Bilateral Trading with Naive Traders*

Rene Saran†

Maastricht University, Maastricht, The Netherlands

Abstract

We introduce naive traders in bilateral trading. These traders report their true types in direct mechanisms and bid/ask their values/costs in auctions. We show that by expropriating naive traders in direct mechanisms, the mechanism designer can subsidize additional trades by strategic traders and improve efficiency ex-post compared to when both traders are surely strategic. In fact, complete expropriation of naive traders is a necessary condition for constrained efficiency. A significant implication is that in the presence of naive traders, ex-post individually rational mechanisms are not constrained efficient. In contrast, if the traders use fixed mechanisms, then efficiency can decrease when naive traders are introduced.

Keywords: Bilateral Trading; Naive Traders; Efficiency; Ex-post Individual Rationality

JEL: C78; D82

1 Introduction

Asymmetric information generates inefficiencies in bilateral trading (Myerson and Satterthwaite (1983)). Private information is misrepresented by strategic traders and therefore, sometimes traders don't trade even though gains from trade exist. However, there is evidence that people do not always misrepresent private information. For instance, although

*This paper is based on Chapter 3 of my Ph.D dissertation (Brown University, 2007) that was circulated earlier under the titles "In Bargaining we Trust" and "Commitment to Honesty in Bilateral Trading". I am indebted to my advisor Roberto Serrano for his constant guidance. I also thank Pedro Dal Bó, Allan Feldman, Glenn Loury, Sergio Turner, seminar audience at Brown, Maastricht, 17th International Conference on Game Theory (Stony Brook), LAMES 2006 (Mexico City) and SED 2008 (Ann Arbor) for their suggestions.

†*Email address:* r.saran@maastrichtuniversity.nl; *Tel:* +31-43-3883763; *Fax:* +31-43-3884878

consumers are extremely concerned about protecting their privacy, they willingly reveal personal information on the internet (Spiekermann et al. (2001)). Valley et al. (2002) and McGinn et al. (2003) find subjects revealing their private values/costs during preplay communication and submitting bids/asks equal to their values/costs in double auctions.[1] With this motivation, we study a variation of the bilateral trading problem à la Myerson and Satterthwaite (1983) in which the traders can be naive with a positive probability.

A naive trader does not misrepresent her private information lest she loose the opportunity to trade (aversion to lying and bounded rationality are some of the other reasons that can account for such behavior; see Severinov and Deneckere (2006)). We formalize this behavioral description by making the following assumptions about the strategies of the naive traders. In direct mechanisms, we assume that the naive trader reports her true type. In k-double and buyer's bid auctions, we assume that the naive buyer bids her value.[2] Similarly, the naive seller asks her cost in k-double and seller's ask auctions. In posted-price mechanisms (Hagerty and Rogerson (1987)) and seller's ask auctions, the naive buyer accepts the price/ask if and only if it is at most equal to her value. Likewise, in posted-price mechanisms and buyer's bid auctions, the naive seller accepts the price/bid if and only if it is at least equal to her cost. Finally, we assume that the naive traders possess limited rationality: if they can access their outside option, they never choose a strategy that is worse than their outside option.

Our first set of results concern the situation when the mechanism designer has the flexibility to tailor the mechanism to ϵ, the probability of naive types. In this scenario, the presence of naive traders improves efficiency *ex-post*. For any allocation rule that is incentive compatible and interim individually rational when $\epsilon = 0$, there exists an incentive compatible and interim individually rational direct mechanism when $\epsilon > 0$ such that the latter mechanism generates weakly higher ex-post gains from trade and if the supports of the traders' value and cost intervals overlap, the ex-post gains are strictly higher over an event of positive measure. This is achieved by expropriating surplus from the naive traders and using it to subsidize additional trades by the strategic traders. Indeed, when ϵ is sufficiently large, the designer

[1] For instance, McGinn et al. (2003) find over 50% of individual communications in which subjects revealed their values/costs when they were allowed to communicate before bargaining. They also find subjects bidding/asking their values/costs both when allowed to communicate (30% of the individual bids/asks) and when not allowed to communicate (44% of the individual bids/asks) before playing the $\frac{1}{2}$-double auction.

[2] k-double auctions have been extensively studied both theoretically and experimentally. See Chatterjee and Samuelson (1983), Williams (1987), Farrell and Gibbons (1989), Mathews and Postlewaite (1989), Leininger et al. (1989), Satterthwaite and Williams (1989), Radner and Schotter (1989), Gresik (1991b), Valley et al. (2002) and McGinn et al. (2003).

can construct incentive compatible, interim individually rational and ex-post efficient direct mechanisms.

We show that complete expropriation of the naive traders is in fact a necessary condition for constrained efficiency over the range of ϵ where ex-post efficiency is unattainable. This necessity of expropriating the naive traders has a significant implication for a large class of mechanisms that generate *ex-post* individually rational allocation rules in equilibrium, e.g., buyer's bid auction, seller's ask auction, k-double auctions, posted-price mechanisms, veto mechanisms (Mathews and Postlewaite (1989), Forges (1999), Compte and Jehiel (2009)). Myerson and Satterthwaite (1983) prove that the linear-equilibrium of the $\frac{1}{2}$-double auction, which is ex-post individually rational, is also constrained efficient when $\epsilon = 0$ and the traders' value and cost are distributed uniformly on the same interval. Gresik (1991a) generalizes this result to show that for a large class of distributions, there exist incentive compatible allocation rules that are both ex-post individually rational and constrained efficient when $\epsilon = 0$ and the value and cost are distributed on the same interval. He concludes that in these environments, imposing the stronger ex-post rationality condition generates no welfare loss. In a sharp contrast, we show that for all $\epsilon > 0$, any incentive compatible, ex-post individually rational allocation rule is *not* constrained efficient whenever the traders' value and cost are distributed over the same interval.

We next consider the situation in which the traders use a *given* mechanism irrespective of the value of ϵ. We provide three examples to show that in this case, the introduction of naive traders could increase, have no impact or decrease efficiency. In the buyer's bid (seller's ask) auction, efficiency increases when $\epsilon > 0$ since the naive buyer (seller) bids (asks) her value. In the posted-price mechanism, the introduction of naive traders has no impact on efficiency since for any ϵ, trade occurs if and only if the posted price is in between the buyer's value and seller's cost. In these mechanisms, the introduction of naive traders does not alter the behavior of the strategic traders. However, in k-double auctions, the presence of naive traders can change the strategic incentives in a way that reduces efficiency. We provide an example in which every k-double auction has an ex-post efficient equilibrium outcome when $\epsilon = 0$. However, it is impossible to attain ex-post efficiency in any equilibrium outcome of any k-double auction when $\epsilon > 0$. The reason is the weighted-average pricing rule, which induces the strategic buyer (seller) to capitalize on the low asks (high bids) of the naive seller (buyer) by increasing the amount of exaggeration (shading).

In a related paper, Saran (2009) shows that the presence of traders who naively announce their true type during preplay communication improves efficiency in double auctions. Earlier,

Severinov and Deneckere (2006) introduced consumers who honestly report their values in the monopolistic screening problem and showed that in this case, the take-it-or-leave-it offer is not optimal for the seller. Erard and Feinstein (1994) assume that a proportion of taxpayers always honestly report their incomes to the tax authority. In contrast, Alger and Renault (2006) and Alger and Renault (2007) study conditional honesty: in the former, a player is honest only if the probability of lying in equilibrium is not sufficiently high while in the latter, a player is honest only if she has pre-committed to act in that manner.

