Preference manipulations lead to the uniform rule *

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Abstract

In the division problem with single-peaked preferences, it is well known that the uniform rule is robust to strategic manipulation. Furthermore, under efficiency and symmetry, it is the unique strategy-proof rule (Sprumont, 1991; Ching, 1994). We conversely analyze the consequences of strategic manipulation for allocation rules. Given a rule, we interpret its associated direct revelation game as a manipulation game, and we characterize its equilibrium allocations. For each rule in a wide class of rules, the uniform allocation (i) is the unique strong Nash equilibrium allocation and the unique Pareto-efficient Nash equilibrium allocation, and (ii) is the unique Nash equilibrium allocation under an additional strict monotonicity condition. Thus, attempts to manipulate any such rule lead to the uniform allocation. A by-product of our results is the identification of a large class of direct mechanisms that doubly implement the uniform rule in Nash and strong Nash equilibria.

Keywords: Uniform rule, Manipulation of preferences, (Double) Implementation, Direct revelation mechanisms, Manipulation games, Existence of a strong Nash equilibrium, Fair allocation.

JEL codes: C72, D63, D61, C78, D71.

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1 Introduction

We study the problem of fairly allocating an amount of a divisible resource to a group of agents when preferences are single-peaked. For instance, suppose a group of agents have agreed to share working hours to complete a task, and a fixed wage per unit of labor prevails. Given this fixed-price procedure, standard assumptions entail that agents have single-peaked preferences over how much labor they are willing to supply for the task. Another instance which typically involves single-peaked preferences is the division without free disposal of a social endowment.

In both examples, the possible imbalance between demand and supply implies that some allocation rule must be chosen.[1] This type of problem is better known in the mechanism design literature as the fair division problem under single-peaked preferences (Sprumont, 1991). In the catalog of appealing rules, a particular rule stands out, the so-called *uniform rule* (Benassy, 1982).[2] The uniform rule retains the flavor of equal division while meeting the requirement of efficiency: each agent receives either his peak, or a common share in such a way that the stock of resource available is fully distributed. This rule is efficient and symmetric, like many other standard rules –e.g., the *proportional rule*. However, an important feature of the uniform rule is its robustness to strategic manipulations. Indeed, it is not only strategy-proof, but also group strategy-proof. Furthermore, under efficiency and symmetry, the uniform rule is the unique strategy-proof rule (Sprumont, 1991; Ching, 1994).

While strategy-proofness is an important requirement of strategic robustness, its violation does not give any insight regarding the consequence of possible strategic manipulations. To analyze this issue, we investigate the consequence of manipulation of allocation rules. For that purpose, we characterize (Nash-)equilibrium allocations in the direct revelation game associated with a given rule. We call such a game, a *manipulation game*.[3]

Our results cover a wide class of efficient and symmetric rules. We denote by Ψ^1 the class of efficient and symmetric rules satisfying certain monotonicity and continuity conditions. It contains the uniform rule, the constrained equal distance rule,

[1] Henceforth allocation rule, or simply rule.

[2] See, for example, Ching (1994), Thomson (1994a,b, 1995, 1997), among many others. Thomson (2005, Ch.11) offers a survey of the literature.

[3] Various authors analyze manipulation games in different economic models. However, not so much effort has been carried out to this approach, in contrast with the "canonical" approach that seeks implementable rules or mechanisms implementing rules. Nash-style equilibrium in manipulation games is first studied by Hurwicz (1978, 1979) in economies with divisible goods. Earlier literature includes Sobel (1981), Otani and Sicilian (1982), and Thomson (1984, 1987, 1888), who also deal with economies with divisible goods. They are briefly surveyed by Thomson (1988). For recent studies on this topic, see Takamiya (2009, Sect 0.2) and references therein, Tadenuma and Thomson (1995) and Fujinaka and Sakai (2007, 2009) for economies with indivisibilities, and Velez and Thomson (2009) for economies with divisible goods. Following our paper, Ashlagi, Karagozoglu, and Klaus (2008) and Yamamura (2008) respectively analyze this issue in a bankruptcy model and a voting model, which share certain similarity to our model, and obtain results somewhat similar to ours.

and the proportional rule (under a weak assumption), among many others.[4] Our first main result is that, in the manipulation game of any rule in Ψ^1, there exists one and only one strong Nash equilibrium allocation and it is the uniform allocation. Moreover, this allocation turns out to be the unique efficient Nash equilibrium allocation.

For several rules $\psi \in \Psi^1$, the manipulation game of ψ may admit Nash equilibrium allocations that are not the uniform allocation. An important example of such a rule is actually the uniform rule itself.[5] Next we narrow down the class to $\Psi^2 \subsetneq \Psi^1$ by imposing an additional strict monotonicity condition. We establish a strong connection between outcomes of manipulation and the uniform rule: in the manipulation game associated with each $\psi \in \Psi^2$, there is no Nash equilibrium allocation other than the uniform allocation.

Our results state that any (coalitional) attempts to manipulate a rule leads to the recommendation made by the uniform rule, thereby strengthening the position of the uniform rule in this model. On a positive side, through strategic manipulation, the distributional objectives of the uniform rule such as efficiency or no-envy are achieved. On a negative side, any distributional objective that the uniform allocation does not posses cannot be met. Our results also have an implementation-theoretic implication: the direct mechanism associated with each $\psi \in \Psi^2$ doubly implements the uniform rule in Nash and strong Nash equilibria.[6] Thus we identify a wide class of direct mechanisms that doubly implement the uniform rule. Since the direct mechanism associated with the uniform rule implements the uniform rule in strong Nash equilibrium but not in Nash equilibrium, it may be considered to use some rule $\psi \in \Psi^2$ instead of the uniform rule itself to realize the uniform allocation.[7] In fact, based on this idea, Bochet et. al. (2008) conduct a laboratory experiment and observe that the proportional rule works, in a sense, better than the uniform rule to realize the uniform allocation.

The remaining of the paper is organized as follows. In Section 2, we introduce the model and the definitions. In Section 3, we state our two main results. We offer some discussions related to our results in Section 4. We conclude in Section 5. Proofs are relegated to the Appendix.

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[4]As far as we know, almost all symmetric and continuous rules discussed in the literature belong to Ψ^1.

[5]This fact was first pointed out by a working paper version of Saijo, Sjöström, and Yamato (2007) that was circulated as RIETI Discussion Paper 03-E-019. The final version does not contain such an argument.

[6]Note that any rule that is doubly implementable in Nash and strong Nash equilibria is also implementable in any solution concept that is in-between these two equilibrium concepts, such as coalition-proof Nash equilibrium (Bernheim, Peleg, and Whinston, 1987). Double implementation in Nash and strong Nash equilibria is introduced by Maskin (1979) and its necessary and sufficient condition is obtained by Suh (1997). In the present model, Thomson (2010) examines Nash implementability of various allocation correspondences and characterizes the uniform rule on the basis of Maskin's monotonicity condition (Maskin, 1999).

[7]A recent paper, Bochet and Sakai (2010), focus on this topic in its relation to "secure implementation" introduced by Saijo, Sjöström, and Yamato (2007).

2 Definitions

2.1 Basic definitions

Let $N \equiv \{1, 2, \ldots, n\}$ be the finite set of *agents*. There is a fixed amount of a divisible resource $\Omega > 0$ to be allocated. An *allotment* for $i \in N$ is $x_i \in [0, \Omega]$. An *allocation* is a vector of allotments $x = (x_1, \ldots, x_n) \in [0, \Omega]^N$ such that $\sum_{i \in N} x_i = \Omega$. Let X be the set of allocations.

A *single-peaked preference* is a transitive, complete, and continuous binary relation R_i over $[0, \Omega]$ for which there exists a "peak" amount $p_i \in [0, \Omega]$ such that, for each $x_i, x'_i \in [0, \Omega]$,

$$x'_i < x_i \leq p_i \Longrightarrow x_i \; P_i \; x'_i,$$
$$p_i \leq x_i < x'_i \Longrightarrow x_i \; P_i \; x'_i,$$

where the symmetric and asymmetric parts of R_i are denoted by I_i, P_i, respectively. Let R be the set of single-peaked preferences. A preference profile is $R \equiv (R_i)_{i \in N}$. Let R^N be the set of preference profiles. For each $R \in R^N$, we let $p \equiv (p_i)_{i \in N} \in [0, \Omega]^N$ be its associated peak-profile. A *rule* is a function $\psi : R^N \to X$ which maps each preference profile R to an allocation $\psi(R) \in X$.

