criterion.” However, she concludes that it is ultimately a “semantic question”.

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Battaglini (2005), Zhang (2009) and Pavan, Segal and Toikka (2009). At a technical level, our paper is most closely related to Zhang’s work on dynamic taxation, which also studies an environment where time is continuous and where there is Markov switching between productivity levels. In terms of the economics, however, Board (2007) seems the closest. Board studies a model with private uncertainty and finds that the fully-optimal mechanism can be implemented with option contracts. Board’s environment is different from ours, however, not only because the stochastic process is different (comprising a fully-persistent initial observation with i.i.d. noise) but because all potential buyers are present at a known date (the optimal mechanism therefore involves auctioning the option contracts). Relative to Board’s paper, we are able to make interesting predictions about how the option contracts offered by the seller evolve with time. Indeed, this paper (and a companion paper (Garrett, 2010)) is the first to study the question of the profit or revenue maximizing mechanism when the buyer’s value changes and his arrival date is uncertain.\footnote{Bergemann and Valimaki (2010) do consider the question of the efficient mechanism when buyers arrive over time. Since the efficiency objective places no weight on minimizing buyer rents, however, the analysis is very different.}

2 Model

We consider a single seller and a single representative buyer. The buyer obtains value from just one unit and the seller’s cost of supplying that unit is normalized to zero. There is no re-sale.

The buyer’s arrival time is distributed exponentially with parameter $\lambda > 0$. The exponential distribution has the advantage of being easy to work with, and the fact the process is memoryless seems appealing, if only because it will imply that (abstracting from any past contractual commitments) the environment the seller faces remains the same at each date, conditional on the buyer having not yet arrived. Assuming that the seller’s capacity is unconstrained, subject to the usual measurability concerns, our analysis will hold true also for (say) a unit measure of consumers, where the proportion of consumers who arrive by date $t \geq 0$ is $1 - e^{-\lambda t}$.\footnote{We will not discuss these measurability difficulties which are a well-known consequence of considering a continuum of consumers. The analysis is nonetheless also valid for any number of ex-ante identical buyers who arrive according to an exponential distribution with parameter $\lambda$.} That is, the model can be given
a “population interpretation”.

Using a standard abuse of notation, at each moment \( t \) after which the buyer has purchased the good, he enjoys flow consumption value \( \omega_t \in \{\omega_L, \omega_H\} \), where \( \omega_H > \omega_L > 0 \). Thus the buyer’s “value” at time \( t \) is either “low” (the flow consumption value equals \( \omega_L \)) or “high” (the flow consumption value equals \( \omega_H \)). This consumption value evolves stochastically over time, possibly as a response to changes in the buyer’s environment such as new information coming to light about the product or changes in personal circumstances which affect his enjoyment of the product. These changes occur at a Poisson rate: the value switches from \( \omega_H \) to \( \omega_L \) at rate \( \alpha_H \geq 0 \) and from \( \omega_L \) to \( \omega_H \) at rate \( \alpha_L \geq 0 \). We assume throughout that at least one of \( \alpha_H \) and \( \alpha_L \) is strictly positive.

The buyer and seller have common discount rate \( r > 0 \). The buyer never has the opportunity to re-sell the good. Denote the buyer’s expected discounted value from owning the good at time \( t \) by \( \theta_t \). Again abusing notation, if the buyer’s flow value at \( t \) is high, this is equal to

\[
\theta_H = \mathbb{E} \left[ \int_t^\infty e^{-rs} \tilde{\omega}_s ds | \tilde{\omega}_t = \omega_H \right].
\]

If his flow value at \( t \) is low, it is equal to

\[
\theta_L = \mathbb{E} \left[ \int_t^\infty e^{-rs} \tilde{\omega}_s ds | \tilde{\omega}_t = \omega_L \right].
\]

Therefore,

\[
\theta_H = \frac{(\alpha_L + r) \omega_H + \alpha_H \omega_L}{r (\alpha_L + \alpha_H + r)}, \text{ and }
\theta_L = \frac{\alpha_L \omega_H + (\alpha_H + r) \omega_L}{r (\alpha_L + \alpha_H + r)}.
\]

(clearly \( \theta_H > \theta_L > 0 \)). The mapping between flow values \( (\omega_L, \omega_H) \) and the expected discounted values \( (\theta_L, \theta_H) \) is bijective and at times it will be convenient to refer to both (we will refer to each simply as the buyer’s “value” whenever this creates no confusion).

We record histories of values between a date \( t \) and \( z \geq t \) by letting, for any \( s \in [t, z] \), \( \theta^{[t, z]} (s) = \theta_s \in \{\theta_L, \theta_H\} \). The properties of \( \theta^{[t, z]} \) are described in Zhang (2009); in particular \( \theta^{[t, z]} (\cdot) \) must be right continuous and contain a finite number of jumps. An infinite history is denoted \( \theta^{[t, \infty]} \).
Upon arrival at date $\tau$, the buyer has a high value with probability $\gamma \in (0, 1)$ and a low value with probability $1 - \gamma$. If $\gamma = \frac{\alpha_L}{\alpha_L + \alpha_H}$ then the initial distribution is “ergodic”, i.e., the marginal probability of the buyer having a high value is time invariant.

3 Posted prices

3.1 Preliminary analysis: optimal prices

In this section, we study the optimal deterministic posted prices. A posted-price mechanism $\Omega_P = \langle p, x \rangle$ is then a price path $p : \mathbb{R}_+ \to \mathbb{R}_+$ and a prescription $x$ of whether to purchase in each state such that $x(\theta_L, \cdot), x(\theta_H, \cdot) : \mathbb{R}_+ \to \{0, 1\}$, where the value 1 indicates that the buyer purchases. Here, without loss of optimality, we are assuming that the prescription $x$ depends only on the buyer’s value, and not on his past history of arrival and values; i.e., we assume that the purchase rule is Markov.\(^8\)

The buyer’s problem is an optimal stopping problem. This problem is Markov in the sense that it depends only on the buyer’s current value and not on the history of past values or time of arrival. For a given (Markov) stopping rule $\sigma(\theta_L, \cdot), \sigma(\theta_H, \cdot) : \mathbb{R}_+ \to \{0, 1\}$, any time by which the buyer has arrived $t$, and any history of values $\theta^{[t, \infty)}$, we may define the stopping time by $\gamma_\sigma(\theta^{[t, \infty)}) \equiv \min \{s \geq t : \sigma(\theta_s, s) = 1\}$. The expected payoff to the buyer from using the stopping rule $\sigma$ at any date $t$ at which he has not yet purchased depends on his value and is given by

$$
\begin{align*}
\Omega^P_t(\theta_H; \sigma) &= \mathbb{E}\left[ e^{-r(\tilde{\gamma}_\sigma - t)} \left( \tilde{\theta}_{\tilde{\gamma}_\sigma} - p(\tilde{\gamma}_\sigma) \right) \mid \tilde{\theta}_t = \theta_H \right] \\
\Omega^P_t(\theta_L; \sigma) &= \mathbb{E}\left[ e^{-r(\tilde{\gamma}_\sigma - t)} \left( \tilde{\theta}_{\tilde{\gamma}_\sigma} - p(\tilde{\gamma}_\sigma) \right) \mid \tilde{\theta}_t = \theta_L \right],
\end{align*}
$$

where, suppressing sample paths, $\tilde{\gamma}_\sigma$ is the stopping time determined by the stopping rule $\sigma$. Incentive compatibility of the posted-price mechanism $\Omega_P = \langle p, x \rangle$ is then the requirement that, for each $t$ and $\theta_t$, and for all stopping rules $\sigma$, $\Omega_t^P(\theta_t; x) \geq \Omega_t^P(\theta_t; \sigma)$.

\(^8\)When the mechanism is a simple price path, only Markov purchase rules are implementable if we assume that a buyer who is indifferent always purchases. This assumption does not harm the seller’s expected revenue because, by purchasing whenever indifferent, the buyer obtains the same rents in expectation as if purchase were delayed, but efficiency is improved.
We define a “sales policy” to be the set of times at which prices are low enough to attract the buyer to purchase if his value is low. Formally, a sales policy is the set \( A \subset \mathbb{R}_+ \) of times \( t \) at which \( x(\theta_L, t) = 1 \). Without any loss of optimality, we restrict attention to sales policies which are at-most countable collections of disjoint closed intervals and points. Let \( K \) be a countable index set and let \((I_k)_{k \in K}\) be a sequence of disjoint intervals such that \( A = \bigcup_{k \in K} I_k \), where \( I_k = [t_k, \tilde{t}_k] \) for all \( k \), except allowing that one \( I_k \) may be the half space \([t_k, \tilde{t}_k)\) with \( \tilde{t}_k = \infty \).

We begin by setting out necessary conditions for a posted-price mechanism to be incentive compatible.

**Lemma 1** Necessary conditions for the mechanism \( \Omega_p = \langle p, x \rangle \) to be incentive compatible are:

(i) For all \( t \), if \( x(\theta_L, t) = 1 \) then \( x(\theta_H, t) = 1 \).

(ii) For each \( k \in K \), and for all \( t \in I_k \),

\[
u^\Omega_p(t; \theta_L) \geq e^{-r(\tilde{t}_k-t)} u^\Omega_p(t_k; \theta_L) + \left(1 - e^{-r(\tilde{t}_k-t)} \right) \frac{\alpha_L(\theta_H - \theta_L)}{r}
\]

(if \( \tilde{t}_k = \infty \), then \( u^\Omega_p(t; \theta_L) \geq \frac{\alpha_L(\theta_H - \theta_L)}{r} \) for all \( t \geq t_k \)).

(iii) If \( t \in \mathbb{R}_+ \setminus A \) and there exists \( t \in A \) which is the next time at which the buyer purchases if his value is low, then

\[
u^\Omega_p(t; \theta_L) \geq e^{-r(t-t)} \left( u^\Omega_p(t_L; \theta_L) + \frac{\alpha_L(1 - e^{-(\alpha_L + \alpha_H)(t-t)})}{\alpha_L + \alpha_H} \right), \quad \text{and}
\]

\[
u^\Omega_p(t; \theta_H) \geq e^{-r(t-t)} \left( u^\Omega_p(t_L; \theta_L) + \frac{\alpha_L + \alpha_H e^{-(\alpha_L + \alpha_H)(t-t)}}{\alpha_L + \alpha_H} \right).
\]

Part (i) of Lemma 1 is a monotonicity condition which states it is never possible to exclude the buyer from purchasing when his value is high whilst inducing him to purchase when his value is low. Parts (ii) and (iii) provide lower bounds on the payoffs that the buyer expects to receive at any date at or before the last date in \( A \). Part (ii) can be derived from the requirement that, at each date \( t \) in an interval \( I_k \subset A \), the buyer does not strictly prefer to follow a strategy of waiting to purchase until either his value turns high or the date reaches \( t_k \). If the inequality (1) holds

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Note that a consecutive ordering may not be possible, but the chosen ordering will be immaterial.
with equality at all dates up to \( t_k \), then the buyer is precisely indifferent between purchasing and waiting when his value is low. Part (iii) can be derived from the requirement that, at dates not in the sales policy, the buyer obtains at least the payoff from waiting to purchase at the next date in the sales policy \( t \).

We now show how to derive the revenue-maximizing price path for a given sales policy \( A \). The revenue the seller expects to earn at date \( \tau \) from a buyer who arrives at that date is

\[
p^{\Omega_P} (\tau) = \mathbb{E} \left[ e^{-r(\bar{\tau} - \tau)} p (x_\lambda) \right],
\]

so that the present value of total expected revenue is

\[
TR^{\Omega_P} = \int_0^\infty \lambda e^{-(\lambda + r)\tau} p^{\Omega_P} (\tau) d\tau.
\]

Consider the problem of choosing \( \Omega_P \) to maximize \( TR^{\Omega_P} \) subject to satisfying Conditions (i)-(iii) of Lemma 1 and to the requirement that \( t \in A \) if and only if \( x(\theta_L, t) = 1 \). Suppose then that \( x_A (\theta_L, \cdot) \) satisfies the requirement of consistency with the sales policy \( A \), i.e. is equal to one if \( t \in A \). Omitting the dependence on the stopping rule (which remains to be fully specified), for each date \( t \), let \( u_{A,t} (\theta_L), u_{A,t} (\theta_H) \geq 0 \) denote the minimum level of buyer payoffs consistent with implementation of the sales policy \( A \). First, \( u_{A,t} (\theta_L) = u_{A,t} (\theta_H) = 0 \) for all times \( t \) that lie above all points in \( A \). Second the inequalities (1), (2) and (3), as applicable, hold with equality at all other times, and \( u_{A,t} (\theta_H) = u_{A,t} (\theta_L) + \theta_H - \theta_L \) for \( t \in A \).

A price path consistent with the specified payoffs \( u_{A,t} \) is \( p_A (t) = \theta_L - u_{A,t} (\theta_L) \) for \( t \in A \) and \( p_A (t) = \theta_H - u_{A,t} (\theta_H) \) for \( t \notin A \). Whilst the latter makes the buyer indifferent between purchasing and not purchasing when his value is high, it is without loss of optimality to specify purchase in case of indifference (since this improves efficiency without changing the buyer’s expected payoff). Hence, put \( x_A (\theta_H, t) = 1 \) for all \( t \in \mathbb{R}_+ \). The next result states that the above policy is indeed optimal.

**Proposition 1** The price path \( p_A \) and stopping rule \( x_A \) is the essentially unique revenue-maximizing policy such that the buyer purchases when his value is low at date \( t \) if and only if \( t \in A \).
This result provides a foundation for our analysis by allowing us to focus on the choice of sales policy. An important corollary of the result is that promises to sell to low values at future dates imply an increase in the rents that a buyer earns at all earlier dates. Consider date \( t \), and suppose a change in the sales policy at dates \( s > t \) implies an increase (decrease) in the buyer’s continuation payoff at \( t \) when his value is low, say \( \Delta u_t (\theta_L) \). Under the revenue-maximizing price path, continuation payoffs of low values at earlier dates \( z < t \) increase (decrease) by \( e^{-r(t-z)} \Delta u_t (\theta_L) \). The prices under the (conditionally) optimal policy at dates \( z < t \) decrease (increase) by \( e^{-r(t-z)} \Delta u_t (\theta_L) \).

3.2 Known arrival time: optimal sales policy

Before turning to the optimal sales policy in the environment described above, we consider the case where the buyer’s arrival time is publicly known. Since the price path can be conditioned on the date of arrival, the problem is equivalent to one where the buyer is known to arrive at date zero. Without loss of generality, we focus on the latter.

**Proposition 2** Suppose that the buyer is known to be present at date zero.

(i) The optimal sales policy is either \( A = \{ t^* \} \), where \( t^* \geq 0 \) is the date which maximizes expected revenue

\[
R(t^*) = \left( \gamma + (1 - \gamma) \frac{\alpha_L}{\alpha_L + r} \right) \theta_H + (1 - \gamma) e^{-(\alpha_L + \alpha_H) t^*} \left( \theta_L - \frac{\alpha_L \theta_H}{\alpha_L + r} \right)
- (1 - \gamma) \frac{\alpha_L}{\alpha_L + \alpha_H} \left( 1 - e^{-(\alpha_L + \alpha_H) t^*} \right) e^{-rt^*} \left( \theta_H - \theta_L \right)
- \gamma \left( \frac{\alpha_L}{\alpha_L + \alpha_H} + \frac{\alpha_H}{\alpha_L + \alpha_H} e^{-(\alpha_L + \alpha_H) t^*} \right) e^{-rt^*} \left( \theta_H - \theta_L \right)
\]

or \( A = \emptyset \) in case \( \gamma \theta_H + (1 - \gamma) \frac{\alpha_L \theta_H}{\alpha_L + r} \) is larger for all values of \( t^* \).