The rest of the paper is organized as follows. Section 2 outlines the bilateral trading problem and the necessary conditions implied by incentive and interim individual rationality constraints. Section 3 characterizes the set of incentive compatible and interim individually rational direct mechanisms. Section 4 shows that the presence of naive traders is beneficial for efficiency when the designer is free to choose a mechanism and provides the necessary conditions for constrained efficiency. Section 5 proves the constrained inefficiency of ex-post individually rational allocation rules. Section 6 shows that efficiency can increase, remain constant or decrease when naive traders are introduced in a given mechanism. Section 7 provides a brief conclusion and Section 8 collects the proofs.

2 Bilateral Trading Problem

A pair of risk-neutral traders, buyer b and seller s, engage in a trading mechanism to trade an indivisible object. Each trader i has two disposition types (denoted by t_i), strategic (denoted by 1) and naive (denoted by 2). The probability that a trader is naive is $\epsilon < 1$, which is independent of the other trader's value and disposition.[3] The buyer's value and the seller's cost are denoted by, respectively, v and c. The values of both disposition types of the buyer are distributed as F_b on the interval $[\underline{v}, \bar{v}]$ independently of the seller's cost and disposition. Similarly, the costs of both disposition types of the seller are distributed as F_s on the interval $[\underline{c}, \bar{c}]$ independently of the buyer's value and disposition. The corresponding density functions f_b and f_s are continuous and positive on their respective domains. To avoid trivialities, we assume that $\bar{v} > \underline{c}$. The outside option for any type of any trader is 0. A trader's type, value/cost and disposition, is her private information. All other information is common knowledge.

We make the following assumptions about the naive traders.

[3]The case $\epsilon = 1$ is uninteresting under our assumptions about the naive traders. The results are robust to different probabilities of naive traders for the buyer and seller.

Assumption 2.1. *Naive Trader* (see Sections 3 and 6 for descriptions of these mechanisms)

1. In a direct mechanism, a naive trader reports her true type (value/cost and disposition).[4]

2. In a k-double auction, the naive buyer (seller) bids (asks) her value (cost).

3. In a buyer's bid auction, the naive buyer bids her value whereas the naive seller accepts to trade if and only if the bid is at least equal to her cost. Similarly, in a seller's ask auction, the naive seller asks her cost whereas the naive buyer accepts to trade if and only if the ask is at most equal to her value.

4. In a posted-price mechanism, a naive buyer (seller) accepts to trade if and only if the price is at most (least) equal to her value (cost).

5. If at any stage of any mechanism, a naive trader can access her outside option, then at that stage she does not choose any strategy that gives her a negative continuation payoff.

Bayesian-Nash equilibrium outcomes of trading mechanisms are allocation rules. An *allocation rule* is a pair of mappings (p, x) defined on the type space such that $p(v, t_b, c, t_s) \in [0, 1]$ is the probability that the object is traded and $x(v, t_b, c, t_s) \in \Re$ is the expected payment from the buyer to the seller when the traders' types are (v, t_b) and (c, t_s). When $\epsilon = 0$, it is simpler to describe the allocation rule (p, x) by mappings (p^0, x^0) over $[\underline{v}, \bar{v}] \times [\underline{c}, \bar{c}]$ such that $p^0(v, c) = p(x, 1, c, 1)$ and $x^0(v, c) = x(v, 1, c, 1)$ for all (v, c).

For any allocation rule, $\bar{p}_b(v, t_b)$ is the probability that the buyer gets the object and $\bar{x}_b(v, t_b)$ is the buyer's expected payment when the buyer's type is (v, t_b). Precisely,

$$\bar{p}_b(v, t_b) = \int_{\underline{c}}^{\bar{c}} \left[(1-\epsilon)\, p(v, t_b, c, 1) + \epsilon\, p(v, t_b, c, 2)\right] f_s(c) dc,$$
$$\bar{x}_b(v, t_b) = \int_{\underline{c}}^{\bar{c}} \left[(1-\epsilon)\, x(v, t_b, c, 1) + \epsilon\, x(v, t_b, c, 2)\right] f_s(c) dc.$$

Similarly, define $\bar{p}_s(c, t_s)$, the probability that the seller looses the object, and $\bar{x}_s(c, t_s)$, the seller's expected payment, when the seller's type is (c, t_s). The expected payoffs are

[4] It is necessary for the revelation principle that the naive traders truthfully report *both* their value/cost and disposition in direct mechanisms. If the naive traders report only their true value/cost but can lie about their disposition in the direct mechanisms, then "password mechanisms" will be constrained efficient instead of the direct mechanisms (see Severinov and Deneckere (2006)). All other results will not change.

$U_b(v,t_b) = v\,\bar{p}_b(v,t_b) - \bar{x}_b(v,t_b)$ for the buyer of type (v,t_b) and $U_s(c,t_s) = \bar{x}_s(c,t_s) - c\,\bar{p}_s(c,t_s)$ for the seller of type (c,t_s).

Allocation rules obtained in equilibria of trading mechanisms satisfy the following incentive and participation constraints.

Definition 2.2. An allocation rule is *incentive compatible for the strategic traders* (IC) if:

(i) IC1: $\begin{array}{l} U_b(v,1) \geq v\,\bar{p}_b(v',1) - \bar{x}_b(v',1),\ \forall v,v' \in [\underline{v},\bar{v}], \\ U_s(c,1) \geq \bar{x}_s(c',1) - c\,\bar{p}_s(c',1),\ \forall c,c' \in [\underline{c},\bar{c}]. \end{array}$

(ii) if $\epsilon > 0$, then IC2: $\begin{array}{l} U_b(v,1) \geq v\,\bar{p}_b(v',2) - \bar{x}_b(v',2),\ \forall v,v' \in [\underline{v},\bar{v}], \\ U_s(c,1) \geq \bar{x}_s(c',2) - c\,\bar{p}_s(c',2),\ \forall c,c' \in [\underline{c},\bar{c}]. \end{array}$

IC1 constraints imply that the strategic traders do not gain by imitating the strategies of other strategic traders. The second set of incentive constraints, IC2, are present whenever $\epsilon > 0$ and imply that the strategic traders do not gain by imitating the naive traders.

Finally, voluntary participation imposes the interim individual rationality constraints. These constraints must also hold for the naive traders because of Assumption 2.1(5).

Definition 2.3. An allocation rule is *interim individually rational* (IIR) if:

(i) $U_b(v,1) \geq 0,\ \forall v \in [\underline{v},\bar{v}]$, and $U_s(c,1) \geq 0,\ \forall c \in [\underline{c},\bar{c}]$.

(ii) if $\epsilon > 0$, then $U_b(v,2) \geq 0,\ \forall v \in [\underline{v},\bar{v}]$, and $U_s(c,2) \geq 0,\ \forall c \in [\underline{c},\bar{c}]$.