We now introduce three basic properties of rules. The first one is the well-known efficiency condition. An allocation y *Pareto dominates* x at R if for each $i \in N$, $y_i \; R_i \; x_i$ and for some $j \in N$, $y_j \; P_j \; x_j$. Then, an allocation x is *efficient* at R if no y Pareto dominates x at R. Note that, by single-peakedness, efficiency of x at R is equivalent to the "same sidedness": $\sum_{i \in N} p_i \geq \Omega$ implies $p_i \geq \psi_i(R)$ for each $i \in N$, and $\sum_{i \in N} p_i \leq \Omega$ implies $p_i \leq \psi_i(R)$ for each $i \in N$. Let $P(R)$ be the set of efficient allocations at R.

Efficiency: For each $R \in R^N$, $\psi(R) \in P(R)$.

The next property is a standard horizontal equity property.

Symmetry: For each $R \in R^N$ and each pair $i, j \in N$ with $R_i = R_j$, $\psi_i(R) \; I_i \; \psi_j(R)$.

Finally, we introduce a strategic property that is central in the mechanism design literature.

Strategy-proofness: For each $R \in R^N$, each $i \in N$, and each $R'_i \in R$, $\psi_i(R) \; R_i \; \psi_i(R'_i, R_{-i})$.

A rule is *manipulable* if it violates *strategy-proofness*.

2.2 Direct revelation games as manipulation games

We fix a true preference profile R^0 and consider the *direct revelation game* $\Gamma(\psi, R^0)$ associated with a given rule ψ at R^0. In this game, each $i \in N$ reports a preference

$R_i \in \mathcal{R}$ to maximize his (true) preference R_i^0 when the outcome is determined by ψ. We simply call it the *game of ψ at R^0*.

A preference profile $R \in \mathcal{R}^N$ is a *Nash equilibrium* in $\Gamma(\psi, R^0)$ if for each $i \in N$ and each $R_i' \in \mathcal{R}$,

$$\psi_i(R)\ R_i^0\ \psi_i(R_i', R_{-i}).$$

Let $Nash_e(\psi, R^0)$ be the set of Nash equilibria and let

$$Nash(\psi, R^0) \equiv \{x \in X : \exists R \in Nash_e(\psi, R^0),\ x = \psi(R)\}$$

be the corresponding set of *Nash equilibrium allocations* in $\Gamma(\psi, R^0)$. Also, let

$$PNash(\psi, R^0) \equiv \{x \in X : \exists R \in Nash_e(\psi, R^0),\ x = \psi(R) \text{ and } x \in P(R^0)\}$$

be the set of *efficient Nash equilibrium allocations* in $\Gamma(\psi, R^0)$.

We also consider a stronger notion of Nash equilibrium that is robust to any coalitional deviation. A preference profile $R \in \mathcal{R}^N$ is a *strong Nash equilibrium* in $\Gamma(\psi, R^0)$ if there exist no $S \subseteq N$ and no $R_S' \equiv (R_i')_{i \in S} \in \mathcal{R}^S$ such that

$$\begin{aligned} &\forall i \in S, \quad \psi_i(R_S', R_{N \setminus S})\ R_i^0\ \psi_i(R), \\ &\exists j \in S, \quad \psi_j(R_S', R_{N \setminus S})\ P_j^0\ \psi_j(R). \end{aligned}$$

Let $SNash_e(\psi, R^0)$ be the set of strong Nash equilibria and let

$$SNash(\psi, R^0) \equiv \{x \in X : \exists R \in SNash_e(\psi, R^0),\ x = \psi(R)\}$$

be the set of *strong Nash equilibrium allocations* in $\Gamma(\psi, R^0)$.[8]

For each $R \in SNash(\psi, R^0)$, there exists no $R' \in \mathcal{R}^N$ such that $\psi(R')$ Pareto dominates $\psi(R)$ at R, since the group deviation by N with R' is not profitable. Therefore,

$$SNash(\psi, R^0) \subseteq PNash(\psi, R^0) \subseteq Nash(\psi, R^0),$$

where the last inclusion always holds by definition. However, the non-emptiness of these equilibrium allocation sets is non-trivial.

2.3 Uniform rule and other rules

The rule that has played a prominent role is the uniform rule (Benassy, 1982):

Uniform rule, U: For each $R \in \mathcal{R}^N$ and each $i \in N$,

$$U_i(R) = \begin{cases} \min\{p_i, \lambda\} \text{ if } \sum_i p_i \geq \Omega, \\ \max\{p_i, \lambda\} \text{ if } \sum_i p_i \leq \Omega, \end{cases}$$

where λ solves $\sum_{j \in N} U_j(R) = \Omega$.

[8] We defined strong Nash equilibrium by weak domination. In terms of establishing the existence of equilibrium, this makes the problem difficult, since the set of strong Nash equilibria under weak domination is contained by the set of strong Nash equilibria under strong domination. See Section 4.4.

This rule is justified by many desirable properties.[9] In particular, it is robust to strategic manipulation: it is not only strategy-proof but also group strategy-proof.[10] Furthermore, under efficiency and symmetry, this rule is the unique strategy-proof rule (Ching, 1994). In this sense, any other efficient and symmetric rule is manipulable. The following rules are such examples.

Constrained equal distance rule, Ce: For each $R \in R^N$ and each $i \in N$,

$$Ce_i(R) = \begin{cases} \max\{0, p_i - \lambda\} \text{ if } \sum_{j \in N} p_j \geq \Omega, \\ p_i + \dfrac{\Omega - \sum_{j \in N} p_j}{n} \text{ if } \sum_{j \in N} p_j \leq \Omega, \end{cases}$$

where λ solves $\sum_{j \in N} \max\{0, p_j - \lambda\} = \Omega$.

Proportional rule, Pro: For each $R \in R^N$ and each $i \in N$,
(i) when $\sum_{j \in N} p_j > 0$, $Pro_i(R) = \dfrac{p_i}{\sum_{j \in N} p_j}\Omega$, and
(ii) when $\sum_{j \in N} p_j = 0$, $Pro_i(R) = \dfrac{1}{n}\Omega$.

The proportional rule is discontinuous around the origin, though it is obviously continuous on the positive orthant: for example, whenever $p_1 > 0$, $Pro_1(0, 0) = \frac{\Omega}{2}$ and $Pro_1(p_1, 0) = \Omega$. Our main results will only apply to continuous rules, so they do not cover the proportional rule. However, the discontinuity of the proportional rule is not significant, and the scope of our analysis essentially covers this rule. We elaborate on the proportion rule in Section 4.1.

We next define a continuously modified version of the proportional rule. This rule coincides with the proportional rule in the case with excess demand, but differs in the case with excess supply: it allocates the resource so that the amount one does not receive is proportional to the amount one does not want:

Symmetricized proportional rule, Sym: For each $R \in R^N$,
(i) when $\Omega \leq \sum_{j \in N} p_j$, $Sym(R) = Pro(R)$, and
(ii) when $\Omega \geq \sum_{j \in N} p_j$, there is $\lambda > 0$ such that for each $i \in N$, $\Omega - Sym_i(R) = \lambda(\Omega - p_i)$.[11]

Notice that the constrained equal distance and the symmetricized proportional rules are manipulable because of their sensitivity with respect to changes in peaks: an agent can profitably increase (resp. decrease) what he gets by over-reporting (resp. under-reporting) his peak. Obviously, the uniform rule does not share this feature.

[9] See Thomson (2005, Ch. 11) for a survey.
[10] See, for example, Sprumont (1991), Ching (1994), and Serizawa (2006).
[11] One can check that $\lambda = \frac{n\Omega - \Omega}{n\Omega - \sum_{j \in N} p_j}$.

3 General equivalence results

In our main results, we restrict our attention to rules that are *peak-only*: for each pair $R, R' \in R^N$ such that $p = p'$, $\psi(R) = \psi(R')$. Hence each rule can be seen as a function from $[0, \Omega]^N$ to X, which maps each peak profile $p \in [0, \Omega]^N$ to an allocation $\psi(R) \in X$. When there is no confusion, we often treat a rule ψ as a function that maps a peak profile to an allocation, and identify a preference profile R with its peak profile p.