(ii) For all \( \gamma \leq \frac{\alpha_L}{\alpha_L + \alpha_H} \), \( A \neq \emptyset \) implies \( A = \{ 0 \} \), i.e. the buyer purchases immediately irrespective of his value.

(iii) A sufficient condition for the optimal sales policy to be non-empty is

\[
\theta_L > \left( \gamma + (1 - \gamma) \frac{\alpha_L}{\alpha_L + r} \right) \theta_H.
\]
If $γ ≤ \frac{\alpha_L}{\alpha_L + \alpha_H}$ and the reverse of (5) holds, then $A = \emptyset$ is uniquely optimal.\(^{10}\)

The expression for revenue in Part (i) when $A = \{t^*\}$ is calculated given optimal prices

$$p(t) = \theta_H - e^{-r(t^*-t)} \left( \theta_H - \theta_L \right) \frac{\alpha_L + \alpha_H e^{-(\alpha_L+\alpha_H)(t^*-t)}}{\alpha_L + \alpha_H}$$

for $t ∈ [0,t^*]$, with $p(t) = \theta_H$ (or any higher price) for $t > t^*$. The first term of $R(t^*)$ is simply the expected revenue generated by inducing the buyer to purchase only if his value is high, i.e. setting $p(t) = \theta_H$ for all $t$. The second term is the expected gain in surplus from inducing purchase if the buyer’s value is still low at $t^*$, rather than only once his value becomes high.\(^{11}\) The final two terms subtract the rent that the buyer expects to earn by waiting to purchase at $t^*$ conditional on initially having a low and high value respectively. If the buyer has a high value, of course, he does not wait, but he receives a payoff equivalent to that from waiting. Part (ii) gives a condition under which, if it is profitable to induce the buyer to purchase with a low value at any date, then it is optimal to do so immediately. The condition (5) in Part (iii) simply states that the revenue from inducing immediate purchase irrespective of the buyer’s value (i.e., by setting an initial price of $\theta_L$), exceeds the expected revenue from inducing the buyer to purchase only if his value is high.

Part (ii) is closely related to Stokey’s (1979) result about the sub-optimality of intertemporal price discrimination when the buyer’s value does not change with time. Consider the case where the initial distribution of values is ergodic. For the optimal price path, the rent the buyer expects to obtain ex-ante (i.e., before he knows his value) when he purchases with a low value at $t^*$ is equal to the rent he obtains in case all purchases must be made at $t^*$, i.e. $e^{-rt^*} \frac{\alpha_L}{\alpha_L + \alpha_H} (\theta_H - \theta_L)$. The expected gain in surplus from a policy of inducing the buyer to purchase if his value is low at $t^*$ rather than only ever inducing purchase when his value is high is $e^{-(\alpha_L+r)t^*} \frac{\alpha_L}{\alpha_L + \alpha_H} (\theta_L - \frac{\alpha_L \theta_H}{\alpha_L+r})$. Therefore, if inducing purchase by the buyer when his value is low at $t^* > 0$ raises more revenue than only ever inducing purchase when his value is high, revenue can always be improved by instead

\(^{10}\)If the inequality (5) instead holds with equality, the sales policy $A = \{0\}$ yields the same expected revenue as for $A = \emptyset$.

\(^{11}\)Note that $\theta_L - \frac{\alpha_L \theta_H}{\alpha_L + r} = \frac{\omega_L}{\alpha_L + r}$, which is the present value of an annuity that pays $\omega_L$ until termination at rate $\alpha_L$, when the discount rate is $r$. 

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setting \( t^* = 0 \). The logic is the same as Stokey’s: a delay leads the buyer’s expected rents to be discounted, but the surplus generated from the buyer purchasing when his value is low is discounted as well. In fact, since the buyer’s value changes in our problem, the gain in surplus from inducing purchase with a low value at \( t^* \) rather than not at all is reduced both due to discounting and because the probability a buyer remains a low value from date zero diminishes with time.

The result in Part (ii) should be compared to the results for the durable goods models of Conlisk (1984), Biehl (2001, Proposition 2) and Deb (2010). The first two authors examine two-period models where the buyer is known to be present in the first period and where values (either high or low) change stochastically between the first and second periods. The initial distribution for values is ergodic.\(^\text{12}\) Biehl reaches the same conclusion we do: it is optimal either to have the buyer purchase only when his value is high, or to have him purchase immediately regardless of his value. In a slightly different model, Conlisk finds a decreasing price path is optimal for certain parameters. The key difference with respect to Biehl’s model is that the buyer uses the good only in the period of purchase (in this sense, the good is not truly durable; Conlisk’s example is a new movie).\(^\text{13}\)

Deb also studies an environment where the buyer is known to be present at the beginning, but time is continuous and the horizon infinite. In the baseline model, the buyer receives a single shock to his valuation. He finds that the seller may optimally set a relatively low price at date zero and a relatively high constant price thereafter. Although this seems consistent with our finding, the exact price set after date zero is payoff relevant, since a buyer with a low enough initial value may choose to delay his purchase with the possibility of purchasing in the future if his value becomes high. The key reason for the difference in our findings is that Deb considers a richer set of values

\(12\) In Conlisk’s model, values are drawn i.i.d. in each period, whilst in Biehl’s values in each period may be correlated.

\(13\) This means that the welfare-maximizing allocation rule in Conlisk’s model may involve allocating the good to the buyer if he has a low value only in the second period (in Biehl’s model, as in ours, there is no efficiency motive for inducing delayed purchase). (The reason such a policy may maximize expected total surplus is that there is a positive option value to withholding the good from the buyer if his value is low in the first period due to the possibility that his value becomes high in the second.) This drives the finding. In Conlisk’s model, a declining price path maximizes revenue only if the corresponding allocation rule, which has the buyer purchase when his value is low only in the second period, also maximizes welfare.
than we do.\textsuperscript{14}

The result in Part (ii) no longer holds if the probability the buyer’s value is initially high sufficiently exceeds the probability under the ergodic distribution (i.e., if $\gamma$ is far enough above $\frac{\alpha_L}{\alpha_L + \alpha_H}$). In this case, the optimal policy may involve selling to the low value at some date $t^* > 0$ and the posted price declines up to $t^*$ and is high thereafter. The following is a simple example.

\textbf{Example 1} Suppose that the buyer is known to be present at date $t = 0$, and that $\alpha_L = 0$, $\alpha_H > 0$ and $\gamma \in (0, 1)$. Expected revenue from the sales policy $A = \{t^*\}$ is given by

$$R(t^*) = \gamma \theta_H + (1 - \gamma) e^{-rt^*} \theta_L - \gamma e^{-(\alpha_H + r)t^*} (\theta_H - \theta_L),$$

where, at an optimum,

$$t^* = \max \left\{ \frac{1}{\alpha_H} \ln \left( \frac{\gamma (\theta_H - \theta_L) (\alpha_H + r)}{(1 - \gamma) \theta_L r} \right), 0 \right\}.$$

In particular, $t^* > 0$ whenever $\gamma$ is sufficiently large.

The intuition is again closely related to the work of Stokey (1979). Stokey shows that intertemporal price discrimination \textit{can be profitable} in case values tend to converge (deterministically) over time. In such cases, having the buyer purchase if his value is low at a date sufficiently far in the future requires sacrificing relatively little rent if his value is instead high. The reason is essentially that, if the buyer’s value is high, he is “impatient” to obtain the good early when his value is at its maximum, which means that he is willing to purchase at a high price to avoid delay (impatience here relates to the change in consumption value from procuring the good rather than to a difference in discount rates).

In Example 1, a buyer who arrives with a high value has an expected payoff from obtaining the good immediately of $\theta_H$, but also anticipates that his value may become low at some time in the future; if it does so, his anticipated value from obtaining the good is $\theta_L$. In particular, if he

\textsuperscript{14}In our model, if the initial price is $\theta_L$, the buyer purchases irrespective of his value. Thus, when the buyer is known to be present at the beginning, there is no chance in equilibrium that the buyer remains “in the market” after date zero.
must wait until \( t^* \) to purchase, he expects his value from obtaining the good at that time to be 
\[ e^{-\alpha_H t^*} \theta_H + (1 - e^{-\alpha_H t^*}) \theta_L. \]
Thus, from the perspective of date zero, the undiscounted expected value of procuring the good at \( t^* \) in case the buyer’s value is initially high drifts downwards towards the expected value if it is initially low. In the example, this value is simply \( \theta_L \). By the same logic as Stokey, we find that intertemporal price discrimination can be profitable.

Example 1 provides a possible rationale for why prices may be non-monotone over time. If the optimal price path has the buyer purchasing when his value is low at \( t^* > 0 \), then the price declines over \([0, t^*]\). Assuming that the seller must commit to a price path for all time, the optimal price jumps from \( \theta_L \) at \( t^* \) to at least \( \theta_H \) at all subsequent dates. This, by itself, however, does not explain periodic discounting; in equilibrium, there are no purchases after \( t^* \), so prices after \( t^* \) are irrelevant provided they are high enough. Moreover, if the distribution of the buyer’s initial value is ergodic (which seems a natural assumption), we have seen that the buyer always purchases immediately. To allow for periodic discounts we therefore introduce the possibility that the buyer’s date of arrival is uncertain. This clearly introduces a possible motive to lower the price more than once, or perhaps continuously. This possibility is the focus of the rest of this section.

### 3.3 Unknown arrival time: simple sales policies

We now suppose the buyer’s arrival date is uncertain and unobserved by the seller. In this subsection we discuss certain simple sales policies which, whilst of interest on their own, provide useful benchmarks for understanding the fully-optimal policy.

First, by choosing constant prices, the seller may implement one of two sales policies: \( A = \emptyset \) (the buyer never purchases if his value is low) or \( A = \mathbb{R}_+ \) (the buyer always purchases if his value is low). A striking observation is the following.

**Remark 1** In case \( A = \emptyset \), the optimal price is constant at \( \theta_H \). In case \( A = \mathbb{R}_+ \), the optimal price is constant at \( \frac{\omega L}{r} \).

This means that, if \( A = \mathbb{R}_+ \), the seller fails to extract any of the future surplus generated from the possibility that the buyer’s consumption value becomes high. The price \( \frac{\omega L}{r} \) is the discounted
value of the buyer’s payoff if his value remains low forever, and raises the same revenue as a rental policy that, by charging rental price $\omega_L$ at each instant, induces him always to rent irrespective of his value.\textsuperscript{15}

One way to understand the observation is to note that the highest price consistent with $A = \mathbb{R}_+$ keeps the buyer indifferent between purchasing immediately and waiting when his value is low. Suppose that a buyer arrives with a low value at date $\tau$ and purchases immediately, and suppose that his value remains low until date $\tau + s$ for $s > 0$. If the additional payment by purchasing at $\tau$ rather than $\tau + s$ exceeds his discounted payoff from holding the good, i.e., if it is greater than $\frac{\omega_L}{\tau} (1 - e^{-rs})$, he regrets his purchase upon reaching date $\tau + s$. Instead, the buyer could avoid such regret by waiting to purchase only if his value turns high. Thus the highest fixed price that can induce purchase when his value is low is $\frac{\omega_L}{\tau}$.

The cost of having the buyer purchase when his value is low in terms of the rents captured by the buyer can be mitigated by only having him purchase with a low value on an initial interval, i.e. by setting $A = [0, \bar{t}]$ for some $\bar{t} > 0$. We term this policy an “introductory offer” (the corresponding price path is increasing on $A$, before jumping to $\theta_H$ above $\bar{t}$). If $\alpha_L > 0$, i.e. the buyer’s value changes from low to high with positive probability, this policy turns out to dominate $A = \mathbb{R}_+$. The following result provides the condition under which an introductory offer policy raises more revenue than having the buyer purchase only when his value is high, and it gives the optimal introductory offer policy under this condition.

\textsuperscript{15} Justifying our focus on selling rather than renting, note that any rental policy is dominated, at least weakly, by a policy in which if the buyer obtains the good, he keeps it forever. This is easiest to see if the process is ergodic. Then, regardless of whether the buyer’s arrival date is known, the optimal rental policy either sets a rental price of $\omega_L$ or $\omega_H$ for all time. This is because the probability of high and low values, conditional on arrival, remains constant. It is easy to see, however, that renting at price $\omega_H$ at all dates yields less revenue than from selling at $\theta_H$ at all dates (the buyer still earns zero rent, but efficiency is improved). As discussed, renting at price $\omega_L$ at all dates is equivalent to selling at $\theta_L$ at all dates. A proof for the case of non-ergodic distributions is available from the author. See Biehl (2001) for further discussion of rental policies in a durable goods model with stochastically changing values.
Proposition 3  Suppose $A = [0, \bar{t}]$ for $\bar{t} \geq 0$. Total expected revenue from this policy is

$$
(1 - e^{-(\lambda + r)\bar{t}}) \frac{\lambda}{\lambda + r} \frac{\omega_L}{r} + e^{-(r+\lambda)\bar{t}} \frac{\lambda}{\lambda + r} \left( \gamma + \frac{\alpha_L}{\alpha_L + r} (1 - \gamma) \right) \theta_H + e^{-r\bar{t}} \left( 1 - e^{-\lambda\bar{t}} \right) \left( \theta_L - \frac{\omega_L}{r} \right).
$$

(7)

The revenue-maximizing choice of $\bar{t}$ is positive if and only if the inequality (5) holds. If $\alpha_L > 0$ (equivalently, if $\theta_L > \frac{\omega_L}{r}$) and (5) holds, then the revenue-maximizing value of $\bar{t}$ is

$$
\frac{1}{\lambda} \ln \left( 1 + \frac{\lambda \left( \theta_L - \left( \gamma + \frac{\alpha_L}{\alpha_L + r} (1 - \gamma) \right) \theta_H \right)}{r \left( \theta_L - \frac{\omega_L}{r} \right)} \right) < \infty.
$$

(8)

The first two terms in the expression for total revenue (7) are a weighted average of the revenue earned under the two fixed-price schemes: a constant price of $\frac{\omega_L}{r}$ and a constant price of $\theta_H$. Assuming $\alpha_L > 0$, the third term represents the additional rent extracted from the buyer by setting the sales policy equal to $[0, \bar{t}]$ rather than to $\mathbb{R}_+$. To understand this term, suppose the buyer arrives at $\bar{t}$ with a low value. From the policy $[0, \bar{t}]$, the seller extracts the entire value the buyer anticipates from holding the good, i.e. $\theta_L$, whereas for the policy $\mathbb{R}_+$, she extracts only $\frac{\omega_L}{r}$.