2.1 Necessary Conditions for IC and IIR Allocation Rules

Before we go on, we note the necessary conditions implied by IC and IIR constraints. For any v, let $\alpha(v) = v - \frac{1-F_b(v)}{f_b(v)}$ and for any c, let $\beta(c) = c + \frac{F_s(c)}{f_s(c)}$. Given p, define

$$\begin{aligned}\xi(p) = &\int_{\underline{v}}^{\bar{v}} \int_{\underline{c}}^{\bar{c}} \Big[(1-\epsilon)^2 \left(\alpha(v) - \beta(c)\right) p(v,1,c,1) + \epsilon^2 (v-c) p(v,2,c,2)\Big] f_s(c) f_b(v) dc dv \\ &+ \epsilon(1-\epsilon) \int_{\underline{v}}^{\bar{v}} \int_{\underline{c}}^{\bar{c}} \Big[\left(\alpha(v) - c\right) p(v,1,c,2) + \left(v - \beta(c)\right) p(v,2,c,1)\Big] f_s(c) f_b(v) dc dv\end{aligned}$$

Lemma 2.4. *An allocation rule satisfies IC1 only if $\bar{p}_b(v,1)$ is weakly increasing in v, $\bar{p}_s(c,1)$ is weakly decreasing in c, $U_b(v,1) = U_b(\underline{v},1) + \int_{\underline{v}}^{v} \bar{p}_b(y,1)dy$, $U_s(c,1) = U_s(\bar{c},1) + \int_{c}^{\bar{c}} \bar{p}_s(y,1)dy$ and*

$$\xi(p) = (1-\epsilon)\Big(U_b(\underline{v},1) + U_s(\bar{c},1)\Big) + \epsilon \left(\int_{\underline{v}}^{\bar{v}} U_b(v,2) f_b(v) dv + \int_{\underline{c}}^{\bar{c}} U_s(c,2) f_s(c) dc \right). \quad (1)$$

Thus, the standard implications of incentive constraints are also true in our model: the probability that the strategic buyer (seller) gets (looses) the object is weakly increasing (decreasing) in her value (cost) and the strategic traders earn information rents ($\int_{\underline{v}}^{v} \bar{p}_b(y,1)dy$ for the strategic buyer of value v and $\int_{c}^{\bar{c}} \bar{p}_s(y,1)dy$ for the strategic seller of cost c). Equation (1) is simply an accounting identity, with $\xi(p)$ being the net surplus of the allocation rule after paying the strategic traders the information rents necessary to satisfy IC1 constraints.

Given p, define

$$\delta_b(p) = \chi \sup_{v,v' \in [\underline{v},\bar{v}]} \left((v - v')\bar{p}_b(v',2) - \int_{\underline{v}}^{v} \bar{p}_b(y,1)dy \right) \geq 0$$
$$\delta_s(p) = \chi \sup_{c,c' \in [\underline{c},\bar{c}]} \left((c' - c)\bar{p}_s(c',2) - \int_{c}^{\bar{c}} \bar{p}_s(y,1)dy \right) \geq 0,$$

where χ is the indicator function with values 1 when $\epsilon > 0$ and 0 when $\epsilon = 0$.[5]

Lemma 2.5. *An IC and IIR allocation rule has $U_b(\underline{v},1) \geq \delta_b(p)$ and $U_s(\bar{c},1) \geq \delta_s(p)$.*

For any IC1 allocation rule, we can satisfy IIR by having $U_b(\underline{v},1) \geq 0$ and $U_s(\bar{c},1) \geq 0$. If we want this allocation rule to also satisfy IC2, then the above lemma says that we must have $U_b(\underline{v},1) \geq \delta_b(p)$ and $U_s(\bar{c},1) \geq \delta_s(p)$. Thus, $\delta_b(p)$ and $\delta_s(p)$ are the minimum information rents that the strategic traders must earn in order to satisfy IC2 constraints. To see this, consider the buyer. IIR implies that the expected payment of the naive buyer of value v' is at most $v'\bar{p}_b(v',2)$. Hence, IC2 constraints imply that $U_b(v,1) \geq (v-v')\bar{p}_b(v',2)$ for all $v, v' \in [\underline{v},\bar{v}]$. The inequality in the lemma follows by substituting $U_b(v,1) = U_b(\underline{v},1) + \int_{\underline{v}}^{v} \bar{p}_b(y,1)dy$.

We close this section with some definitions that are used often in the paper.

Definition 2.6. An allocation rule is *ex-post individually rational* (EIR) if $\forall(t_b,t_s)$ that have a positive probability, the following holds $\forall(v,c) \in [\underline{v},\bar{v}] \times [\underline{c},\bar{c}]$:

$$v\,p(v,t_b,c,t_s) - x(v,t_b,c,t_s) \geq 0 \text{ and } x(v,t_b,c,t_s) - c\,p(v,t_b,c,t_s) \geq 0.$$

Definition 2.7. An allocation rule is *ex-post efficient* if $\forall(t_b,t_s)$ that have a positive prob-

[5] $\delta_b(p)$ and $\delta_s(p)$ are well-defined because the terms inside the brackets are bounded. If $v = v' = \underline{v}$, then $(v - v')\bar{p}_b(v',2) - \int_{\underline{v}}^{v} \bar{p}_b(y,1)dy = 0$ and if $c = c' = \bar{c}$, then $(c' - c)\bar{p}_s(c',2) - \int_{c}^{\bar{c}} \bar{p}_s(y,1)dy = 0$. Hence, $\delta_b(p), \delta_s(p) \geq 0$.

ability, the following holds $\forall(v,c) \in [\underline{v}, \bar{v}] \times [\underline{c}, \bar{c}]$:

$$p(v, t_b, c, t_s) = \begin{cases} 1 & \text{if } v \geq c \\ 0 & \text{if } v < c. \end{cases}$$

If the above holds for *almost all* (v,c), then the allocation rule is *almost surely ex-post efficient.*

Definition 2.8. An allocation rule (p,x) is *constrained efficient* if it satisfies IC and IIR and there does not exist any other IC and IIR allocation rule $(\hat{p}, \hat{x})$ such that

(i) $\forall(t_b, t_s)$ that have a positive probability,

$$(v-c)\hat{p}(v, t_b, c, t_s) \geq (v-c)p(v, t_b, c, t_s), \forall(v,c) \in [\underline{v}, \bar{v}] \times [\underline{c}, \bar{c}].$$

(ii) For some (t_b, t_s) that has a positive probability, there exists a set $R \subseteq [\underline{v}, \bar{v}] \times [\underline{c}, \bar{c}]$ of positive measure such that

$$(v-c)\hat{p}(v, t_b, c, t_s) > (v-c)p(v, t_b, c, t_s), \forall(v,c) \in R.$$

Thus, an allocation rule is constrained efficient if it is IC and IIR, and there does not exist any IC and IIR allocation rule with weakly higher ex-post gains from trade in every event and strictly higher ex-post gains from trade in some event of positive measure. This constrained efficiency criterion is weak as it focuses on ex-post gains from trade; in particular, it includes IC and IIR allocation rules that maximize the ex-ante gains from trade.[6]

3 Direct Mechanisms

A *direct mechanism* is such that the traders simultaneously report their types and for each pair of reports, the allocation is defined by some (p,x). Our interest in the direct mechanisms stems from the revelation principle, which states that for any Bayesian-Nash equilibrium outcome of any trading mechanism there exits an outcome-equivalent IC and IIR direct mechanism. The revelation principle holds since any Bayesian-Nash equilibrium outcome of any trading mechanism must satisfy IC and IIR, and the naive traders report their true types in direct mechanisms.

[6]There are many different constrained efficiency criteria; see Holmström and Myerson (1983). Our criterion is stronger than ex-post incentive efficiency defined in Holmström and Myerson (1983).

We next characterize the set of IC and IIR direct mechanisms.