Before proceeding to our main results, we introduce some properties of rules. Because of the peak-only assumption, we simply define the properties using peaks of preferences.

Own-peak monotonicity: For each $p \in [0, \Omega]^N$, each $i \in N$, and each $p'_i \in [0, \Omega]$ such that $p_i \leq p'_i$, we have $\psi_i(p) \leq \psi_i(p'_i, p_{-i})$.

Others-oriented peak monotonicity: For each $p \in [0, \Omega]^N$, each pair $i, j \in N$ with $i \neq j$, and each $p'_i \in [0, \Omega]$ with $p_i \leq p'_i$, we have $\psi_j(p'_i, p_{-i}) \leq \psi_j(p)$.

Peak continuity: ψ is continuous on $[0, \Omega]^N$.

It is obvious that *others-oriented peak monotonicity* implies *own-peak monotonicity*. Most of existing rules in the literature satisfy these properties.[12] Furthermore, there are many different types of *efficient* and *symmetric* rules satisfying them. For example, in the context of the bankruptcy problem where an amount of a divisible resource is to be allocated according to agents' claims, many allocation rules satisfy the counterparts of these properties. Since such allocation rules can be easily translated into rules in our model by regarding claims as peaks of preferences, rules so obtained in our model also satisfy those properties. An interesting example of such a rule is the Talmud rule (Aumann and Maschler, 1985).[13]

Let Ψ^1 be the set of peak-only rules satisfying *efficiency*, *symmetry*, *others-oriented peak monotonicity* (and hence *own-peak monotonicity*), and *peak continuity*.[14] Our first main result shows that, for each (true) preference profile and each rule in Ψ^1, the sets of efficient Nash and strong Nash equilibrium allocations coincide and in fact contain a single allocation: it is the uniform allocation at the true preference profile. In terms of implementation theory, this result implies that the direct mechanism associated with any such rule implements the uniform rule in strong Nash equilibria.

[12] As already mentioned, an exception is that the proportional rule is not *peak continuous* because of its discontinuity at the origin.

[13] For a survey of the bankruptcy problem and its variants, see, Thomson (2003).

[14] Our main results are about the consequence of strategic manipulations for rules in Ψ^1. Even when one does not totally agree with the normative appeal of some of the properties, we shall emphasize that the most important fact here is that Ψ^1 contains many appealing rules that are well-known in the literature. This is because the main purpose of our study is to analyze consequence of strategic manipulations for a wide class of rules, not to axiomatize rules satisfying appealing requirements.

Theorem 1. *Let $R^0 \in R^N$ be any (true) preference profile. For each peak-only rule ψ satisfying efficiency, symmetry, others-oriented peak monotonicity, and peak continuity,*

$$\emptyset \neq SNash(\psi, R^0) = PNash(\psi, R^0) = \{U(R^0)\}.$$

Proof. See the Appendix. □

In view of Theorem 1, it is natural to ask whether the set of Nash equilibrium allocations is also the singleton of the uniform allocation. The answer is no in general. In particular, the game of the uniform rule has inefficient Nash equilibria as shown in the next example.

Example 1 (*Inefficient Nash equilibria in the game of the uniform rule*)**.** Let $N = \{1,2,3\}$, $\Omega = 6$, and $R^0 \in R^N$ be such that $p^0 = (1,2,4)$. Then, any R with $p = (2,2,2)$, which realizes $U(R) = (2,2,2)$, is a Nash equilibrium in $\Gamma(U, R^0)$. By definition of the uniform rule, no one can change this allocation by any unilateral deviation from R. Notice that $U(R^0) = (1,2,3)$ Pareto dominates $U(R)$ at R^0. Agents 1 and 3 have the joint profitable deviation that simply consists of their true preferences, realizing the true uniform allocation is obtained. By the same token, (i) R^1 with $p^1 = (1.5, 2, 2.5)$ and $U(R^1) = (1.5, 2, 2.5)$, (ii) R^2 with $p^2 = (2,2,1)$ and $U(R^2) = (2,2,2)$, and (iii) R^3 with $p^3 = (3,2,2)$ and $U(R^3) = (2,2,2)$ are also Nash equilibria in this game. One can verify that there are infinitely many inefficient Nash equilibria in this example. They are described by the following sets, $\{R \in R^N : 1 < p_1 \leq 2, p_2 = 2, 2 \leq p_3 < 4 \text{ such that } p_1 + p_2 + p_3 = 6\}$, $\{R \in R^N : p_1 = 2, p_2 = 2, p_3 \leq 2\}$, and $\{R \in R^N : p_1 \geq 2, p_2 = 2, p_3 = 2\}$. ◇

We next investigate under which conditions the set of Nash equilibrium allocations becomes the singleton of the uniform allocation. We characterize the equilibrium sets for a subclass of Ψ^1. The next condition states that whenever an agent is receiving a positive amount, he can increase what he gets by any over-reporting of his own peak. Note that this condition is *not* implied by *others-oriented peak monotonicity.*

Strict own-peak monotonicity: For each $p \in [0,\Omega]^N$, each $i \in N$, and each $p'_i \in [0,\Omega]$ such that $0 < \psi_i(p)$ and $p_i < p'_i$, we have $\psi_i(p) < \psi_i(p'_i, p_{-i})$.

Let Ψ^2 be the set of rules satisfying the four properties of Theorem 1 and *strict own-peak monotonicity.* Obviously, $\Psi^2 \subsetneq \Psi^1$. The constrained equal distance rule and the symmetricized proportional rule belong to Ψ^2, but the uniform rule does not, since it is not *strictly own-peak monotonic.*[15] The next theorem shows that, once Ψ^1 is narrowed down to Ψ^2, all inefficient Nash equilibria are eliminated: the uniform allocation becomes the unique Nash equilibrium allocation.

Theorem 2. *Let $R^0 \in R^N$ be any (true) preference profile. For each peak-only rule ψ satisfying efficiency, symmetry, others-oriented peak monotonicity, peak continuity, and strict own-peak monotonicity,*

$$\emptyset \neq SNash(\psi, R^0) = PNash(\psi, R^0) = Nash(\psi, R^0) = \{U(R^0)\}.$$

[15] For example, when $\Omega = 2$, $U(1,1) = (1,1) = U(1,2)$.

Proof. See the Appendix. □

We finally remark that the uniqueness of a Nash equilibrium allocation does not imply the uniqueness of a Nash equilibrium:

Example 2 (*Multiple Nash equilibria in the game of the symmetricized proportional rule*)**.** Consider any preference R^0 with $\sum_{i \in N} p_i^0 = \Omega$ and the symmetricized proportional rule. Then, the set of Nash equilibria is characterized as $Nash_e(Sym, R_0) = \{R \in R^N : \exists \mu > 0,\ \forall i \in N,\ p_i = \mu p_i^0\}$. Clearly, this set contains infinitely many Nash equilibria. Importantly, because this set is not a product set, Nash equilibrium strategies are not interchangeable. ◇

4 Discussion

4.1 Proportional rule

We pointed out that the proportional rule is discontinuous at the origin, $\mathbf{0} \equiv (0, 0, \ldots, 0) \in \mathbb{R}^N$, but it is continuous everywhere else. With taking care of the origin, the next proposition summarizes results on manipulation of the proportional rule. It shows that, except for this unique singular point, the same conclusion as Theorem 2 holds for the proportional rule.

Proposition 1. *Let $R^0 \in R^N$ be any (true) preference profile.*
(i) $SNash_e(Pro, R^0) \neq \emptyset$.
(ii) For each $p \in Nash_e(Pro, R^0)$ with $p \neq \mathbf{0}$, $p \in SNash_e(Pro, R^0)$ and $Pro(p) = U(R^0)$.
(ii) $\mathbf{0} \in Nash_e(Pro, R^0)$ if and only if $0\ R_i^0\ \Omega$ for all $i \in N$.

Proof. See the Appendix. □

A simple way to eliminate the case that $\mathbf{0}$ is a Nash equilibrium is to restrict the domain of peaks to be $[\varepsilon, \Omega]$ for a sufficiently small $\varepsilon > 0$. Then the same conclusion as Theorem 2 holds for the proportional rule. Assuming it, Bochet et. al. (2008) confirm the theoretical prediction of Proposition 1 in laboratory experiments. Following our paper, Kawasaki and Yamamura (2008) show that in the game of the proportional rule, any path of best responses converges to the set of Nash equilibria that realize the uniform allocation, thereby showing the dynamic stability of such Nash equilibria.