Secondly, the rent the buyer expects at $\bar{t}$ affects the rent he expects in case of earlier arrival, which has probability $1 - e^{-\lambda\bar{t}}$. Finally, note that $e^{-r\bar{t}} \left( 1 - e^{-\lambda\bar{t}} \right)$ is increasing in $\bar{t}$ at small values but decreasing whenever $\bar{t}$ is sufficiently large. The gradient of this term dominates the gradient of (7) in the limit, implying that the optimal choice of $\bar{t}$ is finite.\(^{16}\)

Finally, note that an introductory-offer policy, with the buyer purchasing with a low value for at least some dates, is optimal if and only if it is optimal to induce purchase with a low value immediately when the arrival date is known, i.e. if and only if the inequality (5) holds. Consider the choice of $\bar{t}$ for an introductory-offer policy. The benefit of increasing $\bar{t}$ to $\bar{t} + \varepsilon$, for some small $\varepsilon > 0$, is that if the buyer arrives between $\bar{t}$ and $\bar{t} + \varepsilon$, he is induced to purchase and is charged approximately $\theta_L$ (since, under an optimal price path, he earns zero rent from delaying purchase.

\(^{16}\)It is interesting to compare this result with that for Deb’s (2010) model with multiple shocks. Deb restricts the class of mechanisms in this environment by permitting purchase only on dates at which the buyer’s value changes. Analogous to the restriction to introductory offers here, this has the implication that the optimal price path in his model is monotone.
until after $\bar{t} + \varepsilon$) rather than being charged $\theta_H$ once his value becomes high. If the inequality (5) holds, then this benefit is positive. The cost of extending the interval of sale to low values (assuming $\alpha_L > 0$) is higher rents in case of an earlier arrival, which has probability $1 - e^{-\lambda \bar{t}}$. Thus, for $\bar{t}$ sufficiently small, the benefit dominates.

### 3.4 Unknown arrival time: optimal sales policies

We now study the fully-optimal sales policy when the buyer’s arrival date is uncertain. Define, for any $z > 0$,

$$
\pi(z) = \int_0^z \lambda e^{-(r+\lambda)s}R(z - s) \, ds,
$$

where $R(\cdot)$ is given by (4). Thus $e^{-(r+\lambda)t} \pi(z)$ gives the component of seller revenue corresponding to arrival in the interval $[t, t + z]$ when the buyer purchases with a low value only at the end of the interval, i.e. at $t + z$, and where the payoff to the buyer purchasing at this time with a low value is zero (i.e., $u_{t+z}^{\Omega_P} (\theta_L; x) = 0$, and $p(t + z) = \theta_L$). Define, for any $z > 0$,

$$
\phi(z) = \int_0^z \lambda e^{-(r+\lambda)s} \left( \theta_L - \left(1 - e^{-r(z-s)}\right) \frac{\alpha_L (\theta_H - \theta_L)}{r} \right) \, ds.
$$

Thus $e^{-(r+\lambda)t} \phi(z)$ gives the component of seller revenue corresponding to arrival in the interval $[t, t + z]$, where the buyer is induced to purchase upon arrival with a low value at any time in the interval, again assuming the buyer obtains zero payoff if buying at the end of the interval when his value is low.

Not captured by the above definitions, the expected revenue earned during any interval depends on the sales policy after that interval (in particular, it depends on $u_{t+z}^{\Omega_P} (\theta_L; x)$, which is strictly positive whenever $\alpha_L > 0$ and the buyer is induced to purchase with a low value at dates after $t + z$). To understand this dependence, note that if the buyer purchases with a low value at $t$ (i.e., if $t \in A$), the difference in total expected rents from a policy of having the buyer purchase with a low value at exactly one future date $t + z$, rather than never having him purchase again with a low value after $t$, is

$$
e^{-rt} \left(1 - e^{-rM}\right) u_{t}^{\Omega_P} (\theta_L; x) = e^{-r(t+z)} \left(1 - e^{-rM}\right) (\theta_H - \theta_L) \frac{\alpha_L (1 - e^{-(\alpha_L + \alpha_H)z})}{\alpha_L + \alpha_H}.
$$
The difference in total expected rents from a policy of having the buyer purchase whenever his value is low at each instant over \([t, t + z]\) and never after, rather than having him purchase with a low value at \(t\) but never after, is

\[
e^{-rt} \left(1 - e^{-\lambda t}\right) u_t^\Omega P(\theta_L; x) = e^{-rt} \left(1 - e^{-\lambda t}\right) \left(1 - e^{-rz}\right) \frac{\alpha_L(\theta_H - \theta_L)}{r}.
\]

Given a date \(t\) at which the buyer purchases with a low value, we can use these two expressions to calculate the total effect of the sales policy after date \(t\) on the rents of buyers arriving before \(t\). This allows us to consider a modified problem which is entirely forward-looking.

Suppose that the sales policy has been determined up to date \(t\) and that this is a date at which the buyer purchases if his value is low. For such \(t\), let

\[
A(t) = \cup_{m \in M(t; A(t))} [t_m, \tilde{t}_m] \subset [t, \infty)
\]

be a sales policy from period \(t\) onwards, where \(M(t; A(t))\) is a set of indices for the intervals comprising \(A(t)\). Let \(A(t)\) be the set of all possible continuation policies, which, as described in Subsection 3.1, we take to be countable unions of closed intervals. Define \(\tilde{t}_m = \min \{A(t) \setminus [t, \tilde{t}_m]\}\) to be the next time the buyer purchases with a low value after an interval \([t_m, \tilde{t}_m]\), or \(\tilde{t}_m = \infty\) if there is no such date. Then the value of the seller’s continuation problem is given by

\[
W^*(t) = \sup_{A(t) \in A(t)} \sum_{m \in M(t; A(t))} \left( e^{-r(t_m - t)} \left( e^{-\lambda m \phi (\tilde{t}_m - t_m)} - (1 - e^{-\lambda m}) \left(1 - e^{-r(t_m - t_m)}\right) \frac{\alpha_L(\theta_H - \theta_L)}{r} \right) + e^{-r(\tilde{t}_m - t)} \left( e^{-\lambda \tilde{t}_m \pi } \left(1 - e^{-\lambda (\tilde{t}_m - t_m)}\right) \left(1 - e^{-\lambda (\tilde{t}_m - t_m)}\right) \left(\frac{\alpha_L(1 - e^{-(\alpha_L + \alpha_H)(t_m - t_m)})}{\alpha_L + \alpha_H}\right) \right) \right).
\]

For \(t = 0\), this gives the supremum of the seller’s expected revenue.

The value function \(W^*\) satisfies the following functional equation, which describes the seller’s
problem recursively:

\[
W^*(t) = \sup_{t \geq t \geq t} \left\{ e^{-\lambda t} \left( 1 - e^{-\lambda t} \right) \left( 1 - e^{-\lambda (t-t)} \right) \frac{\alpha_L (\theta_H - \theta_L)}{\alpha_L + \alpha_H} \right. \\
+ e^{-r(t-t)} \left( e^{-\lambda t} (1 - e^{-\lambda (t-t)} (1 - e^{-\lambda t}) (\theta_H - \theta_L) \right) \frac{\alpha_L (1-e^{-\alpha_L \alpha_H}) (t-t)}{\alpha_L + \alpha_H} \\
+ e^{-r(t-t)} W^*(\hat{t}) \\
\left. e^{-\lambda \frac{t}{r+\Delta}} (\gamma + \frac{\alpha_L (1-\gamma)}{\alpha_L + r}) \theta_H \right\} 
\]

(10)

In this problem, given a date \( t \) at which the buyer purchases if his value is low, a choice is made either to (a) induce the buyer to purchase with a low value at all instants up to \( \hat{t} \geq t \), and thereafter induce purchase by the buyer only if his value his high up to date \( \hat{t} \), the next date at which the buyer purchases with a low value, or (b) induce purchase by the buyer only if his value is high at all dates after \( t \).

We now determine some properties of an optimal sales policy; a proof that an optimum exists is available from the author. Firstly, it is easy to see that, for certain parameter values, \( A = \emptyset \) is optimal and so it is optimal to set prices at the constant level \( \theta_H \). The interesting cases are where \( A = \emptyset \) is not optimal. The following remark is useful for identifying these.

**Remark 2** Optimal posted prices induce purchase by the buyer if his value is low on at least one positive date provided the inequality (5) holds. If (5) does not hold, and if \( \gamma \leq \frac{\alpha_L}{\alpha_L + \alpha_H} \), then \( A = \emptyset \) is an optimal sales policy.\(^{17}\)

That (5) is sufficient for the optimality of inducing purchase with a low value at positive dates is immediate from Proposition 3, since if (5) holds, the policy of inducing purchase only in case the buyer’s value is high is dominated, at least by an introductory-offer policy. If \( \gamma \leq \frac{\alpha_L}{\alpha_L + \alpha_H} \) and (5) fails, Proposition 2 implies that it is optimal to induce purchase by the buyer when his value is high in case arrival times are observed. By charging a constant price equal to \( \theta_H \), the same outcomes can be achieved also when arrival times are unobserved.

\(^{17}\)In case \( A = \emptyset \) is optimal, \( A = \{0\} \) is clearly optimal as well, since the probability of an arrival at date zero is equal to zero.
We now assume that $\alpha_L, \alpha_H > 0$, and show that introductory offer schemes are never optimal. This implies that if the buyer ever purchases with a low value under the optimal policy, then the price path must be non-monotone. In particular, there is at least one interval over which the buyer does not purchase if his value is low and the price path is declining, followed by a date at which the buyer is induced to purchase if his value is low. The analysis also allows us to determine a date after which, in an optimal sales policy, there is no interval of positive length during which the buyer always buys if his value is low.

**Proposition 4** Suppose $\alpha_L, \alpha_H > 0$.

(i) “Introductory offers” are never optimal. If the sales policy includes any date after zero, then the price path is non-monotone.

(ii) Let $T = 0$ if

$$\omega_L \leq \gamma \omega_H,$$

and

$$T = \frac{1}{\lambda} \log \left( 1 + \frac{\omega_L \lambda (1 - \gamma) - \gamma \lambda (\omega_H - \omega_L)}{\alpha_L (\omega_H - \omega_L)} \right) > 0$$

otherwise. If $[t, \bar{t}] \subset A$, with $\bar{t} > T$, then $\bar{t} = \bar{t}$.

Part (i) follows from considering an optimal introductory offer sales policy, as defined in Proposition 3. If $A = [0, \bar{t}]$ is such a policy, with $\bar{t} > 0$, then we consider an alternative policy $A \setminus (t^#, \bar{t})$, and show that this policy improves on $A$ provided $t^#$ is sufficiently close to $\bar{t}$. In so doing, we make use of the fact that $\bar{t}$ was chosen optimally. This guarantees that $\bar{t}$ and thus the probability of arrival before $\bar{t}$ is large enough that the expected reduction in buyer rents from the change in policy exceeds the expected efficiency loss. Part (ii) relies on similar logic, showing that if the sales policy $A$ includes any interval $[t_k, \bar{t}_k]$, with $\bar{t}_k > T$, there must be an alternative policy $A \setminus (t^#, \bar{t}_k)$ with higher revenue for some $t^#$ close enough to $\bar{t}_k$. The choice of $T$ ensures that the probability of prior arrival is sufficiently large that an appropriately chosen alternative policy leads to a reduction in total expected rent that exceeds the loss in surplus.
Proposition 4 may seem surprising in light of Part (ii) of Proposition 2. It shows that, once we allow the buyer's arrival date to be uncertain and unknown to the seller, intertemporal price discrimination can be profitable even if \( \gamma \leq \frac{\alpha_L}{\alpha_L + \alpha_H} \), and in particular when the distribution of buyer initial values is ergodic. Under the optimal policy, the buyer always purchases immediately if his value is high but he may choose to delay purchase when his value is low, purchasing only on certain dates when the price is low enough. As discussed above, the key difference between this environment and the one where the buyer is known to arrive at date zero is that the seller may wish to set a low price at dates after date zero. Indeed, he may wish to set many such dates to attract the buyer to purchase in case he arrives after the previous date with a low price and his value remains low. Since the arrival date is not observed, it is not possible to exclude the buyer from purchasing at these later dates in case his arrival is early. Deterring imitation of later arrival requires the buyer to earn additional rents for every future date with a price low enough to induce purchase with a low value. Choosing to set such prices only intermittently (i.e., at distinct times) is a way to reduce these rents whilst balancing the concern that only having the buyer purchase when his value is high is inefficient (indeed, any delay in purchase represents a loss in surplus).

The reason setting distinct dates at which the buyer purchases with a low value reduces rents in case of an early arrival can be understood by considering the ability of the buyer to obtain rents when purchasing when his value is high. Indeed the only reason the revenue-maximizing prices must leave rent to the buyer when his value is low is the possibility that his value will be high at some future date. If the buyer’s value is high on a date when he is willing to purchase with a low value, he earns a rent \( \theta_H - \theta_L \) in excess of what he earns if his value is low. Suppose for the sake of argument that purchases may only be made on dates at which the buyer purchases with a low value (the optimal price path also induces purchase at all other dates in case the buyer’s value is high, but the buyer’s expected rents under the two policies are the same at all dates). If the buyer must wait longer for the next opportunity to purchase when his value is high, then he expects lower rents both because purchase is delayed and because, given that \( \alpha_H > 0 \), there is a larger probability that his value becomes low whilst waiting.
Another way to understand the results in Proposition 4 is by considering mechanisms that involve both rental and outright sale. Suppose first that the inequality (11) holds. A sales policy that includes an interval \([t_k, t_{\bar{k}}]\), say, is equivalent to the mechanism described as follows. First, the buyer is offered the chance to rent the good at rental price \(\omega_L\) from \(t_k\) up to \(t_{\bar{k}}\), thus inducing the buyer to rent irrespective of his value. Second, the good is offered for outright sale at all other times at the same price as for the original mechanism; this includes choosing the price at \(t_{\bar{k}}\) to attract the buyer to purchase irrespective of his value. The inequality (11) means that setting a rental price of \(\omega_L\) over the interval is sub-optimal. If the seller must rent the good rather than sell it between \(t_k\) and \(t_{\bar{k}}\), a better policy would be to set a rental price \(\omega_H\), inducing the buyer to rent only if his value is high (again offering the good for sale at all other dates at the same price as under the original mechanism). This policy is, however, strictly dominated by the policy of instead selling outright rather than renting between \(t_k\) and \(t_{\bar{k}}\) at a price that induces the buyer to purchase only if his value is high and which makes him indifferent between purchasing immediately and waiting until \(t_{\bar{k}}\). The reason that selling dominates rental in this case is simply that outright selling is more efficient, as the buyer continues to hold the good even if his value turns low, while his expected rents, assuming optimal prices, are the same.

The same intuition can be extended to the case where (11) does not hold by recognizing that a commitment to rent at a price of \(\omega_L\) over an interval of positive length (rather than, say, to sell at a price which induces the buyer to purchase only if his value is high) implies that earlier prices must be lower. That is, rental at price \(\omega_L\) becomes suboptimal at dates sufficiently far from the beginning also if \(\omega_L > \gamma \omega_H\).

As argued above, delaying the next date of purchase by the buyer when his value is low limits the buyer’s rents before this date in case his value is high because, given that \(\alpha_H > 0\), his value may become low by the time this date arrives. This motivation for inducing delay is crucial for the findings in Proposition 4. The next result confirms this by showing that they no longer hold in case \(\alpha_H = 0\).
Proposition 5 Suppose that $\alpha_H = 0$ and $\alpha_L > 0$. If the inequality (5) holds, then the optimal sales policy is $A = [0, \bar{t}]$ with $\bar{t}$ given by (12) (equivalently, given $\alpha_H = 0$, by (8)).