Proposition 3.1. *For any p, there exists a x such that (p, x) is a direct mechanism that satisfies IC and IIR if and only if $\bar{p}_b(v, 1)$ is weakly increasing in v, $\bar{p}_s(c, 1)$ is weakly decreasing in c and*

$$\xi(p) \geq (1-\epsilon)\Big(\delta_b(p) + \delta_s(p)\Big) \tag{2}$$

To prove the "if" part, we construct a payment function x that extracts full surplus from the naive traders and transfers them to the strategic traders. The "only if" part follows from Lemmas 2.4 and 2.5, and IIR for the naive traders. For the case when $\epsilon = 0$, the above proposition is the same as Theorem 1 in Myerson and Satterthwaite (1983). Thus, the only additional requirement when $\epsilon > 0$ is that $\xi(p)$, the net surplus of the mechanism after paying the strategic traders the information rents necessary to satisfy IC1 constraints, must be sufficient to pay the strategic traders the minimum information rents necessary to satisfy IC2 constraints; otherwise, the mechanism will not satisfy IIR.

4 Efficiency when Mechanisms can be Optimally Designed

In this section, we assume that the mechanism designer has the flexibility to design mechanisms for the traders. The next proposition shows that by appropriately designing direct mechanisms, the mechanism designer can increase efficiency ex-post when $\epsilon > 0$ compared to when $\epsilon = 0$.

Proposition 4.1. *Suppose $(\underline{v}, \bar{v}) \cap (\underline{c}, \bar{c}) \neq \emptyset$. Let (p^0, x^0) be an IC and IIR allocation rule when $\epsilon = 0$. For any $\epsilon > 0$, there exists an IC and IIR direct mechanism (p, x) such that*

1. *$\forall (t_b, t_s)$, we have*

$$(v-c)p(v, t_b, c, t_s) \geq (v-c)p^0(v, c), \forall (v, c) \in [\underline{v}, \bar{v}] \times [\underline{c}, \bar{c}].$$

2. *For some (t_b, t_s), there exists a set $R \subseteq [\underline{v}, \bar{v}] \times [\underline{c}, \bar{c}]$ of positive measure such that*

$$(v-c)p(v, t_b, c, t_s) > (v-c)p^0(v, c), \forall (v, c) \in R.$$

Intuitively, to increase efficiency, we have to offer higher information rents to the strategic traders—this is the standard trade-off between efficiency and information rents. The naive traders provide an endogenous source of funding these rents in direct mechanisms. Since the naive traders honestly reveal their types in direct mechanisms, we can expropriate their surplus to subsidize additional trades by the strategic traders.

Remark 4.2. The following related results are worth noting.

1. If $(\underline{v}, \bar{v}) \cap (\underline{c}, \bar{c}) \neq \emptyset$, then there exists an $\epsilon^* \in (0, 0.5]$ such that there exist IC, IIR and ex-post efficient direct mechanisms if and only if $\epsilon \geq \epsilon^*$. In general, the value of ϵ^* is a function of the distributions (F_b, F_s).[7] However, if $[\underline{v}, \bar{v}] = [\underline{c}, \bar{c}]$, then $\epsilon^* = 0.5$ for any (F_b, F_s).[8]

2. If $(\underline{v}, \bar{v}) \cap (\underline{c}, \bar{c}) = \emptyset$, then it is easy to show that there exist IC, IIR and ex-post efficient direct mechanisms for all ϵ.

In fact, as the following proposition shows, completely expropriating the naive traders is necessary for constrained efficiency of an allocation rule unless the allocation rule is almost surely ex-post efficient.

Proposition 4.3. *If an allocation rule is constrained efficient, then either it is almost surely ex-post efficient or it must be such that* $\epsilon \int_{\underline{v}}^{\bar{v}} U_b(v, 2) f_b(v) dv = \epsilon \int_{\underline{c}}^{\bar{c}} U_s(c, 2) f_s(c) dc = 0$.

In the next section, we apply this necessary condition to show that IC and EIR allocation rules are not constrained efficient.

5 Ex-post Individually Rational Allocation Rules

Several trading mechanisms generate EIR allocation rules in equilibrium.

Example 5.1. *Veto Mechanisms*: In a veto mechanism, a trader has the option to quit the mechanism and opt for her outside option at any stage. Such veto rights are quite common in the real-world bargaining protocols. Any equilibrium outcome of any veto mechanism must be such that in the event the object is traded, the payment from the buyer to the seller must be at least equal to the seller's cost but not exceed the buyer's value; whereas, if trade does not occur, then there should be no payment from the buyer to the seller—this is also

[7] ϵ^* is such that $(2\epsilon^* - 1) \int_{\underline{v}}^{\bar{c}} (1 - F_b(y)) F_s(y) dy + (1 - \epsilon^*) \epsilon^* \left(\int_{\underline{c}}^{\underline{v}} F_s(y) dy + \int_{\bar{c}}^{\bar{v}} (1 - F_b(y)) dy \right) = 0$.

[8] Proof available upon request.

true for the naive traders due to Assumption 2.1(5). Thus, equilibrium outcomes of veto mechanisms satisfy EIR.

Example 5.2. *Posted-price mechanisms*: The mechanism designer posts a price drawn according to some distribution and after observing the price, the traders simultaneously announce whether they accept to trade at that price. If both traders accept, then trade occurs at the posted price; otherwise, there is no trade and no payment.[9] It is a dominant strategy for the buyer (seller) to accept the posted price if and only if it is at most (least) equal to her value (cost). We have assumed that the naive traders also follow this dominant strategy (Assumption 2.1(4)). Thus, the equilibrium of the posted-price mechanism satisfies EIR.[10]

Example 5.3. *Buyer's Bid and Seller's Ask Auctions:* In a buyer's bid auction, the buyer submits a bid for the object. After observing the bid, the seller either accepts or rejects to trade. If the seller accepts, then the object is traded at a price equal to the buyer's bid; otherwise, there is no trade and no payment. The strategic and naive types of the seller accept a bid if and only if it is at least equal to their costs. Hence, for any ϵ, the strategic buyer of value v will bid $z^* \leq v$ such that $z^* \in \arg\max_z \int_{\bar{c}}^{z} (v - z) f_s(c) dc$.[11] Moreover, the naive buyer bids equal to her value (Assumption 2.1(3)). Therefore, the equilibria of the buyer's bid auctions satisfy EIR. Seller's ask auction is defined symmetrically and its equilibria also satisfy EIR.

Example 5.4. *k-Double Auctions*: The buyer and seller simultaneously submit, respectively, a bid and an ask for the object. If the buyer's bid z_b is greater than or equal to the seller's ask z_s, then trade takes place at the price $kz_b + (1-k)z_s$, where $k \in [0,1]$; otherwise, there is no trade and no payment. A naive buyer (seller) submits a bid (ask) equal to her value (cost) (Assumption 2.1(2)). It is without loss of generality to consider only EIR equilibria in which the strategic buyer (seller) bids (asks) at most (least) equal to her value (cost).[12]

[9] Formally, a posted-price mechanism is not a veto mechanism since a trader cannot opt out once she has agreed to trade at the posted-price.

[10] Strictly speaking, a strategic buyer (seller) can accept a price greater (less) than her value (cost) in equilibrium if she surely does not trade at that price. However, any such equilibrium has at most as high ex-post gains from trade in every event than the dominant strategy equilibrium.

[11] To be precise, the strategic buyer of value $v < \underline{c}$ can submit any bid up to $\underline{c}$ in equilibrium. However, for any such equilibrium, there exists another equilibrium that satisfies EIR and generates at least as much ex-post gains from trade in every event as the former equilibrium. The latter equilibrium is obtained by altering the strategy of those strategic types of the buyer who bid more than their values to bid equal to their values.