The following example describes a typical Nash equilibrium in the game of the proportional rule:

Example 3. *Manipulation of the proportional rule leads to the uniform allocation* Let $N = \{1, 2, 3, 4\}$, $\Omega = 9$, and $R^0 \in R^N$ be such that $p^0 = (1, 2, 5, 6)$. Note $U(R^0) = (1, 2, 3, 3)$. Consider any R with $p = (p_1, p_2, \Omega, \Omega)$ such that

$$\frac{p_1}{p_1 + p_2 + 2\Omega}\Omega = p_2^0,$$
$$\frac{p_2}{p_1 + p_2 + 2\Omega}\Omega = p_3^0,$$

which can be explicitly written as $p = (3, 6, 9, 9)$. Then, $U(R^0) = Pro(R)$. It is easy to see that R is the unique Nash equilibrium that is not $(0, 0, \ldots, 0)$ in this game. It is also a strong Nash equilibrium. $\diamond$

4.2 Constrained equal-preferred-sets rule

We have focused on *peak-only* rules so far. In general, one might be interested in rules that are sensitive to other features of agents' preferences. We give next an example of such a rule which is in the spirit of the constrained equal distance rule introduced earlier. Here, we measure the sacrifice an agent makes at his allotment by the size of his preferred set at that point. Like for the constrained equal distance rule, an allocation at which sacrifices are equal across agents may not exist. So instead, we require sacrifices to be as equal as possible (Thomson, 1990).

We need some notation. Given any $R_i \in R$ and any $x_i \in [0, \Omega]$, the preferred set of R_i at x_i is given by $\{a \in [0, \Omega] : a\ R_i\ x_i\}$. By single-peakedness, the set $\{a \in [0, \Omega] : a\ R_i\ x_i\}$ is a connected interval, and we denote its length by $d(R_i, x_i)$. Note that, (i) if $a\ I_i\ b$ with $a < b$, then $d(R_i, a) = d(R_i, b) = b - a$, and (ii) if $\Omega\ R_i\ a$, then $d(R_i, a) = \Omega - a$.

Constrained equal-preferred-sets rule, Eps: For each $R \in R^N$, $Eps(R) \in P(R)$ and there exists $\lambda \geq 0$ such that for each $i \in N$, $d(R_i, Eps_i(R)) \leq \lambda$, and $d(R_i, Eps_i(R)) < \lambda$ implies $Eps_i(R) = 0$.

Note that $d(R_i, Eps_i(R)) < d(R_j, Eps_j(R))$ is possible only if $Eps_i(R) = 0$.

Proposition 2. *Let $R^0 \in R^N$ be any (true) preference profile. Then,*

$$\emptyset \neq SNash(Eps, R^0) = PNash(Eps, R^0) = Nash(Eps, R^0) = \{U(R^0)\}.$$

Proof. See the Appendix. $\square$

4.3 Strong Nash equilibrium

The proof of Theorem 1 involves showing the existence of a strong Nash equilibrium in the game of any $\psi \in \Psi^1$. In the proof, we first define a "reduced game" obtained by fixing the strategies of some players at Ω, and then establish the existence of a Nash equilibrium in the reduced game using a standard fixed-point argument. Then, we show that the Nash equilibrium profile of strategies of agents in the reduced game and the fixed strategies of the outside agents constitute in fact a strong Nash equilibrium of the original game. This technique is quite different from the standard technique in the literature that links the existence of a strong Nash equilibrium with the non-emptiness of the core in a related NTU cooperative game (e.g., Ichiishi, 1993, p. 39).[16]

[16] Assuming that the number of outcomes is finite and each agent's payoff depends only on the number of agents who choose the same strategy, Konishi, Le Breton, and Weber (1997) establish the existence of a strong Nash equilibrium. Our games satisfy none of these assumptions.

4.4 Weaker version of strong Nash equilibrium

Our strong Nash equilibrium is defined by weak domination. One can consider a weaker notion of strong Nash equilibrium defined by strong domination. Notice that as long as we consider rules in Ψ^2, the set of allocations realized by such a weaker version of strong Nash equilibrium coincides with the set of Nash equilibrium allocations. Indeed, even the set of (our) strong Nash equilibrium allocations coincides with the set of Nash equilibrium allocations. On rules that are not in Ψ^2, Bochet and Sakai (2010) show that in the game of the uniform rule, the uniform allocation is the unique allocation that is realized by the weaker version of strong Nash equilibrium. Therefore, for the uniform rule, Theorem 1 holds continues to hold under strong domination. We do not have a general characterization of such equilibrium allocations for other rules in $\Psi^1 \setminus \Psi^2$. We leave this question open for future research.

4.5 Coalition-proof Nash equilibrium

A coalition-proof Nash equilibrium is a Nash equilibrium that is robust to any "credible" coalitional deviation.[17] By definition, any strong Nash equilibrium is coalition-proof, and any coalition-proof Nash equilibrium is a Nash equilibrium. However, a coalition-proof Nash equilibrium may be Pareto inefficient and an efficient Nash equilibrium is not coalition-proof in general. Under which conditions these two notions coincide or are related by inclusion is an ongoing topic (e.g., Yi, 1999; Shinohara, 2005).[18] Theorem 2 states that, under a set of conditions that includes *strict own-peak monotonicity*, the set of strong Nash equilibrium allocations coincides with the set of Nash equilibrium allocations. Since the set of coalition-proof Nash equilibrium allocations contains the first set and is contained in the second set, this implies that the three sets are in fact all the same.[19]

4.6 Natural implementation

In words of implementation theory, Theorem 2 implies that the direct mechanism associated with any rule $\psi \in \Psi^2$ doubly implements the uniform rule in Nash and strong Nash equilibria.[20] The fact that the strategy space is $[0, \Omega]$ means that everyone only reports the self-relevant information of his own peak (Hurwicz, 1960) and each agent can always find a best response because of its compactness. Furthermore, the fact that the outcome function ψ is continuous means that a small mistake in choosing strategies does not lead to a big change of outcomes (Postlewaite

[17] We refer to the seminal work by Bernheim, Peleg, and Whinston (1987) for its precise definition.

[18] Yi and Shinohara study games satisfying strategic substitutability and certain independence conditions. Our games are not such games, and hence we cannot apply their results to the present model.

[19] Since the uniform rule violates *strict own-peak monotonicity*, the argument here does not cover this rule. However, in a companion paper, Bochet and Sakai (2010) show that the same equivalence holds in the direct revelation game of the uniform rule.

[20] See, for example, Jackson (2001) for a survey of implementation theory.

and Wettstein, 1989). That is, any such mechanism satisfies many of the properties attributed to "natural" mechanisms (Dutta, Sen, and Vohra, 1995; Saijo, Tatamitani, and Yamato, 1996). This suggests that, when the problem of inefficient Nash equilibria of the uniform rule is serious and pre-play communication to exclude them is not allowed, these rules can be a good tool to realize the uniform allocation.

4.7 Tightness

We imposed several properties in Theorems 1 and 2. They are shown to be sufficient conditions to guarantee equilibrium existence and allow us to characterize equilibrium outcomes. Unlike axiomatic studies characterizing rules satisfying a set of properties, it is not easy to check the tightness of the properties in our study. However, the following facts regarding the equilibrium set of outcomes can be easily verified.

(i) The equal division rule satisfies all the properties in Theorem 1 except for *efficiency*, and of course, the only Nash equilibrium allocation is the equal division allocation regardless of reported preferences.

(ii) The proportional rule satisfies all the properties in Theorem 2 but *peak continuity*, and it does not meet the conclusion of the theorem since $p = (0, 0)$ is a Nash equilibrium at p^0 satisfying $p_1^0 = 0$, $\frac{\Omega}{2} < p_2^0 < \Omega$, and $\frac{\Omega}{2}\ P_2^0\ \Omega$.

(iii) A priority rule –which allocates in a serial dictatorship way the stock of resource– satisfies all the properties but *symmetry*, and neither conclusion of the two theorems are met.