Proposition 5 gives us a sense in which the intuition from Example 1 is important for our results about the optimality of cycling prices in Proposition 4. Driving that finding is the observation that, by spacing the dates at which the buyer purchases with a low value, the seller can exploit the “impatience” of the buyer when his value is high (as discussed with respect to the example) by charging higher prices. This also means that the rents the buyer expects when his value changes from low to high are less, and therefore that the rents early arrivers must be promised to implement the sales policy in question are reduced. This intuition clearly only applies in case $\alpha_H > 0$, and Proposition 5 confirms that it is crucial to our finding that the optimal price path involves cycles.

A more detailed understanding of the optimal price path is available through numerical analysis. Numerical examples are approximated by discretizing the dates at which the buyer may be induced to purchase with a low value and using value function iteration.\(^{18}\) We also use the fact, mentioned in the Introduction, that, provided $\alpha_L > 0$, then the optimal sales policy is bounded. This follows for the same reasons as discussed with respect to Proposition 3, namely that the cost of setting low prices in terms of the effect on the buyer’s ex-ante expected rents, when compared to the effect on total surplus, grows without bound.\(^{19}\)

Example 2 Suppose that $\omega_L = 1\frac{1}{4}$ and that $\omega_H = 2$, $\alpha_L = \alpha_H = 1\frac{1}{4}$ and that $\lambda = r = \frac{1}{10}$. It follows that $\theta_L = 15\frac{5}{8}$ and $\theta_H = 16\frac{7}{8}$. We approximate $A$ by considering policies with sales at each date $n/100$, $n \in \mathbb{N}$.\(^{20}\) The optimal sales policy and price path is depicted in Figure 1.

Although we have been unable to find counterexamples, it seems difficult to guarantee analytically that the durations between dates of sale, at least after the date $T$ defined in Part (ii) of

\(^{18}\)Note that, as the discretization becomes fine, the policy of selling at every discrete date over an interval approximates the revenue obtained by selling continuously over that interval. Thus it is sufficient to consider a program where sales are made only at discrete dates, where the time between dates is taken to be small.

\(^{19}\)A proof is available from the author.

\(^{20}\)An approximation with $n/1000$ yields approximately the same results. The only notable difference is that there are additional discrete dates in the sales policy at the end of the initial interval, suggesting that discrete dates cluster from the right at the end of the interval for the truly optimal policy.
Proposition 4, become monotonically greater with time. One might hope to reason, for example, that if \(4, 5, 6\) are consecutive dates in an optimal sales policy \(A\), then 1 and 3 must not be consecutive dates in \(A\). If date 5 is optimally included in the sales policy, then the increase in revenue over the interval \((4, 6]\) as a result of including this date must be greater than the additional expected rents earned by buyers arriving before date 4. This implies, however, that the increase in revenue earned in \((1, 3]\) from including also date 2, must exceed the change in expected rents for buyers arriving before date 1. This logic is difficult to extend more generally, however, because consecutive dates of sale are endogenously determined and may not be easily comparable.\(^{21}\)

\(^{21}\)Progress can be made by considering a policy that includes adjacent dates of sale to low types \(t', t' + z\) and \(t''\), with \(z \in \left(\frac{t'' - t'}{2}, t''\right)\). An alternative to such a policy discards the date \(t' + z\) and replaces it with \(t'' - z\). By directly comparing the expected revenue under the two policies, we can verify that the change in policy yields an improvement in total revenue. However, the inequality to be verified is cumbersome, and it is unclear whether it holds uniformly over all possible parameter values.
3.5 Diminishing values from consumption

As mentioned in the Introduction, the implication of our model that prices tend to rise over time is not borne out in many markets where the evolution of fashion and technology play an important role. One reason this seems to be true is that, as fashions and technology are replaced, consumers derive less and less value from the older products. As this occurs, we would expect the price also to decline. Our model can be extended to accommodate this possibility, and it is easy to see that the key tradeoffs in determining when and whether to reduce prices remain.

One particularly tractable possibility is to posit that the flow value derived from the good vanishes with the time since date zero at a constant rate $f > 0$. In this case, $\omega_H(t) = \omega_H(0) e^{-ft}$ and $\omega_L(t) = \omega_L(0) e^{-ft}$ for all $t \geq 0$, where we assume $\omega_H(0) > \omega_L(0) > 0$. This implies that the expected values from obtaining the good in the high and low states are

\[
\theta_H(t) = e^{-ft} \theta_H(0) \quad \text{and} \quad \theta_L(t) = e^{-ft} \theta_L(0),
\]

where

\[
\theta_H(0) = \frac{(\alpha_L + r + f) \omega_H(0) + \alpha_H \omega_L(0)}{(r + f) (\alpha_L + \alpha_H + r + f)} \quad \text{and} \quad \theta_L(0) = \frac{\alpha_L \omega_H(0) + (\alpha_H + r + f) \omega_L(0)}{(r + f) (\alpha_L + \alpha_H + r + f)}.
\]

Using the same arguments as in the previous subsection, it is straightforward to check that the optimal sales policy is identical for the problem where the discount rate is given by $r + f$. Prices are the same as if the true discount rate were $r + f$, except that they must be scaled by $e^{-ft}$. The following example illustrates one possibility for the pattern of prices.

**Example 3** Suppose that $\omega_L = \frac{1}{4}$, $\omega_H = 2$, $\alpha_L = \alpha_H = \frac{1}{4}$, $\lambda = \frac{1}{10}$, $r = \frac{4}{50}$ and $f = \frac{1}{50}$ (hence $r + f = \frac{1}{10}$). It follows that $\theta_L(0) = 15\frac{5}{8}$ and $\theta_H(0) = 16\frac{7}{8}$. The optimal sales policy is the same as in Example 2. The optimal sales policy and price path is depicted in Figure 2.

In Example 3, after sufficient time, the buyer purchases only once his value turns high. At such a date $t$, the price is equal to the expected value from acquiring the good at date $t$, $\theta_H(t)$, which is declining in $t$. Thus, by combining the hypothesis of private uncertainty with the hypothesis of an anticipated and predictable decline in values, we can account for both discounting both early
and late in the product life cycle, which, as mentioned in the Introduction, is consistent with price patterns found in fashion retail (see Pashigian and Bowen (1991)).

3.6 Buyer attrition

Another important feature of the model considered so far is that the buyer anticipates opportunities to buy the good into the infinite future. This implies that for a given sales policy $A$ with $\max A > 0$, including an additional date $t^* > \max A$ (and maintaining conditionally optimal prices) leads to an increase in the date-zero value of the buyer's rent which is the same for all dates of arrival before $\max A$. It is easy to see that this implication drives the observation that the seller optimally chooses a bounded sales policy so that the price eventually remains at $\theta_H$.

Suppose therefore that the buyer, having arrived in the market at some rate $\lambda > 0$, leaves it...
again at some rate \( k > 0 \). There are two possible interpretations: either the buyer leaves the market at a time when he can no longer take delivery of the good, though continues to have a use for it, or he leaves the market at a time when he foresees no future use for it (equivalently, when the buyer dies). Which interpretation is preferred does not affect the qualitative properties of the optimal price path, although to fix ideas we consider the latter.

It is easy to see that if \( k > \lambda \), then the ex-ante probability the buyer is “alive” (i.e., has arrived and still finds a use for the good), namely \( e^{-\lambda t} \frac{\lambda}{k-\lambda} (1 - e^{-(k-\lambda)t}) \), eventually becomes proportional to the arrival rate \( \lambda e^{-\lambda t} \). This implies, in the long run, that the pattern of prices becomes stationary. Either the prices remain fixed forever after a certain time, or they cycle with a constant period length. Further details will be provided in the next version of the paper.

### 3.7 Towards a theory of countercyclical markups

As mentioned in the Introduction, our theory provides a possible explanation for countercyclical markups that has gone unnoticed. The hypothesis of countercyclical markups was proposed originally by Pigou (1927) and Keynes (1939), who observed that prices tend to change little in response to increases in aggregate demand. Murphy, Shleifer and Vishny (1989) report countercyclical movements in the prices of finished goods relative to intermediate inputs in the post-war period, with the finding most pronounced for consumer durables. At the micro level, Pashigian and Bowen (1991) document the evolution of prices for men’s shirts and Warner and Barsky (1995) document the prices of a wider range of durables and “semi-durables”. Both find evidence that prices are often lower in the pre-Christmas rather than post-Christmas periods, as well as on weekends, when consumers have more time for shopping. Chevalier et al. (2003) find markups tend to fall during high demand periods in supermarket data.

One possible explanation for countercyclical markups seems stem from the simple trade-off made by sellers when deciding their path of prices. At any date, a seller trades off between two objectives: (i) extracting as much revenue as possible from buyers arriving at (or just before) a given time, which may require setting a low price, and (ii) lowering the option value of waiting for
earlier arrivers, which requires setting a high price at the time in question. The first objective obviously becomes much more important if there is a rush of buyers into the market at (or just before) this time. Thus, we might expect to see lower prices at times when the rate of buyer entry is high. However, this intuition is incorrect in a model where buyer values do not change because of Stokey’s (1979) result; i.e., because a constant price is optimal provided the distribution of values among arriving buyers does not change with time. It is, however, consistent with our model provided that low values may turn high.

Suppose, for example, that it is known ex-ante that the rate of arrival jumps at some time $t^*$. For instance, suppose that $Pr(\tau \leq t) = 1 - e^{-\lambda t}$ for all $t \leq t^*$ and $Pr(\tau \leq t) = 1 - e^{-\lambda'(t-t^*)-\lambda t^*}$ for $t > t^*$, where $\lambda' > \lambda$. The intuition above and in the previous sub-sections suggests that the seller will be more inclined to induce purchase by the buyer when his value is low after $t^*$ than if the rate remained constant. For instance, for $\lambda'$ sufficiently large and $\lambda$ sufficiently small, it may be that the buyer never purchases with a low value after $t^*$ if the rate remains constant at $\lambda$, but that he does purchase with a low value at such times if it instead increases to $\lambda'$. The same kind of result should apply also if the date $t^*$ is not known ex-ante but is publicly observable, in which case a contingent price path can be chosen. Although this theory does not predict uniformly lower prices after $t^*$, it is reasonable to conjecture that the effect holds once aggregated across a range of goods, possibly with different initial dates and demand characteristics. Such a conclusion would seem consistent with the evidence.

Although our theory applies to durable goods, it seems likely also to extend, at least to some extent, to many goods that are usually considered non-durables, but for which consumers find there is discretion as to the timing of purchase. For example, a leading example in the study of Chevalier et al. (2003) is canned tuna, which has a seasonal demand spike during Lent, reflecting that tuna is not a consumer staple, but rather a commodity that many purchase only occasionally. Many consumers might be expected to purchase only once they have a craving for tuna (become a “high” type) or once the price drops sufficiently low. As for durables, since an immediate purchase may satiate the consumer’s desire for tuna for some time, failure to purchase has an option value to
instead purchase in the future. Commitments to low prices in the future can therefore be expected to raise this option value.

A number of theories for countercyclical markups have already been proposed in the literature. Closest to ours, Bils (1989) suggests that, for experience goods, firms lower prices in order to attract new consumers and raise them to extract rent from consumers who have previously purchased and know that the product is effective or to their liking. Periods associated with the greatest influx of new buyers are ones where the trade-off between extracting rent from previous purchasers and enticing new buyers to experiment is resolved in favor of the latter. Assuming the seller cannot commit to a path of prices, Bils shows that prices are lowest during these periods. One difficulty with this theory, however, is that it requires experimentation to play a key role in the purchase decision, which may be appropriate for some goods, but inappropriate for purchases that are essentially one-off (e.g., motor vehicles and other durables) and goods for which learning takes place quickly compared to the number of purchases, or where the relevant information is available without purchase.

Another theory is that the elasticity of demand is pro-cyclical, for example because of economies of scale in consumer search. For instance, Warner and Barsky (1995) suggest that times when consumers are most active are also times when consumer search is most efficient; for example, during the weekend when consumers have time to go from store to store. Chevalier et al. (2003), however, find these stories unconvincing because, in their data set, demand is typically found to be no more elastic at the times associated with high consumer demand and low prices. Because the timing of sales in our theory is not driven by changes in demand elasticity, it seems immune to this criticism.

Another possibility, proposed by Rotemberg and Saloner (1986) is that countercyclical markups result from firms having less ability to sustain tacit collusion during periods of high demand. Chevalier et al. suggest this is unconvincing because countercyclical markups are also associated with increases in demand with certain goods in isolation, e.g. the price of tuna falls at Lent when demand tends to be high due to the dietary restrictions of many Christians. Our theory, however,
seems not to be subject to this criticism because it has implications for the pricing of individual products, rather than necessarily only the joint pricing of all products a seller offers.

Finally, Chevalier et al. suggest loss-leader advertising, where certain items are given a low price and a high profile (e.g., in advertising leaflets) to attract shoppers, can explain many of the patterns they find. The idea is that because advertising is costly and consumer attention scarce, it is optimal to lower the prices of the most popular products. A time of high demand for a given product thus ought to be a time at which its price is relatively low. This theory also has its weaknesses, however. For instance, it is unclear how it can explain the, albeit weaker, countercyclical nature of intermediate goods prices. An important test will also be whether the predictions remain true in on-line selling environments, where consumer advertising is much more targeted to individuals.

4 The fully-optimal mechanism

We now turn to consider the fully-optimal mechanism. For ease of exposition, we restrict attention to deterministic mechanisms, although our analysis can be easily extended to show that this restriction is without loss of optimality (see footnote 29 in Appendix B for a discussion). Our chief interest is again in the environment where the buyer’s arrival is unobserved. In this environment, the seller can commit to a mechanism that allows the buyer to reveal himself potentially at any date after his arrival by signing a pre-specified contract. A key part of the analysis is therefore to ensure not only that the buyer finds it individually rational to sign a contract, but also that he finds it incentive compatible to sign the one that is meant for him, i.e. that he prefers to sign this contract, rather than to, say, wait for the possibility of signing a different one.

We assume initially that the buyer can make long-term contractual commitments and thus commit himself to always continuing under the terms of a given contract. We show however, that, by ensuring the buyer always expects non-negative continuation payoffs, a revenue-maximizing policy can always be implemented without relying on his commitment ability. Another consideration that becomes relevant when considering the population version of the model is whether the seller can
recognize buyers’ identities. Our analysis for this version remains unchanged provided we assume that the seller can recognize a buyer’s identity upon signing a contract, and that each buyer can assume only one identity, which precludes, in particular, the signing of multiple contracts.

The fully-optimal mechanism turns out to be a collection of option contracts. In equilibrium, if the buyer arrives at date $\tau$, then he immediately purchases an option to buy at a sequence of date-contingent posted prices. These posted prices are determined to induce immediate purchase if the buyer’s value is high or becomes high, and to induce purchase when his value is low only on the date at which the option to purchase expires; we denote by $t_L(\tau)$ the expiry date for the option acquired at $\tau$. As for the contracts in Subsection 3.2 where the buyer’s arrival date is public information, the date of purchase with a low value is specific to the buyer’s arrival date. However, there are two key differences in the analysis with respect to Subsection 3.2. First, by selling the option to purchase at a posted price, the seller can hope to extract some of the surplus that the buyer expects if his value turns high before the option expires. Second, since the buyer’s arrival time is unobserved, ensuring that it is incentive compatible for the buyer to sign a contract and reveal himself at the appropriate moment (as discussed above) is non-trivial.