[12] This is because for any equilibrium that is not EIR, there exists another equilibrium that satisfies EIR

The main result of this section is presented next. Here we assume that the value and cost are distributed on the same interval. Under this assumption, Gresik (1991a) has shown that for a large class of distributions (viz. $\frac{F_b(v)-1}{f_b(v)}$ and $\frac{F_s(c)}{f_s(c)}$ are increasing), there exist IC and EIR allocation rules that are constrained efficient when $\epsilon = 0$. We prove that this result is not robust to the introduction of naive traders.

Proposition 5.5. *Suppose* $[\underline{v}, \bar{v}] = [\underline{c}, \bar{c}]$. *If* $\epsilon \in (0, 1)$, *then any IC and EIR allocation rule is not constrained efficient.*

We use Proposition 4.3 in the proof. Firstly, we show that there does not exist an IC and EIR allocation rule that is almost surely ex-post efficient. In the final step, we show that any IC and EIR allocation rule that extracts full surplus from the naive traders is not constrained efficient.

Remark 5.6. We make two related clarifications.

1. The above result is true whenever the intervals of value and cost overlap and ϵ is not too large. Precisely, if $(\underline{v}, \bar{v}) \cap (\underline{c}, \bar{c}) \neq \emptyset$ and $\epsilon \in (0, \epsilon^*)$, then any IC and EXIR allocation rule is not constrained efficient—recall ϵ^* from Remark 4.2.[13]

2. If $(\underline{v}, \bar{v}) \cap (\underline{c}, \bar{c}) = \emptyset$, then it is easy to show that there exist IC, EIR and ex-post efficient allocation rules for any ϵ.

6 Efficiency when Mechanisms are Fixed

The previous sections assumed that the mechanism designer had the flexibility to tailor the mechanism to the value of ϵ in order to improve efficiency. In the absence of this flexibility, how does the introduction of naive traders impact efficiency of any *given* mechanism? The following three examples show that depending on the mechanism, the presence of naive traders could increase, have no effect or even decrease efficiency.

Example 6.1. Suppose the traders use the buyer's bid auction. The bid of the strategic buyer does not depend on ϵ (see Example 5.3). However, the naive buyer bids equal to her value. Therefore, compared to the equilibrium outcome when $\epsilon = 0$, the equilibrium outcome

and generates at least as much ex-post gains from trade in every event. The latter equilibrium is obtained by altering the strategy of those strategic types of the buyer who bid more than their values to bid equal to their values and those strategic types of the seller who ask less than their costs to ask equal to their costs.

[13] Proof available upon request.

of the buyer's bid auction when $\epsilon > 0$ achieves higher ex-post gains from trade whenever the buyer is naive and the same ex-post gains from trade whenever the buyer is strategic. A similar result holds for the seller's ask auction.

Example 6.2. Suppose the traders use the posted-price mechanism. The presence of naive traders has no effect on efficiency since both strategic and naive types of the buyer (seller) accept to trade at the posted price if and only if it is at most (least) equal to their values (costs).

In the above examples, the introduction of naive traders does not change the incentives of the strategic traders. The strategic traders have dominant strategies in the posted-price mechanisms; whereas in the buyer's bid auction, the strategic buyer's optimal bid is fixed since the seller's strategy does not depend on ϵ. Saran (2009) provides an example of a $\frac{1}{2}$-double auction with preplay communication in which the introduction of the naive traders changes the incentives of the strategic traders in a way that increase efficiency irrespective of the ex-post dispositions of the traders. However, the presence of naive traders can also alter the strategic incentives in a way that reduces efficiency.

Example 6.3. Suppose $\bar{c} = \underline{v} = r$.[14] Traders use a k-double auction as the trading mechanism. A naive buyer (seller) bids (asks) equal to her value (cost).

If $\epsilon = 0$, then for *any* k-double auction, there exists an ex-post efficient equilibrium outcome (each trader's strategy is to bid/ask equal to r in this equilibrium). However, if $\epsilon \in (0, 1)$, then there does not exist *any* equilibrium outcome of *any* k-double auction that is almost surely ex-post efficient (see Section 8 for the proof). Intuitively, if $\epsilon > 0$ and $k > 0$, then the strategic buyer whose value is very close to r prefers to bid less than r to take advantage of the low asks of the naive seller. Although bidding at least r would increase her probability of trade, these additional trades generate very little profit—since her value is very close to r—and she ends up paying a high price on all other trades. Thus, she is better off bidding less than r, resulting in ex-post inefficiency. Similarly, if $\epsilon > 0$ and $k < 1$, then the strategic seller whose value is very close to r asks greater than r to take advantage of the high bids of the naive buyer.

[14] This assumption is not necessary to prove our point. Even if $\bar{c} < \underline{v}$, we can construct examples in which every k-double auction has an ex-post efficient equilibrium outcome when the traders are surely strategic but there does not exist any equilibrium outcome of any k-double auction that is almost surely ex-post efficient when ϵ is high enough.

7 Conclusion

Our results suggest two broad conclusions. Firstly, the results of Section 5 imply that several indirect bargaining mechanisms used in the real world are not constrained efficient. This underscores the importance of using optimally designed direct mechanisms. Secondly, although naive behavior in itself is beneficial for efficiency, the presence of such naiveté alters strategic incentives differently in different mechanisms. As Example 6.3 shows, the consequence could well be a fall in efficiency.

8 Proofs

Proof of Lemma 2.4: Using the same arguments as in Theorem 1 in Myerson and Satterthwaite (1983), we can obtain the results that $\bar{p}_b(v,1)$ is weakly increasing in v, $\bar{p}_s(c,1)$ is weakly decreasing in c, $U_b(v,1) = U_b(\underline{v},1) + \int_{\underline{v}}^{v} \bar{p}_b(y,1)dy$ and $U_s(c,1) = U_s(\bar{c},1) + \int_{c}^{\bar{c}} \bar{p}_s(y,1)dy$. So we only prove (1). Any allocation rule that satisfies IC1 must be such that

$$\begin{aligned}
&\int_{\underline{v}}^{\bar{v}} \int_{\underline{c}}^{\bar{c}} (v-c)\Big[(1-\epsilon)^2 p(v,1,c,1) + \epsilon(1-\epsilon)\big(p(v,2,c,1) + p(v,1,c,2)\big) + \epsilon^2 p(v,2,c,2)\Big] f_s(c) f_b(v) dc dv \\
&= (1-\epsilon)\left(\int_{\underline{v}}^{\bar{v}} U_b(v,1) f_b(v) dv + \int_{\underline{c}}^{\bar{c}} U_s(c,1) f_s(c) dc\right) + \epsilon\left(\int_{\underline{v}}^{\bar{v}} U_b(v,2) f_b(v) dv + \int_{\underline{c}}^{\bar{c}} U_s(c,2) f_s(c) dc\right) \\
&= (1-\epsilon)\left(U_b(\underline{v},1) + \int_{\underline{v}}^{\bar{v}} \int_{\underline{v}}^{v} \bar{p}_b(y,1) f_b(v) dy dv + U_s(\bar{c},1) + \int_{\underline{c}}^{\bar{c}} \int_{c}^{\bar{c}} \bar{p}_s(y,1) f_s(c) dy dc\right) \\
&\quad + \epsilon\left(\int_{\underline{v}}^{\bar{v}} U_b(v,2) f_b(v) dv + \int_{\underline{c}}^{\bar{c}} U_s(c,2) f_s(c) dc\right)
\end{aligned}$$