(iv) Finally *strict own-peak monotonicity* cannot be relaxed to *peak own monotonicity* in Theorem 2 since the uniform rule has an inefficient Nash equilibrium, as observed in Example 3.

5 Concluding remark

In terms of implementation theory, we observed that the direct mechanism associated with any rule in Ψ^2 doubly implements the uniform rule in Nash and strong Nash equilibrium, while the uniform rule itself fails. In relation to this rather surprising result, we close the discussion by mentioning some insight we found. As we observed, *strict own-peak monotonicity* is important to Nash implement the uniform rule but the uniform rule violates this property. In fact, the uniform rule is quite rigid to changes in peak announcements. This rigidity is, however, the main reason for the uniform rule to satisfy *strategy-proofness* and Maskin's monotonicity condition for Nash implementation (Maskin, 1999). On the other hand, the direct mechanism associated with a strictly own peak monotonic rule Nash implements the uniform rule because the rule as an outcome function is sensitive with respect to changes in peaks reported. Although we do not have general relations between a rule to be implemented using a direct mechanism and an implementing outcome function, our finding seems to suggest that there may be certain sensitivity-insensitivity relations between them. We leave this question open for future research.

6 Appendix

In this appendix –exception made of the proof of Proposition 2– we throughout assume that ψ is *peak-only* and regard it as a function from $[0,\Omega]^N$ to X. We also identify each $R \in R^N$ with its peak profile p when there is no confusion.

6.1 Preliminaries

We use the following properties that do not appear in the statements of the theorems.

Non-bossiness: For each $p \in [0,\Omega]^N$, each $i \in N$, and $p'_i \in [0,\Omega]$ such that $\psi_i(p'_i, p_{-i}) = \psi_i(p)$, $\psi(p'_i, p_{-i}) = \psi(p)$.

Peak order preservation: For each $p \in [0,\Omega]^N$ and each $i, j \in N$ such that $p_i \leq p_j$, we have $\psi_i(p) \leq \psi_j(p)$.

Non-bossiness is introduced by Satterthwaite and Sonnenschein (1981) and states that no one can change someone else's allotment unless he changes his own. Obviously, *others-oriented peak monotonicity* implies *non-bossiness*. The next simple lemma on *peak order preservation* is a version of Lemma 4 in Ashlagi, Karagozoglu, and Klaus (2008):

Lemma 1. *If ψ satisfies efficiency, symmetry, and others-oriented peak monotonicity, then it satisfies peak order preservation.*

Proof. Let $p \in [0,\Omega]^N$ and $i, j \in N$ be such that $p_i < p_j$. Let $p'_j \equiv p_i$. Then, by *efficiency* and *symmetry*, $\psi_i(p'_j, p_{-j}) = \psi_j(p'_j, p_{-j})$. By *others-oriented peak monotonicity* and *own-peak monotonicity*, $\psi_i(p_j, p_{-j}) \leq \psi_i(p'_j, p_{-j})$ and $\psi_j(p'_j, p_{-j}) \leq \psi_j(p_j, p_{-j})$. Thus $\psi_i(p_j, p_{-j}) \leq \psi_j(p_j, p_{-j})$. □

6.2 Proofs of Theorems 1 and 2

Hereafter, for each $i \in N$, we denote agent i's true single-peaked preference by R^0_i, the profile of true preferences by $R^0 \equiv (R^0_i)_{i\in N}$, and its peak profile by $p^0 \equiv (p^0_i)_{i\in N}$. These true preference are arbitrary chosen, and we fix them throughout the rest of the proof. We assume the case with excess demand, that is, $\Omega \leq \sum_{i\in N} p^0_i$. The case with excess supply can be dealt with in a similar fashion.[21]

Lemma 2. *If ψ satisfies own-peak monotonicity, peak continuity, and non-bossiness, then for each $p \in Nash_e(\psi, R^0)$, each $i \in N$, and each $p'_i \in [0,\Omega]$ such that $\psi_i(p) < p^0_i$ and $p_i \leq p'_i$,*

$$\psi(p) = \psi(p'_i, p_{-i}).$$

[21] The proof is available from the authors upon request.

Proof. Let $p \in Nash_e(\psi, R^0)$, $i \in N$, and $p'_i \in [0, \Omega]$ be such that $\psi_i(p) < p_i^0$ and $p_i \le p'_i$.

By *own-peak monotonicity*, $\psi_i(p) \le \psi_i(p'_i, p_{-i})$. If $\psi_i(p) < \psi_i(p'_i, p_{-i})$, then by *peak continuity*, there is $p''_i \in [0, \Omega]$ such that

$$\psi_i(p) < \psi_i(p''_i, p_{-i}) < \psi_i(p'_i, p_{-i}) \text{ and } \psi_i(p''_i, p_{-i})\ P_i^0\ \psi_i(p).$$

However, this contradicts $p \in Nash_e(\psi, R^0)$. Therefore, $\psi_i(p) = \psi_i(p'_i, p_{-i})$. By *non-bossiness*, $\psi(p) = \psi(p'_i, p_{-i})$. □

Lemma 3. *If ψ satisfies own-peak monotonicity, peak continuity, and non-bossiness, then for each $p \in Nash_e(\psi, R^0)$, each $i \in N$, and each $p'_i \in [0, \Omega]$ such that $p_i^0 < \psi_i(p)$ and $p'_i \le p_i$, we have $\psi(p) = \psi(p'_i, p_{-i})$.*

Proof. This can be shown in a way parallel to the proof of Lemma 2. □

To denote i's peak that is Ω, we let $\bar{p}_i \equiv \Omega$. This notation is useful since simply writing Ω does not explain whose peak is Ω.

Lemma 4. *If ψ satisfies own-peak monotonicity, peak continuity, and non-bossiness, then for each $p \in Nash_e(\psi, R^0)$ with $\psi(p) \in P(R^0)$, whenever,*

$$N_1 \equiv \{i \in N : \psi_i(p) < p_i^0 \text{ and } p_i < \Omega\} \text{ and } N_2 \equiv \{i \in N : \psi_i(p) = p_i^0 \text{ or } p_i = \Omega\},$$

we have

$$\psi(p) = \psi((\bar{p}_i)_{i \in N_1}, (p_i)_{i \in N_2}) \text{ and } ((\bar{p}_i)_{i \in N_1}, (p_i)_{i \in N_2}) \in Nash_e(\psi, R^0).$$

Proof. Let $p \in Nash_e(\psi, R^0)$ with $\psi(p) \in P(R^0)$ and let N_1, N_2 be defined as above. Note that $N_1 \cup N_2 = N$ and $N_1 \cap N_2 = \emptyset$.

Step 1: For each $\boldsymbol{i \in N_1}$, $\boldsymbol{\psi(\bar{p}_i, p_{-i}) = \psi(p)}$. For each $i \in N_1$, by Lemma 2, $\psi_i(p) = \psi_i(\bar{p}_i, p_{-i})$, and then by *non-bossiness*, $\psi(p) = \psi(\bar{p}_i, p_{-i})$.

Step 2: For each $\boldsymbol{i \in N_1}$, $\boldsymbol{(\bar{p}_i, p_{-i}) \in Nash_e(\psi, R^0)}$. Let $i \in N_1$. Let us verify that $(\bar{p}_i, p_{-i}) \in Nash_e(\psi, R^0)$. Since $\psi_i(\bar{p}_i, p_{-i}) = \psi_i(p) < p_i^0$, for i to profitably deviate at $\psi_i(\bar{p}_i, p_{-i})$, i needs to increase what he gets. However, since i is already reporting Ω, this is impossible by *own-peak monotonicity.* Clearly, no $j \in N_2$ with $\psi_j(p) = p_j^0$ can profitably deviate. Also, no $j \in N_2$ with $p_j = \Omega$ can profitably deviate at $(\bar{p}_i, p_{-i})$, since $p_j = \Omega$ and ψ is *own-peak monotonic.*

It remains to show that no $j \in N_1 \setminus \{i\}$ can profitably deviate at $(\bar{p}_i, p_{-i})$. Suppose, by contradiction, that there exists $j \in N_1 \setminus \{i\}$ such that for some $p'_j \in [0, \Omega]$,

$$\psi_j(\bar{p}_i, p_j, p_{-ij}) < \psi_j(\bar{p}_i, p'_j, p_{-ij}). \tag{1}$$

By *own-peak monotonicity*, $p_j < p'_j$. By Step 1,

$$\psi_j(p_i, \bar{p}_j, p_{-ij}) = \psi_j(p_i, p_j, p_{-ij}) = \psi_j(\bar{p}_i, p_j, p_{-ij}). \tag{2}$$

By *others-oriented peak monotonicity*,

$$\psi_j(\bar{p}_i, p'_j, p_{-ij}) \leq \psi_j(p_i, p'_j, p_{-ij}). \tag{3}$$

By (1), (2), and (3),

$$\psi_j(p_i, \bar{p}_j, p_{-ij}) < \psi_j(p_i, p'_j, p_{-ij}), \tag{4}$$

which contradicts *own-peak monotonicity*.