Our solution to the problem of designing the fully-optimal mechanism proceeds as follows. Without loss of optimality, we focus on mechanisms where the buyer signs a contract and thus reveals himself immediately upon arrival at date $\tau$. We require that the buyer is willing to sign when his initial value is low, but we ignore the constraint that he is also willing to do so when his value is high. Thus we study a “relaxed program”. For the relaxed program, we show (Lemma 2) how any mechanism is dominated (at least weakly) in terms of the revenue it generates by a collection of option contracts, where, (i) the expiry date of each option $t_L(\tau)$ is set equal to the date at which the buyer acquires the good under the original mechanism when he arrives at date $\tau$ and his value remains low, (ii) once the buyer has purchased an option, at all dates before the expiry date when his value is high, he is indifferent between buying the good immediately and waiting, and (iii) if the buyer arrives and purchases an option at date $\tau$ when his value is low, he receives the same expected payoff as under the original mechanism. We then determine (Lemma 3)
3), given fixed expiry dates \((t_L(\tau))_{\tau \geq 0}\), the smallest possible values of the rents expected by the buyer when his initial value is low. These are the values that make the buyer, for a low initial value, precisely indifferent between signing the contract upon arrival and waiting to sign at a future date. We also provide a condition on the expiry dates such that the relaxed program is valid; i.e., that the collection of option contracts induces the desired behavior by the buyer, in particular, that the buyer wants to sign immediately when arriving with a high value. Finally (Proposition 6), we compute the optimal expiry dates and verify that they satisfy the aforementioned condition.

The next result documents the optimality of option contracts as described above. The proof follows from considering the direct mechanism, which is introduced in Appendix B.

**Lemma 2** Let \(\Omega\) be any mechanism. Suppose that, if the buyer arrives at date \(\tau\) with a low value, then (a) his expected payoff at that time is \(w_\tau\), and (b) if his value remains low, then, in equilibrium, he receives the good at date \(t_L(\tau)\). Consider the alternative mechanism whereby, upon arrival at date \(\tau\), the buyer is asked to pay an up-front fee

\[
 f^\#_\tau (t_L(\tau), w_\tau) = e^{-\gamma(t_L(\tau)-\tau)} (\theta_H - \theta_L) \frac{\alpha_L \left(1 - e^{-(\alpha_L + \alpha_H) (t_L(\tau) - \gamma)}\right)}{\alpha_L + \alpha_H} - w_\tau
\]

and, conditional on paying the fee, receives the right to purchase at each date \(t \in [\tau, t_L(\tau)]\) at price

\[
 p^\#_\tau (t; t_L(\tau)) = \theta_H - e^{-\gamma(t_L(\tau)-t)} (\theta_H - \theta_L) \frac{\alpha_L + \alpha_H e^{-(\alpha_L + \alpha_H)(t_L(\tau)-t)}}{\alpha_L + \alpha_H}
\]

(the buyer cannot purchase after date \(t_L(\tau)\), which is the expiry date for the option). The buyer is asked to purchase either as soon as his value is high or at date \(t_L(\tau)\) if his value is still low at that time. Provided he is willing to sign the contract on arrival at \(\tau\) when his value is high, then the requested actions are incentive compatible. Expected revenue is at least as high under this mechanism as for the original mechanism \(\Omega\).

The reason the option contracts in the proposition must do as well as any candidate mechanism \(\Omega\) is as follows. First, because these contracts induce the buyer to purchase at least as early as \(\Omega\) does, expected surplus is greater. Second, as mentioned above, prices are chosen so that, after
signing a contract at date \( \tau \) when his value is high, the buyer is indifferent between purchasing immediately and waiting until date \( t_L(\tau) \). This means that the rent he expects at the time of contracting when his value is high is as small as possible. Thus, provided the buyer is willing to act in the prescribed way (we argue in the appendix that this must be the case provided he is willing to sign immediately when his initial value is high), expected rents are no higher than for \( \Omega \).

The posted price function \( p^\#_\tau (\cdot; \cdot) \), is the same as in (6) and is such that, for \( t \geq \tau \), \( p^\#_\tau (t; t) = \theta_L \). In addition to the posted price, the buyer must pay an up-front fee (the price of the option). Note that this fee may be negative; in particular, this is true whenever \( w_\tau \) is large relative to the waiting time \( t_L(\tau) - \tau \). Intuitively, the buyer may need to be subsidized to take the option contract available on arrival at date \( \tau \) in return for giving up the option to sign a different contract available at a later date. That the fee or subsidy occurs at the time of contracting is not essential, however. The result in Lemma 2 remains true also if payments are spread differently across time. The reason for the flexibility in the timing of payments is that both the buyer and seller have payoffs that are linear in money and have the same discount rate. The suggested timing of payments, and in particular the use of an up-front fee, has the advantage that, after signing a contract at date \( \tau \) and paying the fee, there are no further promises the buyer might wish to reneg on. The same is true for any contract that sufficiently front-loads the buyer’s payments.

We now determine the minimal level of rents for the buyer when he arrives with a low value. With a slight abuse of the notation introduced in the proof of Lemma 2 (Appendix B), let \( w_\tau (\theta_\tau) \) be the buyer’s expected payoff if he signs the contract at date \( \tau \) when his value is \( \theta_\tau \in \{ \theta_L, \theta_H \} \). If the buyer’s value is low upon arrival at date \( \tau \), he is willing to sign the date-\( \tau \) contract only if his expected payoff exceeds that from the alternative strategy of signing a contract if and only if his value turns high; thus, he signs the date-\( \tau \) contract only if

\[
  w_\tau (\theta_L) \geq \int_\tau^\infty \alpha_L e^{-(\tau + \alpha L)(z - \tau)} w_z (\theta_H) dz.
\]

Under the option contracts of Lemma 2, if the buyer contracts at date \( \tau \) and receives the good after a history of low values at \( t_L(\tau) \), then the difference in payoffs at the time of contracting
for high and low initial values is exactly $e^{-(r+\alpha_L+\alpha_H)(t_L(\tau)-\tau)}(\theta_H - \theta_L)$. Therefore, if the values $(w_\tau(\theta_L), w_\tau(\theta_H))_{\tau \geq 0}$ are the payoffs at contracting under a mechanism which induces the buyer to sign on arrival, it must be that

$$w_\tau(\theta_L) \geq \int_{\tau}^{\infty} \alpha_L e^{-(r+\alpha_L)(z-\tau)} \left( w_z(\theta_L) + e^{-(r+\alpha_L+\alpha_H)(t_L(z)-z)}(\theta_H - \theta_L) \right) dz. \quad (13)$$

The payoff $w_\tau(\theta_L)$ therefore assumes its minimum value for each $\tau$ if (13) holds with equality at all dates. Taking derivatives with respect to $\tau$, we obtain the linear first-order differential equation

$$rw_\tau(\theta_L) = \frac{\partial w_\tau(\theta_L)}{\partial \tau} + \alpha_L e^{-(r+\alpha_L+\alpha_H)(t_L(\tau)-\tau)}(\theta_H - \theta_L). \quad (14)$$

The unique solution consistent with $w_\tau(\theta_L)$ remaining bounded as $\tau$ grows large is

$$w^\min_{\tau}(\theta_L; t_L) = \int_{\tau}^{\infty} e^{-r(s-\tau)} \alpha_L e^{-(r+\alpha_L+\alpha_H)(t_L(s)-s)}(\theta_H - \theta_L) ds.$$

Suppose we restrict attention to mechanisms that implement allocations consistent with a given function $t_L(\cdot)$. A mechanism that solves the relaxed program given $t_L(\cdot)$ is the collection of option contracts specified in Lemma 2 with $w_\tau = w^\min_{\tau}(\theta_L; t_L)$ for each $\tau$ (as argued above, this mechanism minimizes the buyer’s expected rents and is as efficient as possible given the choice of expiry dates $t_L(\cdot)$). It turns out that, to guarantee this mechanism is also a solution to the problem of interest, all that is required is a simple condition on $t_L(\cdot)$ given in the following result.

**Lemma 3** Suppose $t_L(\cdot)$ is continuously differentiable and non-decreasing. Consider the class of mechanisms such that, if the buyer arrives at date $\tau$, and if his value remains low, then he receives the good on date $t_L(\tau)$. The collection of option contracts described by $\left( f^\#_{\tau}(t_L(\tau), w_\tau), p^\#_{\tau}(\cdot; t_L(\tau)) \right)_{\tau \geq 0}$, with $w_\tau = w^\min_{\tau}(\theta_L; t_L)$ for all $\tau$ (i) induces the buyer to purchase an option upon arrival, and (ii) is revenue maximizing in this class.

The condition on $t_L(\cdot)$ requires that the earlier the buyer contracts with the seller, the earlier he receives the good when his value remains low. The reason this condition plays a crucial role is that, as is clear from the discussion above, the shorter the buyer must wait for receipt of the good when his value remains low, the greater the additional rents the buyer expects when his value is instead
high at the time of contracting. Since the buyer is indifferent between signing and waiting if his value is low, it is clearly sufficient that the waiting time \( t_L (\tau) - \tau \) is non-decreasing. Whilst this property turns out to be satisfied by the optimal mechanism, the monotonicity condition in Lemma 3 is weaker. The reason is that waiting (as opposed to signing immediately) is made unattractive for the buyer when his value is high both because of discounting and because this value then has a chance to turn low before a contract is signed.

Next, we derive the principal’s expected revenue. Using integration by parts, the level of rent the buyer expects to earn from a time-zero perspective when acting as requested by the mechanism is

\[
\gamma \int_0^\infty \lambda e^{-(r+\gamma)\tau} w^\min_T (\theta_H; t_L) d\tau + (1 - \gamma) \int_0^\infty \lambda e^{-(r+\gamma)\tau} w^\min_T (\theta_L; t_L) d\tau
\]

\[
= (\theta_H - \theta_L) \int_0^\infty e^{-r t_L (\tau)} - (\alpha_L + \alpha_H) (t_L (\tau) - \tau) \left( \alpha_L \left( 1 - e^{-r \tau} \right) + \gamma \lambda e^{-r \tau} \right) d\tau.
\]

This allows us to give an expression for the seller’s expected total revenue ((18) in Appendix B), from which we have the following result.

**Proposition 6** Suppose that \( \alpha_H > 0 \). Define for all \( \tau \geq 0 \)

\[
t_L^* (\tau) = \max \left\{ \tau + \frac{1}{\alpha_H} \ln \left( \frac{(\omega_H - \omega_L) (\alpha_L (e^{\lambda \tau} - 1) + \gamma \lambda)}{\omega_L \lambda (1 - \gamma)} \right), \tau \right\}.
\]

The collection of option contracts given by

\[
\left( f^\# (t_L^* (\tau), w_T), p^\# (\cdot; t_L^* (\tau)) \right)_{\tau \geq 0}, \quad \text{with} \quad w_T = w^\min_T (\theta_L; t_L^* (\tau)) \quad \text{for all} \quad \tau,
\]

maximizes the seller’s expected revenue.\(^{23}\)\(^{24}\)

To understand the function \( t_L^* \), consider the contract intended for the buyer if he arrives at date zero. Unlike the contracts available later on, this contract is obviously relevant to the buyer

\(^{23}\)As noted above, the timing of payments is not uniquely determined. However, the optimal choice of \( t_L \) is unique. That \( t_L \) is essentially unique is immediate from the arguments above (in particular, it follows from noting that, for each \( \tau \), \( t_L^* (\tau) \) uniquely maximizes the expression under the integral in (18) in Appendix B). A proof of uniqueness at all dates is available upon request. The argument is based on the failure of the relaxed program in case \( t_L (\cdot) \) contains downward jumps.

\(^{24}\)If, instead, \( \alpha_H = 0 \) and \( \alpha_L > 0 \), then (subject to the non-uniqueness of the timing of payments discussed in the previous footnote) the optimal mechanism is either a constant price of \( \theta_H \), or, in case the inequality (5) holds, the posted-price mechanism specified in Remark 5.
only if he is there at the beginning. This means that the optimal waiting time when the buyer’s value is low, $t^*_L(0)$, coincides with the one for the optimal contract when the buyer’s arrival time is publicly known (i.e., under the same informational assumptions as in Subsection 3.2). This contract induces delayed purchase ($t^*_L(0) > 0$) if and only if $\gamma \omega_H > \omega_L$. The condition states that, in case the seller must rent the good, she would set the rental price equal to $\omega_H$ and thus induce rental only in case the buyer’s value is high. This is the same condition which guarantees the discreteness of the sales policy for posted-price mechanisms, as established in Proposition 4.

There is also an explanation for the optimality of delayed purchase which is closely related to the one for the finding of discrete sales dates with posted prices. The option contract which induces the buyer to purchase at date zero regardless of his value is equivalent to one which rents the good at rate $\omega_L$ for all time, and then charges an up-front fee at date zero for the right to rent equal to the surplus the buyer expects to earn when his initial value is low, i.e. $\theta_L - \frac{\omega_L}{\tau}$, less his outside option from not signing a date-zero contract, call it $w_0$. Now consider the alternative policy which sets the rental price equal to $\omega_H$ at dates $[0, \delta)$ and equal to $\omega_L$ from date $\delta$ onwards. The up-front fee is reduced accordingly to keep the buyer indifferent between paying the fee at date zero and not contracting at date zero (in which case he obtains $w_0$) when his value is low. For small $\delta$, the reduction in the up-front fee is of second-order magnitude. The reduction in rents for the buyer if his value is initially high is of first-order magnitude and approximately equal to $\delta (\omega_H - \omega_L)$ and the loss in surplus conditional on the buyer having a low value is approximately $\delta \omega_L$. Therefore, given that $\gamma (\omega_H - \omega_L) > (1 - \gamma) \omega_L$, expected revenue (conditional on arrival at date zero) is higher under the alternative policy.

The other explanation for the optimality of inducing delayed purchase is the one familiar from the literature on dynamic mechanism design with changing values. Suppose that the buyer arrives at some date $\tau \geq 0$ and determines to purchase the good at date $t_L(\tau)$ irrespective of his value (the mechanism specifies that the buyer purchases as soon as his value becomes high, but his rent is determined by the option of waiting). In case his value is high at the time of contracting, he expects to earn an additional rent by purchasing at $t_L(\tau)$ compared to what he expects if his initial
value is low. This additional rent cannot be extracted by the up-front fee, and so, in choosing \( t_L(\tau) \), the seller trades off the objective of reducing the size of this rent with the inefficiency caused by delaying transfer of the good when the buyer’s value remains low. The later the date \( t_L(\tau) \), the smaller the difference in the distributions of values at that date conditional on the initial values, which means the smaller the additional expected rent when the buyer’s value is initially high. If \( t_L(\tau) \) is sufficiently large, the effect on the buyer’s rent from inducing purchase after low values at \( t_L(\tau) \), rather than, say, never inducing purchase when the buyer’s value is low, is negligible when compared to the expected gain in surplus. This explains why the optimal mechanism always induces purchase by the buyer after a sufficiently long time even if his value remains low (i.e., it explains why \( t^*_L(\tau) - \tau < \infty \) for all \( \tau \)).\(^{25}\) The effect is the same one Battaglini (2005) describes as “vanishing distortions in the long run”.