We obtain (1) by subtracting $(1-\epsilon)\left(\int_{\underline{v}}^{\bar{v}} \bar{p}_b(v,1)(1-F_b(v))dv + \int_{\underline{c}}^{\bar{c}} \bar{p}_s(c,1)F_s(c)dc\right)$ from both sides of the above equation. □

Proof of Lemma 2.5: If $\epsilon = 0$, then $\chi = 0$ and thus, the condition is satisfied because of IIR. If $\epsilon > 0$, then since the allocation rule also satisfies IC, we must have

$$U_b(\underline{v},1) + \int_{\underline{v}}^{v} \bar{p}_b(y,1)dy = U_b(v,1) \geq v\bar{p}_b(v',2) - \bar{x}_b(v',2) \geq v\bar{p}_b(v',2) - v'\bar{p}_b(v',2),\ \forall v, v' \in [\underline{v}, \bar{v}],$$

where the second inequality is because of IIR for the naive buyer. Hence,

$$U_b(\underline{v},1) \geq (v-v')\bar{p}_b(v',2) - \int_{\underline{v}}^{v} \bar{p}_b(y,1)dy,\ \forall v,v' \in [\underline{v},\bar{v}].$$

So the result for the buyer follows. A similar argument works for the seller. □

Proof of Proposition 3.1: It easily follows from Lemmas 2.4 and 2.5, and IIR for the naive types that an IC and IIR direct mechanism satisfies the conditions listed in the proposition.

To prove sufficiency, define

$$\begin{aligned}
x(v,1,c,2) =& \frac{1}{1-\epsilon} c\,\bar{p}_s(c,2); \quad x(v,2,c,1) = \frac{1}{1-\epsilon} v\,\bar{p}_b(v,2); \quad x(v,2,c,2) = 0 \\
x(v,1,c,1) =& \frac{1}{1-\epsilon}\int_{\underline{v}}^{v} y d\bar{p}_b(y,1) + \frac{1}{1-\epsilon}\int_{\underline{c}}^{c} y d\bar{p}_s(y,1) + \frac{1}{1-\epsilon}\underline{v}\,\bar{p}_b(\underline{v},1) - \frac{1}{1-\epsilon}\delta_b(p) \\
& - \frac{1}{1-\epsilon}\int_{\underline{c}}^{\bar{c}} y(1-F_s(y))d\bar{p}_s(y,1) - \frac{\epsilon}{(1-\epsilon)^2}\int_{\underline{c}}^{\bar{c}} y\bar{p}_s(y,2)f_s(y)dy.
\end{aligned}$$

Then $U_b(v,2) = 0, \forall v$ and $U_s(c,2) = 0, \forall c$. Thus, the naive types satisfy their IIR constraints.

Next we check IC1. For all $v' < v$, we have (a similar argument works if $v' > v$)

$$\begin{aligned}
v\,\bar{p}_b(v,1) - \bar{x}_b(v,1) - \Big(v\,\bar{p}_b(v',1) - \bar{x}_b(v',1)\Big) =& v\Big(\bar{p}_b(v,1) - \bar{p}_b(v',1)\Big) - \int_{v'}^{v} y d\bar{p}_b(y,1) \\
=& \int_{v'}^{v} (v-y)d\bar{p}_b(y,1) \geq 0.
\end{aligned}$$

For all $c' > c$, we have (a similar argument works if $c' < c$)

$$\begin{aligned}
\bar{x}_s(c,1) - c\,\bar{p}_s(c,1) - \Big(\bar{x}_s(c',1) - c\,\bar{p}_s(c',1)\Big) =& c\Big(\bar{p}_s(c',1) - \bar{p}_s(c,1)\Big) - \int_{c}^{c'} y d\bar{p}_s(y,1) \\
=& \int_{c}^{c'} (c-y)d\bar{p}_s(y,1) \geq 0.
\end{aligned}$$

It follows from Lemma 2.4 that $U_b(v,1) = U_b(\underline{v},1) + \int_{\underline{v}}^{v} \bar{p}_b(y,1)dy$, $U_s(c,1) = U_s(\bar{c},1) + \int_{c}^{\bar{c}} \bar{p}_s(y,1)dy$ and the constructed mechanism satisfies (1). Since $U_b(\underline{v},1) = \delta_b(p) \geq 0$, we have $U_b(v,1) \geq 0$, $\forall v$. Moreover, $U_b(v,2) = 0$, $\forall v$, $U_s(c,2) = 0$, $\forall c$, and p satisfies (2). Hence, $U_s(\bar{c},1) \geq \delta_s(p) \geq 0$. Therefore, $U_s(c,1) \geq 0$, $\forall c$. So the mechanism satisfies IIR for the strategic types.

Finally, if $\epsilon > 0$, then $\forall v,v' \in [\underline{v},\bar{v}]$, $U_b(v,1) = \delta_b(p) + \int_{\underline{v}}^{v} \bar{p}_b(y,1)dy \geq (v-v')\bar{p}_b(v',2)$, where the inequality follows from the definition of $\delta_b(p)$. Therefore, no strategic buyer will

misreport herself as a naive type. A similar argument works for the seller and thus, the direct mechanism satisfies IC2. □

Proof of Proposition 4.1: Let (p^0, x^0) be an IC and IIR allocation rule when $\epsilon = 0$.

First, suppose $p^0(v, c) > 0$ for a positive measure of (v, c). Then IC and IIR imply that the ex-ante expected payoffs of the buyer and seller are positive. Pick $\epsilon > 0$. For all (v, t_b, c, t_s), define $p(v, t_b, c, t_s) = p^0(v, c)$ and $x(v, t_b, c, t_s) = x^0(v, c)$. It is straightforward to establish that (p, x) is an IC and IIR allocation rule when the probability of naive types is ϵ. In this allocation rule, the ex-ante expected payoff of the naive type of trader i is equal to the ex-ante expected payoff of the strategic type of trader i. Since $\epsilon > 0$, ex-ante, the naive type of trader i obtains positive expected payoff. Moreover, this allocation rule is not almost surely ex-post efficient—otherwise (p^0, x^0) will be an IC, IIR and almost surely ex-post efficient allocation rule when $\epsilon = 0$, contradicting Myerson and Satterthwaite (1983, Corollary 1). Therefore, Proposition 4.3 implies that the allocation rule (p, x) is constrained inefficient when the probability of naive types is ϵ and hence, the result follows.

Next, suppose (p^0, x^0) is such that $p^0(v, c) = 0$ for almost all (v, c). Pick any $r \in (\underline{c}, \bar{c}) \cap (\underline{v}, \bar{v})$ and define p as follows:

$$p(v, t_b, c, t_s) = \begin{cases} 1 & \text{if } v \geq r \geq c \\ p^0(v, c) & \text{otherwise.} \end{cases}$$

It can be shown that p satisfies the conditions in Proposition 3.1 (proof available upon request). Thus, there exist a payment function x such that (p, x) is an IC and IIR direct mechanism. By construction, $\forall (t_b, t_s)$, we have $(v - c)p(v, t_b, c, t_s) \geq (v - c)p^0(v, c)$ for all (v, c), and this inequality is strict for almost all (v, c) such that $v \geq r \geq c$. □

Proof of Proposition 4.3: We prove that if (p, x) is an IC and IIR allocation rule that does not satisfy the conditions in the proposition. Let $\check{p}$ be such that for all (t_b, t_s), we have $\check{p}(v, t_b, c, t_s) = 1$ if $v \geq c$ and $\check{p}(v, t_b, c, t_s) = 0$ if $v < c$. Pick a $\gamma \in (0, 1)$ and define $\hat{p}$ as follows: $\hat{p}(v, t_b, c, t_s) = (1 - \gamma)p(v, t_b, c, t_s) + \gamma\check{p}(v, t_b, c, t_s)$. We argue there exists a small enough γ such that $\hat{p}$ satisfies the sufficient conditions listed in Proposition 3.1.