Step 3: Concluding. Applying Steps 1 and 2 inductively, we obtain

$$\psi(p) = \psi((\bar{p}_i)_{i\in N_1}, (p_i)_{i\in N_2}) \text{ and } ((\bar{p}_i)_{i\in N_1}, (p_i)_{i\in N_2}) \in \mathit{Nash}_e(\psi, R^0).$$

□

Lemma 5. *If ψ satisfies efficiency, own-peak monotonicity, peak continuity, peak order preservation, and non-bossiness, then for each $p \in \mathit{Nash}_e(\psi, R^0)$ with $\psi(p) \in P(R^0)$, we have $\psi(p) = U(p^0)$.*

Proof. Let $p \in \mathit{Nash}_e(\psi, R^0)$ with $\psi(p) \in P(R^0)$. Let

$$\begin{aligned} N_1 &\equiv \{i \in N : \psi_i(p) < p_i^0 \text{ and } p_i < \Omega\}, \\ N_2 &\equiv \{i \in N : \psi_i(p) = p_i^0\}, \\ N_3 &\equiv \{i \in N : p_i = \Omega\}. \end{aligned}$$

Note that $N_1 \cup N_2 \cup N_3 = N$ and these sets are mutually disjoint. By Lemma 4,

$$\psi(p) = \psi((\bar{p}_i)_{i\in N_1}, (p_i)_{i\in N_2}, (\bar{p}_i)_{i\in N_3}) \text{ and } ((\bar{p}_i)_{i\in N_1}, (p_i)_{i\in N_2}, (\bar{p}_i)_{i\in N_3}) \in \mathit{Nash}_e(\psi, R^0).$$

Since $\psi(p) = \psi((\bar{p}_i)_{i\in N_1}, (p_i)_{i\in N_2}, (\bar{p}_i)_{i\in N_3})$, by *peak order preservation*,

$$\forall i, j \in N_1 \cup N_3, \forall k \in N_2, \quad p_k^0 = x_k \leq x_i = x_j < \min\{p_i^0, p_j^0\}.$$

where the last inequality follows from the *efficiency* of the Nash equilibrium outcome $\psi(p)$. This immediately implies that for each $i \in N$, $x_i = \min\{p_i^0, \lambda\}$, where $\lambda = x_j$ with $j \in N_1 \cup N_3$. Hence $x = U(p^0)$. Since $x = \psi(p)$ by its definition, $\psi(p) = U(p^0)$. □

Lemma 5 shows that any efficient Nash equilibrium allocation is the uniform allocation. To establish the existence, we in fact prove a stronger statement: there exists a strong Nash equilibrium that supports the uniform allocation.

Lemma 6. *If ψ satisfies efficiency, own-peak monotonicity, others-oriented peak monotonicity, peak continuity, and peak order preservation, then there exists $p \in \mathit{SNash}_e(\psi, R^0)$ such that $\psi(p) = U(R^0)$.*

Proof. If $\sum_{i\in N} p_i^0 = \Omega$, then *efficiency* implies $\psi(R^0) = p^0$, and $p^0 \in SNash_e(\psi, R^0)$ trivially holds. Hereafter, we assume that $\sum_{i\in N} p_i^0 < \Omega$.

Step 1: Setting up. Let $z \equiv U(R^0)$, $S \equiv \{i \in N : z_i = p_i^0\}$, and $T \equiv \{i \in N : z_i < p_i^0\}$. Note that $S \cup T = N$, $S \cap T = \emptyset$, $\max_{i\in S} p_i^0 < \min_{i\in T} p_i^0$, and

$$\max_{i\in S} z_i \leq \min_{i\in T} z_i. \tag{5}$$

Note that $\sum_{i\in N} p_i^0 < \Omega$ implies $T \neq \emptyset$.

Step 2: Finding a Nash equilibrium in a reduced game. We consider a reduced game played by individuals in S given that each $i \in T$ reports Ω. For each $i \in S$, let $u_i : [0, \Omega] \to \mathbb{R}$ be a continuous representation of R_i^0; also, for each $(p_j)_{j\in S} \in [0, \Omega]^S$, define i's "payoff function" $v_i : [0, \Omega]^S \to [0, \Omega]$ by,

$$v_i(p) = u_i(\psi_i((p_j)_{j\in S}, (\bar{p}_j)_{j\in T})).$$

By *peak continuity* of ψ and continuity of u_i, v_i is continuous on $[0, \Omega]^S$.

Given

$$(p_j)_{j\in S\setminus\{i\}} \in [0, \Omega]^{S\setminus\{i\}},$$

for each $p_i', p_i'' \in [0, \Omega]$ with $p_i' < p_i''$, by *own-peak monotonicity*,

$$\begin{aligned} \psi_i(p_i'', (p_j)_{j\in S\setminus\{i\}}, (\bar{p}_j)_{j\in T}) \geq \\ \psi_i(\alpha p_i' + (1-\alpha)p_i'', (p_j)_{j\in S\setminus\{i\}}, (\bar{p}_j)_{j\in T}) \geq \\ \psi_i(p_i', (p_j)_{j\in S\setminus\{i\}}, (\bar{p}_j)_{j\in T}), \end{aligned}$$

and by single-peakedness of u_i,

$$\begin{aligned} u_i(\psi_i(\alpha p_i' + (1-\alpha)p_i'', (p_j)_{j\in S\setminus\{i\}}, (\bar{p}_j)_{j\in T}) \geq \\ \min\{u_i(\psi_i(p_i', (p_j)_{j\in S\setminus\{i\}}, (\bar{p}_j)_{j\in T})), u_i(\psi_i(p_i'', (p_j)_{j\in S\setminus\{i\}}, (\bar{p}_j)_{j\in T}))\}. \end{aligned}$$

Thus v_i is quasi-concave in $p_i \in [0, \Omega]$.

Consider the game in which the set of players is S, the strategy space of each $i \in S$ is $[0, \Omega]$, and the payoff function of each $i \in S$ is v_i. Since $[0, \Omega]$ is compact and convex and v_i is continuous in $[0, \Omega]^S$ and quasi-concave in $[0, \Omega]$, by a standard fixed-point argument (e.g., Theorem 4.1.1. in Ichiishi, 1983), there exists a Nash equilibrium $(p_i)_{i\in S} \in [0, \Omega]^S$ in this game.

Step 3. Characterizing the Nash equilibrium allocation. Let $x \equiv \psi((p_j)_{j\in S}, (\bar{p}_j)_{j\in T})$. Let $\alpha \equiv x_k$ for $k \in T$. The assumption $\Omega < \sum_{j\in N} p_j^0$ implies $T \neq \emptyset$, and hence by *efficiency*,

$$\forall i \in S, \quad \psi_i(0, (p_j)_{j\in S\setminus\{i\}}, (\bar{p}_j)_{j\in T}) = 0. \tag{6}$$

We first claim that for each $i \in S$, $x_i = p_i^0$. If there is $i \in S$ with $p_i^0 < x_i$, then (6) and *peak continuity* imply that i could decrease what he gets by reporting some

$p_i' \in (0, p_i)$, a contradiction. Hence for each $i \in S$, $x_i \leq p_i^0$. Let $S_1 \equiv \{i \in S : x_i = p_i^0\}$ and $S_2 \equiv \{i \in S : x_i < p_i^0\}$. We shall show that $S_2 = \emptyset$ by contradiction. Suppose that $S_2 \neq \emptyset$. For each $i \in S_2$, since $x_i = \psi_i((p_j)_{j \in S}, (\bar{p}_j)_{j \in T})$ by definition, *own-peak monotonicity* and the fact that p_S is a Nash equilibrium in the reduced game imply

$$x_i = \psi_i(\bar{p}_i, p_{S \setminus \{i\}}, (\bar{p}_i)_{i \in T}),$$

and by *non-bossiness*, $x = \psi(\bar{p}_i, p_{S \setminus \{i\}}, (\bar{p}_i)_{i \in T})$. Then, by *equal treatment of equals*, $x_i = \alpha$. Thus

$$\begin{aligned} &\forall i \in S_1, \quad x_i = p_i^0 = z_i, \\ &\forall i \in S_2, \quad \alpha = x_i < p_i^0 = z_i. \end{aligned} \tag{7}$$

Since there is $j \in S_2$, by feasibility, there is $k \in T$ such that $z_k < x_k$. But, then $z_k < x_k = \alpha = x_j < z_j$, a contradiction to (5). Thus for each $i \in S$, $x_i = p_i^0$.