A key innovation with respect to Battaglini and the rest of the dynamic mechanism design literature is to shed light on the way optimal long-term contracts evolve with the date they are signed. Examining the formula \( t^*_L \) in Proposition 6 yields the following corollary.

**Corollary 1** Suppose \( \alpha_L, \alpha_H > 0 \). The length of time the buyer waits to purchase when his value remains low, \( t^*_L(\tau) - \tau \), is increasing without bound.

The reason for this result is the same one emphasized for posted-price mechanisms. By reducing the rent the buyer expects when signing the contract late, the seller reduces the option value to the buyer of waiting to contract whenever he arrives early. As discussed above, rent can be reduced by inducing delay in purchase by the buyer when his value remains low. Thus whilst we still find, as expected, that there is less reason to distort at dates later on in a contractual relationship, contractual relationships are expected to be less efficient the later they are formed.

The pattern of prices and payments also reflects that the buyer is treated less favorably the later he arrives. This is made clear by the following example.

\(^{25}\)This is in contrast to the optimal posted-price mechanism, where the buyer never purchases with a low valuation if he arrives late enough. The reason for this difference in the two mechanisms is that, if the seller sets a low posted price, the buyer expects to earn a rent due to the chance of having a high valuation at that date, but never has to pay the up-front fee associated with the option contracts.
Example 4 (Example 2 continued) Consider the parameters of Example 2. For the fully-optimal mechanism, and for selected arrival times (t = 0, 1, 2, 3), Figure 3 gives the execution price and present value at time of purchase of the buyer’s total payment.

![Execution prices and total payments for Example 4](image)

Figure 3: Execution prices and total payments for Example 4

Finally, an important question is the extent to which the principles developed in this section hold for more general stochastic processes. Although our simple continuous-time framework has the advantage of tractability, and, for example, allows us to obtain the elegant closed-form expression for the expiry dates in Proposition 6, the fully-optimal mechanism can also be characterized for a richer class of first-order Markov processes. A companion paper (Garrett (2010)) achieves this in a discrete-time setting with a continuum of values by adapting techniques developed by Pavan, Segal and Toikka (2011). It demonstrates, in particular, the usefulness of the same kind of relaxed program as the one used here. A solution to this relaxed program involves the buyer being
indifferent between signing a contract upon arrival and waiting to do so in a future period when his initial value is equal to the minimum. Whether the relaxed program is valid is more difficult to check, although it can be done for particular examples. Under natural restrictions on the evolution of values which formalize diminishing persistence over time, and whenever the relaxed program is valid, we verify both that distortions tend to vanish over the course of a given contract and that distortions in optimal long-term contracts are greater the later they are signed. The idea that intertemporal price discrimination can be achieved without varying contractual terms remains true as well, provided the environment is stationary in an appropriate sense. On the other hand, the option contracts no longer take the simple form considered here. Instead, the optimal mechanism often requires the buyer to constantly update and adjust his option to purchase as new information about his value arrives, requiring ongoing communication with the seller rather than a one-off decision to purchase.

4.1 Applications and comparison to posted prices

Waiting lists and corruption

Although somewhat less sophisticated than the option contracts we propose, waiting lists and other delays in service appear to be used in practice to achieve similar objectives. For example, corrupt bureaucrats often extract bribes in return for early service (the so-called “Myrdal effect”, Myrdal (1968)) and retailers of luxury products sometimes require customers to join a waiting list with the possibility of obtaining the same or a similar good sooner at a higher price.

To be more specific, consider the fashion designer Hermès, who, until recently, would place customers on a lengthy waiting list for its famous Birkin bags. Both the opportunity to join the waiting list, as well as to skip it altogether, appear to have been associated with fees implicit in the requirement of buying other Hermès items (Tonello (2008) and Heit (2009, October 14)). This scheme, in addition to other possible advantages (e.g., creating an impression of scarcity), seems to have allowed intertemporal price discrimination to be applied more uniformly across customers.

We expect that fluctuations in posted prices would also remain optimal in such an environment, although characterizing optimal posted prices may prove intractable.
deciding to enter the market at different times than the alternative possibility of offering occasional
discounts. The implicit fee for joining the waiting list is also consistent with an attempt to extract
rent expected by customers from the option to purchase. In particular, this may have allowed
Hermès to extract more rent if customers expected their value for the bag might receive positive
shocks. It is not clear, however, how a customer was treated in case changing their mind and
wanting the bag sooner. It seems likely that the mechanism used by Hermès was less sophisticated
than the fully-optimal mechanism that we characterize here.

It is also true that, at least in retail settings, sophisticated attempts to achieve intertempo-
rnal price discrimination, such as offering different prices for different waiting times, are relatively
infrequent and sometimes subject to controversy (see, e.g., Tonello (2008)), perhaps because it is
viewed as unfair for customers. However, the theory suggests that, in equilibrium, customers do
not necessarily fare worse when sellers use more sophisticated mechanisms. On the face of it, a
simple posted price mechanism with periodic discounts may leave more rent to the buyer. However,
precisely this may mean that the seller chooses not to offer discounts at all. In that case, the buyer
in our model expects zero rent. Under the fully-optimal mechanism, the buyer always expects
to earn at least some rent, since the buyer is always induced to buy with a low value eventually
if he waits long enough. On the other hand, since the fully-optimal mechanism gives the seller
more flexibility to extract rent, one can show that the direction of the change in expected buyer
rent is ambiguous (Example 2 provides an example where buyer expected rent is higher under the
posted-price mechanism).27

Price stickiness in intermediate goods markets

Finally, that intertemporal price discrimination can be administered more uniformly across
arrival times with more sophisticated mechanisms suggests a possible reason why prices might
change less frequently in intermediate good markets where sophisticated contracts are more common
(see Bils and Klenow (2004), who suggest that the less frequent price changes in the study of Blinder

27We conjecture, although have not been able to prove, that the fully-optimal mechanism is always more efficient
than the optimal posted price mechanism (in Example 2, the fully-optimal mechanism is more efficient).
et al. (1998) may relate to a focus on intermediate good industries).

For instance, note that the optimal option contracts evolve smoothly with the date they are signed. This is immediate from Proposition 6: Since the optimal choice of waiting time $t^*_L(\tau)$ varies smoothly with the contracting date $\tau$, so too do the up-front fee and the available prices. On the other hand, optimal posted prices often fluctuate dramatically. As argued in the Introduction, this suggests a reason why price fluctuations might be more common when sophisticated contingent contracts such as options are not feasible. Indeed, with option contracts, price discrimination is possible without changing the contractual terms over time. If $t_L(\tau) - \tau$ is constant in $\tau$, then the option contracts in Lemma 3 remain constant; i.e., both the fixed price $f^{\#}_\tau(t_L(\tau), w_{\tau}^{\min}(\theta_L; t_L))$ and the price path $p^{\#}_\tau(\tau + h; t_L(\tau))$ are the same for all contracting dates $\tau$ and waiting times $h \in [0, t_L(\tau) - \tau]$. Intertemporal price discrimination still occurs if $t_L(\tau) - \tau > 0$, as the buyer pays a high price when he has a high value and waits to purchase when his value is low. When the seller is restricted to using posted prices, this is only possible if prices fluctuate.

5 Conclusion

[To come.]
References


Appendix A: Proofs of results on posted prices

Proof of Lemma 1. Part (i). Let \( t \geq \tau \) and suppose \( x(\theta_L, t) = 1 \). Incentive compatibility of \( x \) implies that, for any stopping rule \( \sigma \), we must have
\[
\mathbb{E} \left[ e^{-r(\bar{\tau}_x-t)} \left( \bar{\theta}_{\bar{\tau}_x} - \varphi(\bar{\tau}_x) \right) | \bar{\theta}_t = \theta_H \right] \geq \theta_H - p(t)
\]
\[= u_t^{Q_H}(\theta_L; x) + \theta_H - \theta_L \]
\[\geq \mathbb{E} \left[ e^{-r(\bar{\tau}_\sigma-t)} \left( \bar{\theta}_{\bar{\tau}_\sigma} - \varphi(\bar{\tau}_\sigma) \right) | \bar{\theta}_t = \theta_H \right] + \theta_H - \theta_L. \quad (16)\]

Suppose that \( (\bar{\psi}_s)_{s \geq t} \) is determined independently and identically to \( (\bar{\theta}_s)_{s \geq t} \). Then (16) and independence of the two stochastic processes imply that
\[
\mathbb{E} \left[ e^{-r(\bar{\tau}_x(\bar{\theta}[r,\infty]))-t)} \left( \bar{\theta}_{\bar{\tau}_x(\bar{\theta}[r,\infty))} - \bar{\psi}_{\bar{\tau}_x(\bar{\theta}[r,\infty))} \right) | \bar{\theta}_t = \theta_H, \bar{\psi}_t = \theta_L \right] \geq \theta_H - \theta_L.
\]
This is possible only if \( x(\theta_H, t) = 1 \).

Part (ii). Consider any interval indexed \( k \in K \), and let \( t \in I_k \). Consider the buyer’s feasible strategy of purchasing if and only if his value turns high, unless he has not purchased by date \( \bar{t}_k \) (if finite), at which point he purchases regardless of his value. Since the buyer must earn at least the same expected payoff by purchasing immediately,
\[
u_t^{Q_H}(\theta_L, x) \geq e^{-(r+\alpha_L)(\bar{t}_k-t)} u_{\bar{t}_k}^{Q_H}(\theta_L, x) + \int_t^{\bar{t}_k} \alpha_L e^{-(r+\alpha_L)(s-t)} u_s^{Q_H}(\theta_H, x) ds
\]
\[= e^{-(r+\alpha_L)(\bar{t}_k-t)} u_{\bar{t}_k}^{Q_H}(\theta_L, x) + \int_t^{\bar{t}_k} \alpha_L e^{-(r+\alpha_L)(s-t)} (u_s^{Q_H}(\theta_L, x) + \theta_H - \theta_L) ds.
\]
The smallest possible expected payoff consistent with incentive compatibility, call it \( u_t^{\min} \), is therefore the one that satisfies the equality
\[
u_t^{\min} = e^{-(r+\alpha_L)(\bar{t}_k-t)} u_{\bar{t}_k}^{\min} + \int_t^{\bar{t}_k} \alpha_L e^{-(r+\alpha_L)(s-t)} (u_s^{\min} + \theta_H - \theta_L) ds
\]
for all \( t \in I_k \), with \( u_{\bar{t}_k}^{\min} = u_{\bar{t}_k}^{Q_H}(\theta_L, x) \). Taking derivatives with respect to \( t \), we obtain the first-order differential equation
\[
r u_t^{\min} = \frac{du_t^{\min}}{dt} + \alpha_L (\theta_H - \theta_L),
\]
which, given the terminal condition, has a unique solution

\[ u_t^{\min} = e^{-r(i_k - t)} u_{i_k}^{\Omega_P} (\theta_L, x) + \left( 1 - e^{-r(i_k - t)} \right) \frac{\alpha_L (\theta_H - \theta_L)}{r}. \]

This establishes the result.

**Part (iii).** Let \( t \in \mathbb{R}_+ \backslash A \) and suppose \( t \in A \) is the next time of sale to the buyer when his value is low. Note that

\[ \Pr \left( \theta_t = \theta_H | \theta_t = \theta_L \right) = \frac{\alpha_L}{\alpha_L + \alpha_H} \left( 1 - e^{-(\alpha_L + \alpha_H)(t-t)} \right) \]

and that

\[ \Pr \left( \theta_t = \theta_H | \theta_t = \theta_H \right) = \frac{\alpha_L}{\alpha_L + \alpha_H} + \frac{\alpha_H}{\alpha_L + \alpha_H} - e^{-(\alpha_L + \alpha_H)(t-t)}. \]

The result then follows by considering the strategy of not purchasing until date \( t \) and observing that \( u_t^{\Omega_P} (\theta_H; x) = u_t^{\Omega_P} (\theta_L; x) + \theta_H - \theta_L. \]

**Proof of Proposition 1.** As explained in the text, \( (p_A, x_A) \) maximizes total expected revenue subject to (i)-(iii) of Lemma 1, and \( x_A \) satisfies \( x_A (\theta_L, t) = 1 \) iff \( t \in A \). Suppose that \( u_{A,t} \) is as defined in the text. It is enough to verify \( x_A \) is indeed an optimal stopping rule for the buyer given \( p_A \), and that the associated payoffs are given by \( u_{A,t} \). The argument is standard. For any stopping rule \( \sigma \),

\[
\mathbb{E}_t \left[ e^{-r(\tilde{\tau}_\sigma - t)} \left( \tilde{\theta}_{\tilde{\tau}_\sigma} - p_A (\tilde{\tau}_\sigma) \right) | \theta_t \right] \\
\leq \mathbb{E}_t \left[ e^{-r(\tilde{\tau}_\sigma - t)} u_{A,\tilde{\tau}_\sigma} (\tilde{\theta}_{\tilde{\tau}_\sigma}) | \theta_t \right] \\
\leq u_{A,t} (\theta_t) \\
+ \mathbb{E}_t \int_{\tilde{\tau}_\sigma}^{\tau_\sigma} e^{-r(s-t)} \left( \begin{array}{l} 1 \left( \tilde{\theta}_s = \theta_L \right) \left( \begin{array}{l} -r u_{A,s} (\theta_L) + \frac{\partial u_{A,s}(\theta_L)}{\partial s} \\ + \alpha_L \left( u_{A,s} (\theta_H) - u_{A,s} (\theta_L) \right) \end{array} \right) \\ + 1 \left( \tilde{\theta}_s = \theta_H \right) \left( \begin{array}{l} -r u_{A,s} (\theta_H) + \frac{\partial u_{A,s}(\theta_H)}{\partial s} \\ + \alpha_H \left( u_{A,s} (\theta_L) - u_{A,s} (\theta_H) \right) \end{array} \right) \end{array} \right) ds | \theta_t \right] \\
\leq u_{A,t} (\theta_t).
\]

The first inequality follows by choice of the functions \( p_A (t) \) and \( u_{A,t} \). The second inequality follows by applying the Dynkin formula and the fact that \( u_{A,t} \) contains only downward discontinuities. The
third follows because, as is readily checked,

\[-ru_{A,t} (\theta_L) + \frac{\partial u_{A,t} (\theta_L)}{\partial t} + \alpha_L (u_{A,t} (\theta_H) - u_{A,t} (\theta_L)) = 0\]

and

\[-ru_{A,t} (\theta_H) + \frac{\partial u_{A,t} (\theta_H)}{\partial t} + \alpha_H (u_{A,t} (\theta_L) - u_{A,t} (\theta_H)) = 0\]

for all \(t\), except at the countable number of points \(t_k\) and \(\bar{t}_k\) with \(k \in K\) where the derivative \(\frac{du_{A,t}(\theta_H)}{dt}\) does not exist. When \(\sigma = x_A\), all inequalities hold with equality (the second because stopping is either immediate or is before any discontinuity in \(u_A\)). Thus \(u_{A,t}\) is the value function associated with the buyer’s problem and this payoff, which is the one associated with \(x_A\), exceeds the payoff attainable with any other stopping rule \(\sigma\). 