It is easy to check that $\bar{\hat{p}}_b(v, t_b) = (1 - \gamma)\bar{p}_b(v, t_b) + \gamma\bar{\check{p}}_b(v, t_b), \forall v \in [\underline{v}, \bar{v}]$ and $\bar{\hat{p}}_s(c, t_s) = (1 - \gamma)\bar{p}_s(c, t_s) + \gamma\bar{\check{p}}_s(c, t_s), \forall c \in [\underline{c}, \bar{c}]$. Since both $\bar{p}_b(v, 1)$ and $\bar{\check{p}}_b(v, 1)$ are weakly increasing in v, $\bar{\hat{p}}_b(v, 1)$ is weakly increasing in v. Similarly, $\bar{\hat{p}}_s(c, 1)$ is weakly decreasing in c.

Moreover, $\forall v, v' \in [\underline{v}, \bar{v}]$, we have

$$
\begin{aligned}
&(v-v')\bar{\hat{p}}_b(v',2) - \int_{\underline{v}}^{v} \bar{\hat{p}}_b(y,1)dy \\
=&(1-\gamma)\left((v-v')\bar{p}_b(v',2) - \int_{\underline{v}}^{v} \bar{p}_b(y,1)dy\right) + \gamma\left((v-v')\bar{\check{p}}_b(v',2) - \int_{\underline{v}}^{v} \bar{\check{p}}_b(y,1)dy\right) \\
\leq&(1-\gamma)\left((v-v')\bar{p}_b(v',2) - \int_{\underline{v}}^{v} \bar{p}_b(y,1)dy\right),
\end{aligned}
$$

where the inequality follows since $\bar{\check{p}}_b(y,1) = \bar{\check{p}}_b(y,2), \forall y \in [\underline{v}, \bar{v}]$, and $\bar{\check{p}}_b(y,1)$ is weakly increasing in v. Therefore, $\delta_b(\hat{p}) \leq (1-\gamma)\delta_b(p)$. Similarly, $\delta_s(\hat{p}) \leq (1-\gamma)\delta_s(p)$. Hence, using (1) and Lemma 2.5, we obtain

$$
\begin{aligned}
\xi(\hat{p}) =&(1-\gamma)\xi(p) + \gamma\xi(\check{p}) \\
\geq&(1-\gamma)\left((1-\epsilon)\Big(\delta_b(p)+\delta_s(p)\Big) + \epsilon\Big(\int_{\underline{v}}^{\bar{v}} U_b(v,2)f_b(v)dv + \int_{\underline{c}}^{\bar{c}} U_s(c,2)f_s(c)dc\Big)\right) + \gamma\xi(\check{p}).
\end{aligned}
$$

Since $\xi(\check{p})$ is bounded and $\epsilon\left(\int_{\underline{v}}^{\bar{v}} U_b(v,2)f_b(v)dv + \int_{\underline{c}}^{\bar{c}} U_s(c,2)f_s(c)dc\right) > 0$, there exists a γ small enough such that $\xi(\hat{p}) \geq (1-\gamma)(1-\epsilon)\Big(\delta_b(p)+\delta_s(p)\Big) \geq (1-\epsilon)\Big(\delta_b(\hat{p})+\delta_s(\hat{p})\Big)$.

Thus, for a small enough γ, $\hat{p}$ satisfies the sufficient conditions listed in Proposition 3.1. Therefore, there exist a $\hat{x}$ such that $(\hat{p}, \hat{x})$ is an IC and IIR direct mechanism. The result follows since $\check{p}$ is ex-post efficient but p is not almost surely ex-post efficient. □

Proof of Proposition 5.5: Let (p, x) be an IC and EIR allocation rule that is almost surely ex-post efficient. By Lemma 2.4, it must satisfy (1). The right-hand side of (1) is equal to

$$
\begin{aligned}
&(1-\epsilon)\Big(U_b(\underline{v},1) + U_s(\bar{c},1)\Big) + \epsilon^2 \int_{\underline{v}}^{\bar{v}} \int_{\underline{c}}^{\bar{c}} (v-c)p(v,2,c,2)f_s(c)f_b(v)dcdv \\
&\quad + \epsilon(1-\epsilon)\int_{\underline{v}}^{\bar{v}} \int_{\underline{c}}^{\bar{c}} \Big[\big(v\,p(v,2,c,1) - x(v,2,c,1)\big) + \big(x(v,1,c,2) - c\,p(v,1,c,2)\big)\Big] f_s(c)f_b(v)dvdc \\
&\geq \epsilon^2 \int_{\underline{v}}^{\bar{v}} \int_{\underline{c}}^{\bar{c}} (v-c)p(v,2,c,2)f_s(c)f_b(v)dcdv,
\end{aligned}
$$

where the inequality follows from EIR. Since p is almost surely ex-post efficient, the left-hand

side of (1) is equal to

$$
\begin{aligned}
&(1-\epsilon)^2 \int_{\underline{v}}^{\bar{v}} \int_{\underline{c}}^{\min\{v,\bar{c}\}} (\alpha(v)-\beta(c)) f_s(c) f_b(v) dc dv + \epsilon^2 \int_{\underline{v}}^{\bar{v}} \int_{\underline{c}}^{\bar{c}} (v-c) p(v,2,c,2) f_s(c) f_b(v) dc dv \\
&\quad + \epsilon(1-\epsilon) \left(\int_{\underline{v}}^{\bar{v}} \int_{\underline{c}}^{\min\{v,\bar{c}\}} (\alpha(v)-c) f_s(c) f_b(v) dc dv + \int_{\underline{v}}^{\bar{v}} \int_{\underline{c}}^{\min\{v,\bar{c}\}} (v-\beta(c)) f_s(c) f_b(v) dc dv \right) \\
&= -(1-\epsilon)^2 \int_{\underline{v}}^{\bar{c}} (1-F_b(y)) F_s(y) dy + \epsilon^2 \int_{\underline{v}}^{\bar{v}} \int_{\underline{c}}^{\bar{c}} (v-c) p(v,2,c,2) f_s(c) f_b(v) dc dv \\
&\quad + \epsilon(1-\epsilon) \left(\int_{\underline{c}}^{\underline{v}} F_s(y) dy + \int_{\bar{c}}^{\bar{v}} (1-F_b(y)) dy \right) \\
&= -(1-\epsilon)^2 \int_{\underline{v}}^{\bar{c}} (1-F_b(y)) F_s(y) dy + \epsilon^2 \int_{\underline{v}}^{\bar{v}} \int_{\underline{c}}^{\bar{c}} (v-c) p(v,2,c,2) f_s(c) f_b(v) dc dv \\
&< \epsilon^2 \int_{\underline{v}}^{\bar{v}} \int_{\underline{c}}^{\bar{c}} (v-c) p(v,2,c,2) f_s(c) f_b(v) dc dv,
\end{aligned}
$$

where the second equality follows since $\underline{v} = \underline{c}$ and $\bar{v} = \bar{c}$. Hence, we have a contradiction.