By *symmetry* to $\psi((p_j)_{j \in S}, (\bar{p}_j)_{j \in T})$,

$$\forall i \in T, \quad \psi_i((p_j)_{j \in S}, (\bar{p}_j)_{j \in T}) = \frac{\Omega - \sum_{j \in S} p_i^0}{|T|} \tag{8}$$

and by *peak order preservation* to $\psi((p_j)_{j \in S}, (\bar{p}_j)_{j \in T})$,

$$\max_{j \in S} p_j^0 \leq \frac{\Omega - \sum_{j \in S} p_i^0}{|T|}. \tag{9}$$

Overall, by (8) and (9),

$$\forall i \in N, \quad \psi_i((p_j)_{j \in S}, (\bar{p}_j)_{j \in T}) = \min\{p_i^0, \frac{\Omega - \sum_{j \in S} p_i^0}{|T|}\}.$$

Thus $\psi((p_j)_{j \in S}, (\bar{p}_j)_{j \in T}) = U(R^0)$.

Step 4. Concluding. Let $p \equiv ((p_j)_{j \in S}, (\bar{p}_j)_{j \in T})$. It remains to show that, in the direct revelation game of ψ, p is a strong Nash equilibrium.

Suppose, by contradiction, that there exist $N' \subseteq N$ and $p'_{N'} \equiv (p_i')_{i \in N'} \in [0, \Omega]^{N'}$ such that

$$\forall i \in N', \quad \psi_i(p'_{N'}, p_{N \setminus N'}) \; R_i \; \psi_i(p), \tag{10}$$

$$\exists j \in N', \quad \psi_j(p'_{N'}, p_{N \setminus N'}) \; P_j \; \psi_j(p). \tag{11}$$

Since $\psi(p) = U(R^0)$,

$$\forall i \in N', \quad \psi_i(p'_{N'}, p_{N \setminus N'}) \geq \psi_i(p), \tag{12}$$

$$\psi_j(p'_{N'}, p_{N \setminus N'}) > \psi_j(p). \tag{13}$$

Since $U(R^0) = \psi(p)$, (11) implies $U_j(R^0) \neq p_j^0$. This in turn implies $j \in T$, and thus $p_j = \Omega$. Let $A \equiv \{i \in N' : p_i < p_i'\}$ and $B \equiv \{i \in N' : p_i' \leq p_i\}$. Note that $A \cup B = N'$ and $A \cap B = \emptyset$. Also by $p_j = \Omega$,

$$j \in B. \tag{14}$$

By repeatedly applying *others-oriented peak monotonicity*,

$$\forall i \in N \setminus B, \;\; \psi_i(p) \leq \psi_i(p_A, p'_B, p_{N \setminus N'}), \tag{15}$$

$$\forall i \in N \setminus A, \;\; \psi_i(p'_A, p'_B, p_{N \setminus N'}) \leq \psi_i(p_A, p'_B, p_{N \setminus N'}). \tag{16}$$

For each $i \in B$, if $\psi_i(p_A, p'_B, p_{N \setminus N'}) < \psi_i(p)$, then by (16), $\psi_i(p'_A, p'_B, p_{N \setminus N'}) < \psi_i(p)$, which contradicts (10). Hence for each $i \in B$, $\psi_i(p) \leq \psi_i(p_A, p'_B, p_{N \setminus N'})$, and so by (15),

$$\forall i \in N, \;\; \psi_i(p) = \psi_i(p_A, p'_B, p_{N \setminus N'}). \tag{17}$$

By (16) and (17),

$$\forall i \in B, \;\; \psi_i(p'_A, p'_B, p_{N \setminus N'}) \leq \psi_i(p).$$

However, this contradicts (13) and (14). $\square$

Proof of Theorem 1. By Lemma 5, $PNash(\psi, R^0) \subseteq \{U(p^0)\}$. By Lemma 6, $\emptyset \neq \{U(p^0)\} \subseteq SNash(\psi, R^0)$. Since $SNash(\psi, R^0) \subseteq PNash(\psi, R^0)$ by *efficiency*,

$$\emptyset \neq SNash(\psi, R^0) = PNash(\psi, R^0) = \{U(p^0)\}.$$

$\square$

Lemma 7. *If ψ satisfies efficiency, strict own-peak monotonicity, others-oriented peak monotonicity, peak continuity, and peak order preservation, then*

$$PNash(\psi, R^0) = Nash(\psi, R^0).$$

Proof. It suffices to show that for each $p \in Nash_e(\psi, R^0)$ and each $i \in N$, $\psi_i(p) \leq p_i^0$. Suppose, on the contrary, that there is $j \in N$ such that $p_j^0 < \psi_j(p)$. Since $\Omega \leq \sum_{i \in N} p_i^0$, there is $k \in N$ such that $\psi_k(p) < p_k^0$. Note that $0 < \psi_j(p)$ and $0 < p_k^0$. If $0 < p_j$, then by *strict own-peak monotonicity* and *peak continuity*, there exists $p'_j \in (0, p_j)$ such that $\psi_j(p'_j, p_{-j}) \; P_j^0 \; \psi_j(p)$, a contradiction to $p \in Nash_e(\psi, R^0)$. Hence, $p_j = 0 < \psi_j(p)$. This fact and *efficiency* of ψ together imply $p_k \leq \psi_k(p)$. Summarizing,

$$0 \leq p_k \leq \psi_k(p) < p_k^0 \leq \Omega. \tag{18}$$

If $0 < \psi_k(p)$, then by *strict own-peak monotonicity* and *peak continuity*, k could profitably deviate by announcing some $p'_k \in (p_k, \Omega]$, a contradiction. Thus,

$$0 = p_k = \psi_k(p). \tag{19}$$

Note that, by *symmetry* and *others-oriented peak monotonicity*,

$$\frac{\Omega}{n} \leq \psi_k(\Omega_k, p_{-k}). \tag{20}$$

By (18), (19), and (20), $\psi_k(p) = 0 < \frac{\Omega}{n} \leq \psi_k(\Omega_k, p_{-k})$ and $p_k = 0 < p_k^0$. Therefore, by *peak continuity*, there exists $p'_k \in (0, \Omega)$ such that $\psi_k(p'_k, p_{-k}) \; P_k^0 \; \psi_k(p)$, a contradiction to $p \in Nash_e(\psi, R^0)$. $\square$

Proof of Theorem 2. Implied by Theorem 1 and Lemma 7. $\square$

6.3 Proofs of Propositions 1 and 2

Proof of Proposition 1. Let $R^0 \in R^N$ be any (true) preference profile.

(i) If $p^0 = \mathbf{0}$, then obviously $\mathbf{0} \in \mathit{Nash}_e(\mathit{Pro}, R^0)$. Consider the case that $p^0 \neq \mathbf{0}$. We only deal with the subcase $\Omega \leq \sum_{i \in N} p_i^0$, since the other subcase can be parallely shown. Let $N_1 \equiv \{i \in N : U_i(R^0) < p_i^0\}$ and $N_2 \equiv \{i \in N : U_i(R^0) = p_i^0\}$. For each $i \in N_1$, let $p_1 \equiv \Omega$. We define $(p_i)_{i \in N_2}$ so as to satisfy

$$p_i^0 = \frac{p_i}{|N_1|\Omega + \sum_{j \in N_2} p_j}\Omega.$$

The existence of such $(p_i)_{i \in N_2}$ can be established by a fixed-point argument, as we did at Step 2 in the proof of Lemma 6. We omit the easy proof that $p \equiv ((p_i)_{i \in N_1}, (p_i)_{i \in N_2}) \in \mathit{SNash}_e(\mathit{Pro}, R^0)$.