**Proof of Proposition 2. Part (i).** Since the date of arrival is \(t = 0\), no more sales will be made after any date \(t^*\) at which the buyer purchases with a low value. Given the sales policy \(A = \{t^*\}\), and the price path given in the text (i.e., \(p(t)\) given by (6) for \(t \leq t^*\), and \(p(t) = \theta_H\) otherwise), expected revenue is

\[R(t^*) = \gamma p(0) + (1 - \gamma) \left( e^{-rt^*} p(t^*) + \int_0^{t^*} \alpha_L e^{-\alpha Lt} \left( e^{-r\bar{t}} p(\bar{t}) - e^{-rt^*} p(t^*) \right) d\bar{t} \right), \]

which is equal to the quantity (4). The expected revenue from setting \(A = \emptyset\) and \(p(t) = \theta_H\) for all \(t\) is \((\gamma + (1 - \gamma) \frac{\alpha_L}{\alpha_L + r}) \theta_H\).

**Part (ii).** Note that

\[R(t^*) = \gamma \theta_H + (1 - \gamma) \frac{\alpha_L \theta_H}{\alpha_L + r} + e^{-rt^*} \left( e^{-\alpha L t^*} \left( \frac{1 - \gamma}{\alpha_L + \alpha_H} \left( \theta_L - \frac{\alpha_L \theta_H}{\alpha_L + r} \right) \right) \right). \tag{17} \]

Under the assumption \(\gamma \leq \frac{\alpha_L}{\alpha_L + \alpha_H}\), it is then easy to see that the final term is strictly decreasing whenever it is non-negative. Therefore, if it is optimal to induce the buyer to purchase with a low value at any time, it is optimal to do so at date zero.
Part (iii). The inequality (5) simply states that inducing purchase by the buyer with a low value at time zero yields a greater revenue than by setting a constant price $\theta_H$ at which the buyer only purchases if his value is high. The result is therefore immediate.

Proof of Proposition 3. The expression for total revenue (7) is calculated given the optimal price path of Proposition 1. The optimal choice of $\bar{t}$ is immediate from the first-order condition.

Proof of Proposition 4. Part (i). Suppose with a view to contradiction that $A = [0, \bar{t}]$ is an optimal sales policy, where $\bar{t} > 0$. Consider an alternative policy $[0, t#] \cup \{\bar{t}\}$ for $t# \in [0, \bar{t}]$. In order for such a policy to be optimal, it must be that the profit-maximizing choice of $t#$ be $t# = \bar{t}$.

We show that this is never the case.

For $\xi > 0$, using a Taylor expansion about $\xi = 0$, the buyer’s ex-ante expected total rents are greater for $t# = \bar{t}$ than for $t# = \bar{t} - \xi$ by the amount

$$e^{-r(\bar{t} - \xi)} \left( 1 - e^{-\lambda(\bar{t} - \xi)} \right) \left( 1 - e^{-r\xi} \right) \frac{\alpha_L (\theta_H - \theta_L)}{r} - e^{-r\xi} (\theta_H - \theta_L) \frac{\alpha_L (1 - e^{-(\alpha_L + \alpha_H)\xi})}{\alpha_L + \alpha_H}$$

$$= \frac{1}{2} \xi^2 e^{-r\bar{t}} \left( 1 - e^{-\lambda\bar{t}} \right) \alpha_L \left( r + \alpha_L + \alpha_H \right) (\theta_H - \theta_L) + o(\xi^2),$$

where, as is standard, $o(\xi^2)$ represents higher order terms. Similarly, the difference in the revenue earned between $\bar{t} - \xi$ and $\bar{t}$, as evaluated at time $\bar{t} - \xi$, is

$$e^{-(r+\lambda)(\bar{t} - \xi)} (\phi(\xi) - \pi(\xi)) = \frac{1}{2} \xi^2 e^{-(r+\lambda)\bar{t}} \alpha_L \left( \theta_L (1 - \gamma) - (\theta_H - \theta_L) \left( \frac{\alpha_L}{r} + \gamma + \frac{\alpha_H \gamma}{r} \right) \right) + o(\xi^2).$$

From (7), the first-order condition for the optimality of $\bar{t}$ is

$$e^{-\lambda\bar{t}} \alpha_L \left( (1 - \gamma) \theta_L - \left( \frac{\alpha_L}{r} + \gamma \right) (\theta_H - \theta_L) \right) - \left( 1 - e^{-\lambda\bar{t}} \right) \alpha_L \left( \alpha_L + r \right) (\theta_H - \theta_L) = 0.$$ 

Thus the difference in expected revenue between the two policies is

$$\frac{1}{2} \xi^2 e^{-r\bar{t}} \left( e^{-\lambda\bar{t}} \alpha_L \left( (1 - \gamma) \theta_L - \left( \frac{\alpha_L}{r} + \gamma \right) (\theta_H - \theta_L) \right) - \left( 1 - e^{-\lambda\bar{t}} \right) \alpha_L \left( \alpha_L + r \right) (\theta_H - \theta_L) \right) + o(\xi^2)$$

$$= -\frac{1}{2} \xi^2 e^{-r\bar{t}} \alpha_H (\theta_H - \theta_L) \alpha_L (1 - e^{-\lambda\bar{t}}) + o(\xi^2),$$

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which is negative provided $\xi$ is sufficiently small.

**Part (ii).** Suppose with a view to contradiction that $A$ is an optimal policy with $[t_k, \tilde{t}_k] \subset A$ an interval during which a low-value buyer is induced to purchase at each instant and for which $\tilde{t}_k > T$. As in the proof of Part (ii), we may consider a modified policy $A \setminus (\tilde{t}_k - \xi, \tilde{t}_k)$ where $\xi \in (0, \tilde{t}_k - t_k)$. Using the same calculation as for Part (ii), the difference in expected revenue between the two policies is

$$
\frac{1}{2} \xi^2 e^{-r\tilde{t}_k} \left( 1 - e^{-\lambda \tilde{t}_k} \right) \frac{\alpha_L (\alpha_L + \alpha_H + r) (\theta_H - \theta_L)}{\alpha_L \gamma + \alpha_H \gamma} + o(\xi^2).
$$

By choice of $T$, this is positive whenever $\xi$ is sufficiently small. $\blacksquare$

**Proof of Proposition 5.** Suppose with a view to contradiction that $\alpha_H = 0$ but that there is an optimal sales policy $A$ which includes dates $t$ and $t + z$, but no dates in $(t, t + z)$. We will show that there is an alternative policy $A \cup \{t + z - \xi, t + z\}$ which, for $\xi$ sufficiently small, yields a higher expected revenue. The change in expected revenue from adopting the modified policy is

$$
\Delta(\xi) = e^{-(\lambda + r)(t + z - \xi)} \left( \phi(\xi) - \pi(\xi) \right) + e^{-\lambda t - r(t + z - \xi)} \int_{t}^{t + z - \xi} \lambda e^{-\lambda \tau - r} \left( \exp \left( \frac{\alpha_L \theta_H - \theta_L}{r} \right) \frac{1 - e^{-\lambda \tau}}{1 - e^{-\lambda(z - \xi)}} \right) d\tau 
$$

$$
- e^{-\lambda t - r(t + z - \xi)} \left( 1 - e^{-r\xi} \right) \frac{\alpha_L \theta_H - \theta_L}{r} \left( 1 - e^{-\lambda(z - \xi)} \right) 
$$

$$
- e^{-\lambda t - r(t + z)} \int_{t}^{t + z} \lambda e^{-\lambda \tau - r} \left( \exp \left( \frac{\alpha_L \theta_H - \theta_L}{r} \right) \frac{1 - e^{-\lambda \tau}}{1 - e^{-\lambda(z - \xi)}} \right) d\tau 
$$

$$
- \left( 1 - e^{-\lambda t} \right) \left( \exp \left( \frac{\alpha_L \theta_H - \theta_L}{r} \right) \frac{1 - e^{-r(t + z - \xi)}}{1 - e^{-r(z - \xi)}} \right) \frac{\alpha_L \theta_H - \theta_L}{r} \left( 1 - e^{-\lambda(t + z - \xi)} \right) 
$$

$$
- e^{-r(t + z)} \left( \theta_H - \theta_L \right) \frac{\alpha_L \left( 1 - e^{-\lambda(z + \alpha_H)(t - z)} \right)}{\alpha_L + \alpha_H} 
$$

The first term is the change in revenue earned in case the buyer arrives in $[t + z - \xi, t + z]$. The second, third and fourth terms together are the change in expected revenue in case the buyer arrives between $t$ and $t + z - \xi$, in particular taking into account that, under the modified policy, the buyer
then purchases either once his value is high, or upon reaching date \( t + z - \xi \). The final term is the change in expected rents associated with arrival before \( t \).

Note that

\[
\Delta'(0) = r e^{-\lambda t - r(t+z-\xi)} \int_t^{t+z} \lambda e^{-\lambda(\tau-t)} \left( e^{-\alpha_L(t+z-\tau)} \left( \begin{array}{c} (1-\gamma) \left( \theta_L - \frac{\alpha_L \theta_H}{\alpha_L + \alpha_H} \right) \\ \frac{-e^{-\alpha_H(t+z-\tau)} (\theta_H - \theta_L)}{\alpha_H} \end{array} \right) \right) d\tau \\
+ e^{-\lambda(t+z)} \int_t^{t+z} \lambda e^{-\lambda(\tau-t)} \left( \begin{array}{c} (1-\gamma) \left( \theta_L - \frac{\alpha_L \theta_H}{\alpha_L + \alpha_H} \right) \\ -\alpha_H e^{-(\alpha_L+\alpha_H)(t+z-\tau)} \left( \theta_H - \theta_L \right) \left( \gamma \alpha_H - (1-\gamma) \alpha_L \right) \end{array} \right) d\tau \\
- re^{-\lambda(t+z)} \frac{\alpha_L (\theta_H - \theta_L)}{r} \left( 1 - e^{-\lambda z} \right) \\
- e^{-r(t+z)} \left( \alpha_L + \alpha_H + r \right) \left( \theta_H - \theta_L \right) \frac{\alpha_L \left( 1 - e^{-(\alpha_L+\alpha_H)z} \right)}{\alpha_L + \alpha_H}.
\]

Assuming \( \alpha_H = 0 \),

\[
\Delta'(0) = (\alpha_L + r) \left( e^{-\lambda t - r(t+z)} \int_t^{t+z} \lambda e^{-\lambda(\tau-t)} \left( e^{-\alpha_L(t+z-\tau)} \left( (1-\gamma) \left( \theta_L - \frac{\alpha_L \theta_H}{\alpha_L + \alpha_H} \right) \right) \right) d\tau \\
- e^{-r(t+z)} \left( 1 - e^{-\lambda z} \right) \left( 1 - e^{-\alpha_L(t+z-\tau)} \right) \left( \theta_H - \theta_L \right) \right).
\]

Consider the dynamic program specified by (10) and note that \( W^*(t + z) \), is non-increasing in \( z \), and that \( e^{-r z} W^*(t + z) \) is strictly decreasing. Since \( A \) is an optimal policy, \( \bar{t} = t \) and \( \hat{t} = t + z \) must be optimal in the problem of (10) at \( t \). This implies that \( \Delta'(0) \) is positive and therefore that the change in policy leads to an improvement in expected revenue provided \( \xi \) is sufficiently small.

\section*{Appendix B: Proofs of results on the fully-optimal mechanism}

This Appendix provides proofs of the results on the fully-optimal mechanism. For the first result, Lemma 2, we use the fact that, since the seller can fully commit, the revelation principle applies in this environment. We therefore need only consider incentive-compatible direct mechanisms. Such a
mechanism $\Omega_D$ is a collection of measurable functions, for each date of contracting $\tau$, 
$y_{\tau} = (y_{\tau,t})_{t \geq \tau}$ and 
$q_{\tau} = (q_{\tau,t})_{t \geq \tau}$. Each $y_{\tau,t}$ specifies, for each feasible history of reports 
$\theta[\tau,t]$, 
$y_{\tau,t} \left( \theta[\tau,t] \right) = 1$ if the buyer, having not yet received the good, receives it at date $t$, and 
$y_{\tau,t} \left( \theta[\tau,t] \right) = 0$ otherwise. Each $q_{\tau,t}$ specifies, for each feasible history of reports 
$\theta[\tau,t]$, the payment made by the agent, 
$q_{\tau,t} \left( \theta[\tau,t] \right) = m_{\tau,t} \left( \theta[\tau,t] \right) + n_{\tau,t} \left( \theta[\tau,t] \right) \kappa_{\tau,t} \left( \theta[\tau,t] \right)$, where 
$\kappa_{\tau,t} (\cdot)$ is the Dirac delta function indicating whether a positive payment is given at date $t$, 
and $n_{\tau,t} (\cdot)$ specifies the size of any positive payment and $m_{\tau,t} (\cdot)$ is the flow rate of payment.

The buyer’s problem is to choose an optimal reporting strategy. A reporting strategy for the buyer, 
given arrival at $\tau$, is a collection of functions $(\zeta_{\tau}, \iota_{\tau}) = (\zeta_{\tau \tau}, (\iota_{\tau,t})_{t \geq \tau})$ where 
$\zeta_{\tau} (\theta[t,\infty]) \in [\tau, \infty]$ specifies for each history $\theta[t,\infty]$ the time at which the buyer participates 
(i.e., reveals himself by signing the contract; $\zeta_{\tau} (\theta[t,\infty]) = \infty$ if he never does so) and 
$\iota_{\tau,t} (\theta[t,d]) \in \{\theta_L, \theta_H\}$ specifies the report of value at each date $t \geq \tau$ at and after participation. 
For each $\tau$, the function, $\zeta_{\tau} (\cdot)$ defines a stopping time. With a slight abuse of notation, for any left-closed interval $E$, 
we denote a continuation history of reports from $\min E$ by $\iota_{\tau}^E (\theta^E) = (\iota_{\tau,s} (\theta[\min E, s]))_{s \in E}$. 
(in general, when referring to an arbitrary continuation strategy, we will often omit the date of contracting $\tau$). 
As described in Zhang (2009), we can restrict attention to reports of values that are feasible in the sense that 
the histories of reports satisfy the same properties as the actual histories of values, i.e. that they exhibit right continuity and finitely many jumps over bounded intervals.

For any $t \geq \tau$, let $h[\tau,t]$ describe a feasible history of reports from $\tau$ up to, but not including, $t$. 
We denote collections of histories using concatenations; for example, $h[\tau,t]h[t,\infty]$ denotes the infinite history 
from $\tau$. For any feasible history of reports $h[\tau,\infty]$, let $\gamma_{y_{\tau}} (h[\tau,\infty]) = \inf \{ s \geq \tau : y_{\tau,s} (h[\tau,s]) = 1 \}$, so that $\gamma_{y_{\tau}} (\cdot)$ defines a stopping time.