(ii) This can be shown using Steps 3 and 4 in the proof of Lemma 6.

(iii) If $\mathbf{0} \in \mathit{Nash}_e(\mathit{Pro}, R^0)$, then by definition of Nash equilibrium, for each $i \in N$ and $p_i' > 0$, $0 = \mathit{Pro}_i(\mathbf{0})\ R_i^0\ \mathit{Pro}_i(p_i', 0, \ldots, 0) = \Omega$. Conversely, if for each $i \in N$, $0\ R_i^0\ \Omega$, then for each $i \in N$ and each $p_i' > 0$, $0 = \mathit{Pro}_i(\mathbf{0})\ R_i^0\ \mathit{Pro}_i(p_i', 0, \ldots, 0) = \Omega$. Thus, $\mathbf{0} \in \mathit{Nash}_e(\mathit{Pro}, R^0)$. $\square$

Proof of Proposition 2. Let $R^0 \in R^N$ be any (true) preference profile.

Step 1: $\{U(R^0)\} \subseteq \mathit{SNash}(\mathit{Eps}, R^0)$. Let $x \equiv U(R^0)$. We show that there is $R \in R^N$ such that $x = \mathit{Eps}(R)$ and $R \in \mathit{SNash}_e(\mathit{Eps}, R^0)$.

If $\Omega = \sum_{i \in N} p_i^0$, then obviously, $x = \mathit{Eps}(R^0)$ and $R^0 \in \mathit{SNash}_e(\mathit{Eps}, R^0)$. We consider the case with $\Omega < \sum_{i \in N} p_i^0$. Let $\bar{x} \equiv \max_{i \in N} x_i$. Note that $\Omega < \sum_{i \in N} p_i^0$ implies $\bar{x} < \Omega$. For each $i \in N$, let $R_i \in R$ be such that

$$x_i < p_i \text{ and } x_i\ I_i\ \Omega + x_i - \bar{x}.$$

Then, for each $i \in N$, $d(R_i, x_i) = \Omega - \bar{x}$, and hence $x = \mathit{Eps}(R)$. We shall verify $R \in \mathit{SNash}_e(\mathit{Eps}, R^0)$.

Suppose, by contradiction, that there is $S \subseteq N$ and $R_S' \equiv (R_i')_{i \in S} \in R^S$ such that, letting $y \equiv \mathit{Eps}(R_S', R_{N \setminus S})$,

$$\begin{aligned} &\forall i \in S, \quad y_i\ R_i^0\ x_i, \\ &\exists j \in S, \quad y_j\ P_j^0\ x_j. \end{aligned} \tag{21}$$

Since $x \in P(R^0)$, by (21), for each $i \in S \setminus \{j\}$, $y_i \geq x_i$, and $y_j > x_j = \bar{x}$. Hence, there is $k \in N \setminus S$ with $y_k < x_k$, and then $y_k < x_k \leq p_k$ by $x \in P(R)$. Since $y \in P(R_S', R_{N \setminus S})$ and $y_k < p_k$, we have $y_j \leq p_j'$. Summarizing the obtained facts on agents j, k,

$$\bar{x} = x_j < y_j \leq p_j' \text{ and } y_k < x_k \leq p_k. \tag{22}$$

Then (22) and the definition of R_k imply

$$d(R_j', y_j) \leq \Omega - y_j < \Omega - \bar{x} = d(R_k, x_k) < d(R_k, y_k),$$

but $d(R_j', y_j) < d(R_k, y_k)$ with $y_j > 0$ contradicts $y \in \mathit{Eps}(R_S', R_{N \setminus S})$.

The proof for the case $\sum_{i\in N} p_i < \Omega$ is similar.

Step 2: $\boldsymbol{Nash(Eps, R^0) \subseteq \{U(R^0\}}$. Let $R \in Nash_e(Eps, R^0)$ and $x \equiv Eps(R)$. We shall show that $x = U(R^0)$. We first give three facts.

Fact 1: For each $\boldsymbol{i \in N}$ with $\boldsymbol{x_i = 0}$, $\boldsymbol{p_i^0 = 0}$. Consider any $i \in N$ with $x_i = 0$. Then, i can receive a positive amount by reporting R'_i such that $p'_i = p_i$ and $\Omega\ P'_i\ 0$, and hence $R \in Nash_e(Eps, R^0)$ implies $p_i^0 = 0$.

Fact 2: For each $\boldsymbol{i \in N}$ with $\boldsymbol{x_i < p_i^0}$, $\boldsymbol{x_i < p_i}$ and $\boldsymbol{\Omega\ R_i\ x_i}$. Consider any $i \in N$ with $x_i < p_i^0$. If $p_i \leq x_i$, then i can increase the amount he gets by reporting R'_i such that $p'_i = p_i$ and $d(R'_i, x_i) < d(R_i, x_i)$, a contradiction to $R \in Nash_e(Eps, R^0)$. Therefore, $x_i \leq p_i$.

If $x_i\ P_i\ \Omega$, then i can increase the amount he gets by reporting R'_i such that $p'_i = p_i$ and $d(R'_i, x_i) > d(R_i, x_i)$, a contradiction to $R \in Nash_e(Eps, R^0)$. Therefore, $\Omega\ R_i\ x_i$.

Fact 3: For each $\boldsymbol{i \in N}$ with $\boldsymbol{p_i^0 < x_i}$, $\boldsymbol{p_i < x_i}$ and $\boldsymbol{0\ R_i\ x_i}$. Similarly shown as Fact 2.

Case 1. $\boldsymbol{\sum_{i\in N} p_i^0 < \Omega}$.

Claim 1-1: For each $\boldsymbol{i \in N}$, $\boldsymbol{x_i \leq p_i}$. Since $\Omega < \sum_{i\in N} p_i^0$, there exists $j \in N$ such that $x_j < p_j^0$. By Fact 2, $x_j < p_j$, and so by $x \in P(R)$, for each $i \in N$, $x_i \leq p_i$.

Claim 1-2: There exists $\boldsymbol{\lambda > 0}$ such that for each $\boldsymbol{i \in N}$ with $\boldsymbol{x_i < p_i^0}$, $\boldsymbol{x_i = \lambda}$. Confider any pair $i, j \in N$ with $x_i < p_i^0$ and $x_j < p_j^0$. By Fact 1, $x_i > 0$ and $x_j > 0$, and so $d(R_i, x_i) = d(R_j, x_j)$. Thus by Fact 2, $\Omega - x_i = d(R_i, x_i) = d(R_j, x_j) = \Omega - x_j$, so that $x_i = x_j$. Hence, there is $\lambda > 0$ such that for each $i \in N$, $x_i = \lambda$.

Claim 1-3: For each $\boldsymbol{i \in N}$ with $\boldsymbol{x_i = p_i^0}$, $\boldsymbol{p_i^0 \leq \lambda}$. Suppose, by contradiction, that there is $i \in N$ such that $x_i = p_i^0$ and $\lambda < p_i^0$. By Claim 1-1, $\lambda < p_i^0 = x_i \leq p_i$. This means that $d(R_i, x_i) < \Omega - \lambda = d(R_j, x_j)$ for j with $x_j = \lambda$, a contradiction.

Claim 1-4: $\boldsymbol{U(R^0) = x}$. By Claims 1-2 and 1-3, for each $i \in N$, $x_i = \min\{p_i^0, \lambda\}$. Thus $U(R^0) = x$.

Case 2. $\boldsymbol{\sum_{i\in N} p_i^0 = \Omega}$. If there is $i \in N$ with $x_i < p_i^0$, then there is $j \in N$ with $p_j^0 < x_j$, but Facts 2 and 3 imply $x_i < p_i$ and $p_j < x_j$, a contradiction to *efficiency*. Thus for each $i \in N$, $p_i^0 \leq x_i$, which implies $p^0 = x$.

Case 3. $\boldsymbol{\sum_{i\in N} p_i^0 < \Omega}$. The proof is similar to Case 1. □

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