The buyer’s continuation payoff at date $t$, given that he contracted at date $\hat{\tau} \leq t$, depends only 
on the contract he signed at date $\hat{\tau}$ (as given by $(y_{\hat{\tau}}, q_{\hat{\tau}})$), his value $\theta_{\hat{\tau}}$ at date $t$ (and, since the process is Markov, not on earlier values), 
the history of reports $h[\hat{\tau},t]$, and his continuation reporting
strategy \( v_{t,\infty} \). In particular,
\[
    u^\Omega_t \left( \theta_t; h_{t,\infty}, v_{t,\infty} \right) = \mathbb{E} \left[ e^{-r(\gamma_{y_{t,\infty}}(h_{t,\infty}, t_{t,\infty}, \hat{\theta}_{t,\infty}) - t)} \hat{\theta}_{y_{t,\infty}}(h_{t,y_{t,\infty}}(\theta_{[t,\infty]}(\hat{\theta}_{t,\infty}))) - \int_t^\infty e^{-r(s-t)} q_{t,s} (h_{t,s} L_{t,s} (\hat{\theta}_{t,s})) ds \mid \theta_t \right].
\]

The value function for the reporting problem at date \( t \geq \hat{t} \) after he has contracted with the seller is
\[
    w^\Omega_t \left( \theta_t; h_{[t,\infty]} \right) = \sup_{\theta_t \mid [t,\infty]} u^\Omega_t \left( \theta_t; h_{[t,\infty]} \right).
\]

The mechanism \( \Omega_D \) induces incentive-compatible reporting conditional on contracting at date \( \hat{t} \) if and only if, for all \( t \), all feasible histories \( h_{t,\infty} \), and each \( \theta_t \in \{ \theta_L, \theta_H \} \), \( w^\Omega_t \left( \theta_t; h_{[t,\infty]} \right) = w^\Omega_t \left( \theta_t; h_{[t,\infty]} \right) \), where \( h_{[t,\infty]} \) is the continuation strategy that specifies truthful reporting at every instant.

The following result provides necessary conditions for the incentive compatibility of truthful reporting once the buyer has revealed himself by contracting with the seller. To state it, we introduce the notation \( \theta_L[t,s] \) to mean the history which includes only \( \theta_L \) between date \( t \) and date \( s \).

**Lemma 4** Suppose \( \Omega_D \) is an incentive-compatible mechanism. Let \( t, \tau \geq 0 \), with \( t \geq \tau \), and let \( h_{[\tau,\infty]} \) be any history.

(i) \( y_{t,\tau} \left( h_{[\tau,\infty]} \right) = 1 \) implies \( y_{t,\tau} \left( h_{[\tau,\infty]} \theta_H \right) = 1 \).

(ii) Let \( t^* = \inf \left\{ s : y_{t} \left( h_{[\tau,\infty]} \theta_L[t,s] \right) = 1 \right\} \). If \( t^* \in [t,\infty) \), then
\[
    w^\Omega_t \left( \theta_L; h_{[\tau,\infty]} \right) \geq e^{-r(t^* - t)} \left( w^\Omega_t \left( \theta_L; h_{[\tau,\infty]} \theta_L[t,t^*] \right) + (\theta_H - \theta_L) \frac{\alpha_L \left( 1 - e^{-(\alpha_L + \alpha_H)(t^* - t)} \right)}{\alpha_L + \alpha_H} \right) - \int_{[t,t^*]} e^{-r(s-t)} q_{t,s} \left( h_{[\tau,\infty]} \theta_L[t,s] \right) ds,
\]
and
\[
    w^\Omega_t \left( \theta_H; h_{[\tau,\infty]} \right) \geq w^\Omega_t \left( \theta_L; h_{[\tau,\infty]} \right) + e^{-(r+\alpha_L+\alpha_H)(t^* - t)} (\theta_H - \theta_L).
\]

If \( t^* = \infty \), then \( w^\Omega_t \left( \theta_L; h_{[\tau,\infty]} \right) \geq - \int_{[t,\infty)} e^{-r(s-t)} q_{t,s} \left( h_{[\tau,\infty]} \theta_L[t,s] \right) ds \) and \( w^\Omega_t \left( \theta_H; h_{[\tau,\infty]} \right) \geq w^\Omega_t \left( \theta_L; h_{[\tau,\infty]} \right). \)

**Proof.** Part (i). Let \( t \geq \tau \), with \( h_{[\tau,\infty]} \) a history of reports, and suppose that \( y_{t,\tau} \left( h_{[\tau,\infty]} \theta_L \right) = 1 \).

Then, by definition of the value function \( w^\Omega_t \) and assumption of incentive compatibility, for any
continuation reporting strategy \( \bar{\theta} \),
\[
\mathbb{E}\left[-r(\gamma_{\tau}(h^{[\tau,\infty]}(t^{[\tau,\infty]})) - t) \bar{\theta}_{\gamma_{\tau}(h^{[\tau,\infty]}(t^{[\tau,\infty]}))} - \int_{t}^{\infty} e^{-r(s-t)} q_{\tau,s}(h^{[\tau,\infty]}(t^{[\tau,\infty]})) ds \mid \theta_H \right]
\]
\[= \omega_{t \Delta} \left( \theta_H; h^{[\tau,\infty]}(t^{[\tau,\infty]}) \right)
\]
\[\geq \theta_H - \theta_L - \left( \int_{t}^{\infty} e^{-r(s-t)} q_{\tau,s}(h^{[\tau,\infty]}) ds \mid \theta_L \right)
\]
\[= \omega_{t \Delta} \left( \theta_L; h^{[\tau,\infty]}(t^{[\tau,\infty]}) \right) + \theta_H - \theta_L.
\]

Using the same reasoning as in Part (i) of Lemma 1, this is only possible in case \( y_{\tau,t} (h^{[\tau,\infty]}(t^{[\tau,\infty]})) = 1 \).

**Part (ii).** As discussed in the text, this follows because the buyer can necessarily achieve at least the payoff obtained from reporting \( \theta_L \) from period \( t \) onwards until date \( t^* \) at which he receives the good and has continuation payoff, given an optimal continuation reporting strategy, of \( \omega_{t \Delta} \left( \theta_L; h^{[\tau,\infty]}(t^{[\tau,\infty]}) \right) \).

Lemma 4 may be viewed as the analog of Lemma 1 for unrestricted mechanisms. Part (i) states that awarding the good exclusively to a low value following a given history is inconsistent with incentive compatibility. Part (ii) provides a lower bound on payoffs after any history such that the buyer has not yet received the good. This lower bound is the payoff that would be obtained by reporting a sequence of low values until he receives the good (if ever).

Suppose the mechanism must ensure the buyer expects to earn a payoff at least \( w_{\tau} \geq 0 \) when contracting at date \( \tau \) when his value is low. Part of the usefulness of Lemma 4 is that it provides a lower bound on the rent a buyer expects to earn at date \( \tau \) also when his value is high. For each date \( \tau \), consistent with the definition in Lemma 2, let \( t_{\Delta}(\tau) = \inf \left\{ s : y_{\tau,s} \left( \theta_L^{[\tau,s]} \right) = 1 \right\} \). In particular, if it is

\[\text{To be precise, as in Part (i) of Lemma 1, we may suppose that } \left( \tilde{\psi}_{s} \right)_{s \geq t} \text{ is determined independently and identically to } \left( \bar{\theta}_{s} \right)_{s \geq t}. \text{ Then (16) and independence of the two stochastic processes imply that }
\]
\[
\mathbb{E}\left[-r(\gamma_{\tau}(h^{[\tau,\infty]}(t^{[\tau,\infty]}))) \bar{\theta}_{\gamma_{\tau}(h^{[\tau,\infty]}(t^{[\tau,\infty]}))} - \int_{t}^{\infty} e^{-r(s-t)} q_{\tau,s}(h^{[\tau,\infty]}(t^{[\tau,\infty]})) ds \mid \theta_H \right]
\]
\[= \omega_{t \Delta} \left( \theta_H; h^{[\tau,\infty]}(t^{[\tau,\infty]}) \right)
\]
\[\geq \theta_H - \theta_L - \left( \int_{t}^{\infty} e^{-r(s-t)} q_{\tau,s}(h^{[\tau,\infty]}) ds \mid \theta_L \right)
\]
\[= \omega_{t \Delta} \left( \theta_L; h^{[\tau,\infty]}(t^{[\tau,\infty]}) \right) + \theta_H - \theta_L.
\]

which is clearly only possible if \( x_{\tau,t} \left( h^{[\tau,\infty]}(t^{[\tau,\infty]}), \theta_H \right) = 1 \).
required that \( w^\Omega_D (\theta_L; \emptyset) \geq w^\tau \), then necessarily \( w^\Omega_D (\theta_H; \emptyset) \geq w^\tau + e^{-(r+\alpha_L+\alpha_H)(t_L(\tau)-\tau)} (\theta_H - \theta_L) \) (in case \( t_L(\tau) = \infty \), the latter is simply \( w^\tau \)). This allows us to prove Lemma 2.

**Proof of Lemma 2.** The proof involves verifying that the buyer arriving at date \( \tau \) finds it optimal to act in accordance with the proposed mechanism, that, by doing so, he obtains a payoff equal to the lower bounds specified above, and that, given the buyer receives the good at date \( t_L(\tau) \) when his value remains low, the allocation rule implemented is as efficient as possible.

The allocation rule being as efficient as possible requires the buyer to receive the good (almost surely) as soon as his value turns high. In terms of the direct mechanism, it is enough to put \( y^\#_{\tau,t} (\theta^\tau_L) = 0 \) for all \( t \in [\tau, t_L(\tau)) \), \( y^\#_{\tau,t_L(\tau)} (\theta^\tau_L) = 1 \), and \( y^\#_{\tau,t} (\theta^\tau_L, \theta_H) = 1 \) for all \( t \in [\tau, t_L(\tau)) \). This is precisely the allocation rule specified in the lemma.

That the buyer receives payoffs equal to \( w^\tau \) and \( w^\tau + e^{-(r+\alpha_L+\alpha_H)(t_L(\tau)-\tau)} (\theta_H - \theta_L) \) is easily verified. Indeed, as in Subsection 3.1, the price path is chosen so that, whenever the buyer’s value is high, he expects the same payoff by purchasing at that date as from waiting until date \( t_L(\tau) \). Since \( p^\#_{\tau,t} (t_L(\tau); t_L(\tau)) = \theta_L \), the payoff he obtains at \( t_L(\tau) \) is zero if his value is low at that moment, and \( \theta_H - \theta_L \) if it is high. If his value is low at date \( \tau \), he expects it to become high and thus to capture the associated rent with probability \( \frac{\alpha_L (1-e^{-(\alpha_L+\alpha_H)(t_L(\tau)-\tau)})}{\alpha_L+\alpha_H} \). The choice of fixed fee therefore ensures an expected payoff of \( w^\tau \). If the buyer’s value is high rather than low at \( \tau \), the probability it is also high at \( t_L(\tau) \) is higher by \( e^{-(\alpha_L+\alpha_H)(t_L(\tau)-\tau)} (\theta_H - \theta_L) \). The buyer’s expected payoff is therefore equal to \( w^\tau + e^{-(r+\alpha_L+\alpha_H)(t_L(\tau)-\tau)} (\theta_H - \theta_L) \).

Finally, consider whether the buyer finds it incentive compatible to act in the prescribed manner. Suppose first that the buyer has signed the contract at date \( \tau \) and thus committed himself to the up-front fee. As in Subsection 3.1, the price path is chosen to make the buyer indifferent between purchasing and waiting when his value is low at \( t_L(\tau) \) — he strictly prefers to purchase at that date if his value is high. It is also chosen to ensure that, at dates in \([\tau, t_L(\tau))\), he is indifferent between purchasing immediately and waiting when his value is high, and strictly prefers to wait rather than purchase when his value is low.

Now, suppose the buyer is willing to sign the contract on arrival at date \( \tau \) when his value is
high. From signing the contract when his value is instead low, he expects a payoff $w_\tau$. Consider the alternative strategy of waiting to sign a later contract. The payoffs associated with any later contract, say at date $s > \tau$, are, by construction, no greater than were available under the original mechanism for a buyer arriving at date $s$. It follows that if $w_\tau$ is the greatest attainable payoff under the original mechanism, it must also be under the proposed alternative. \[ \square \]

**Proof of Lemma 3.** Since the mechanism solves the relaxed program described in the text, it is enough to establish that it remains valid once all relevant constraints are considered. Thus, we need consider only Part (i) of the proposition. Suppose the buyer arrives at date $\tau$ with value $\theta_\tau$ and employs an arbitrary participation strategy $\zeta_\tau$. Using Dynkin’s formula, his expected payoff in the mechanism \( (f_\tau^\#(t_L(\tau), w_\tau^\min(\theta_L; t_L)), p_\tau^\#(\cdot; t_L(\tau))) \) $= \tau$ is then

\[
\mathbb{E}_\tau \left[ e^{-r(\zeta_\tau(\bar{\theta}_r, \infty) - \tau)} w_\tau^\min(\theta_\tau) \zeta_\tau(\bar{\theta}_r, \infty) \left( \theta_\tau - \zeta_\tau(\bar{\theta}_r, \infty) \right) \right] = w_\tau^\min(\theta_\tau) + \mathbb{E}_\tau \int_\tau^{\zeta_\tau(\bar{\theta}_r, \infty)} e^{-r(s - \tau)} \left( \begin{array}{c} 1(\bar{\theta}_s = \theta_L) \left( -rw_\tau^\min(\theta_L) + \frac{\partial w_\tau^\min(\theta_L)}{\partial s} \right) \\ + \alpha_L \left( w_\tau^\min(\theta_H; \theta) - w_\tau^\min(\theta_L) \right) \\ + \alpha_H \left( w_\tau^\min(\theta_L) - w_\tau^\min(\theta_H) \right) \end{array} \right) ds. \]

Since $w_\tau^\min(\theta_L; t_L)$ solves (14), the first term of the integrand is zero. Moreover, it is easily checked that, for each $\tau$,

\[
-rw_\tau^\min(\theta_H) + \frac{\partial w_\tau^\min(\theta_H)}{\partial \tau} + \alpha_H \left( w_\tau^\min(\theta_L; t_L) - w_\tau^\min(\theta_H) \right) = - (\alpha_L + \alpha_H + r) e^{-(r+\alpha_L+\alpha_H)(t_L(\tau)-\tau)} (\theta_H - \theta_L) t'_L(\tau), \]

which is non-positive provided that $t'_L(\cdot) \geq 0$. Therefore, the payoff expected from contracting immediately, $w_\tau^\min(\theta_\tau)$, exceeds that from any alternative strategy involving delay. \[ \square \]

**Proof of Proposition 6.** Using the expression for the buyer’s time-zero expected rent (15), the
seller’s expected total revenue is
\[
\int_0^\infty \left[ \lambda e^{-(r+\lambda)\tau} \left( \gamma \theta_H + (1 - \gamma) \left( \frac{\alpha_L}{r+\alpha_L} \left( 1 - e^{-(r+\alpha_L)(t_L(\tau)-\tau)} \right) \theta_H + e^{-(r+\alpha_L)(t_L(\tau)-\tau)} \theta_L \right) \right) - (\theta_H - \theta_L) \left( \alpha_L \left( 1 - e^{-\lambda\tau} \right) + \gamma \lambda e^{-\lambda\tau} e^{-rt_L(\tau)-(\alpha_L+\alpha_H)(t_L(\tau)-\tau)} \right) \right] d\tau.
\]

Maximizing pointwise under the integral yields the result.\(^{29}\)