Next, suppose (p, x) is such that $\epsilon \int_{\underline{v}}^{\bar{v}} U_b(v,2) f_b(v) dv = \epsilon \int_{\underline{c}}^{\bar{c}} U_s(c,2) f_s(c) dc = 0$. Since $\epsilon > 0$, we must have $U_b(v,2) = 0$ for almost all v and $U_s(c,2) = 0$ for almost all c. Then EIR implies that $p(v,2,c,2) = 0$ for almost all (v,c).

Case 1: $\forall (t_b, t_s) \in \{(1,1),(1,2),(2,1)\}$, $p(v,t_b,c,t_s) = 0$ for almost all (v,c). Pick any $r \in (\underline{c}, \bar{c}) \cap (\underline{v}, \bar{v})$ and define $\hat{p}$ as follows:

$$
\hat{p}(v,t_b,c,t_s) = \begin{cases} 1 & \text{if } v \geq r \geq c \\ p(v,t_b,c,t_s) & \text{otherwise.} \end{cases}
$$

It can be shown that $\hat{p}$ satisfies the conditions in Proposition 3.1 (proof available upon request). Hence, there exists $\hat{x}$ such that $(\hat{p}, \hat{x})$ is an IC and IIR direct mechanism. By construction, $\forall (t_b, t_s)$, we have $(v-c)\hat{p}(v,t_b,c,t_s) \geq (v-c)p(v,t_b,c,t_s)$ for all (v,c) and this inequality is strict for almost all (v,c) such that $v \geq r \geq c$. Thus, (p,x) is not constrained efficient.

Case 2: $\forall (t_b, t_s) \in \{(1,1),(1,2)\}$, $p(v,t_b,c,t_s) = 0$ for almost all (v,c), but $p(v,2,c,1) > 0$ for a positive measure of (v,c). Then $\bar{p}_b(v,1) = 0, \forall v < \bar{v}$ but $\bar{p}_b(v',2) > 0$ for some $v' < \bar{v}$. Hence, $\delta_b(p) > 0 = U_b(\underline{v},1)$, a contradiction ($U_b(v,1) = 0$ because of EIR).

Case 3: $\forall (t_b, t_s) \in \{(1,1),(2,1)\}$, $p(v,t_b,c,t_s) = 0$ for almost all (v,c), but $p(v,1,c,2) > 0$ for a positive measure of (v,c). We can obtain a contradiction like in Case (ii).

In all other cases, we have $\bar{p}_b(\tilde{v},1) > 0$ for some $\tilde{v} < \bar{v}$ and $\bar{p}_s(\tilde{c},1) > 0$ for some $\tilde{c} > \underline{c}$. Since $\bar{p}_b(v,1)$ is weakly increasing in v and $\bar{p}_s(c,1)$ is weakly decreasing in c, we have $u_b = \int_{\underline{v}}^{\bar{v}} \bar{p}_b(y,1)dy > 0$ and $u_s = \int_{\underline{c}}^{\bar{c}} \bar{p}_s(y,1)dy > 0$. Since $\int_{\underline{v}}^{v} \bar{p}_b(y,1)dy$ and $\int_{c}^{\bar{c}} \bar{p}_s(y,1)dy$ are continuous and, respectively, weakly increasing and weakly decreasing, for any $\gamma \in (0, \min\{u_b, u_s\})$, there exist $\tilde{v}' < \bar{v}$ and $\tilde{c}' > \underline{c}$ such that $\int_{\underline{v}}^{v} \bar{p}_b(y,1)dy \geq u_b - \gamma, \forall v > \tilde{v}'$, and $\int_{c}^{\bar{c}} \bar{p}_s(y,1)dy \geq u_s - \gamma, \forall c < \tilde{c}'$. We can also find $v^* \in (\tilde{v}', \bar{v})$ and $c^* \in (\underline{c}, \tilde{c}')$ such that $\bar{v} - v^* \leq u_b - \gamma$, $c^* - \underline{c} \leq u_s - \gamma$ and $c^* < v^*$.

Define $\hat{p}$ as follows: $\forall (t_b, t_s) \in \{(1,1),(1,2),(2,1)\}$, $\hat{p}(v, t_b, c, t_s) = p(v, t_b, c, t_s), \forall (v,c)$, and

$$\hat{p}(v,2,c,2) = \begin{cases} 1 & \text{if } v \in [v^*, \bar{v}] \text{ and } c \in [\underline{c}, c^*] \\ p(v,2,c,2) & \text{otherwise.} \end{cases}$$

Clearly, $\bar{\hat{p}}_b(v,1) = \bar{p}_b(v,1)$ and and $\bar{\hat{p}}_s(c,1) = \bar{p}_s(c,1)$. Hence, $\bar{\hat{p}}_b(v,1)$ is weakly increasing in v and $\bar{\hat{p}}_s(c,1)$ is weakly decreasing in c.

Moreover, $\forall v \leq v'$, we have $(v - v')\bar{\hat{p}}_b(v',2) - \int_{\underline{v}}^{v} \bar{\hat{p}}_b(y,1)dy \leq 0$. On the other hand, $\forall v > v'$ and $v' < v^*$,

$$(v - v')\bar{\hat{p}}_b(v',2) - \int_{\underline{v}}^{v} \bar{\hat{p}}_b(y,1)dy = (v - v')\bar{p}_b(v',2) - \int_{\underline{v}}^{v} \bar{p}_b(y,1)dy$$

while $\forall v > v^*$ and $v' \in [v^*, v)$,

$$(v - v')\bar{\hat{p}}_b(v',2) - \int_{\underline{v}}^{v} \bar{\hat{p}}_b(y,1)dy \leq u_b - \gamma - \int_{\underline{v}}^{v} \bar{p}_b(y,1)dy \leq 0.$$

Hence, $\delta_b(p) \leq \delta_b(\hat{p})$. Similarly, we can argue that $\delta_s(p) \leq \delta_s(\hat{p})$. However, clearly $\xi(\hat{p}) > \xi(p)$. Therefore, $\hat{p}$ satisfies the sufficient conditions listed in Proposition 3.1. Hence, there exists $\hat{x}$ such that $(\hat{p}, \hat{x})$ is an IC and IIR direct mechanism. By construction, for all (t_b, t_s), we have $(v - c)\hat{p}(v, t_b, c, t_s) \geq (v - c)p(v, t_b, c, t_s)$ for all (v,c), and $(v - c)\hat{p}(v,2,c,2) > (v - c)p(v,2,c,2)$ for almost all (v,c) such that $v \in [v^*, \bar{v}]$ and $c \in [\underline{c}, c^*]$. So (p, x) is not constrained efficient. □

Example 6.3 continued: Suppose there exists an equilibrium outcome of some k-double auction that is almost surely ex-post efficient. Let z_i be the bid/ask of trader i, G_i be the distribution of bids/asks over the support $[\underline{z}_i, \bar{z}_i]$ induced by the strategy of the strategic type of trader i. Then it must be that $\lim_{z_b \nearrow r} G_b(t_b) = 0$, i.e., the probability that the strategic buyer bids less than r is 0. If not, then a positive measure of the cost types of the strategic

seller will trade with a positive probability at prices that are less than their costs, which cannot happen in equilibrium. Similarly, we must have $G_s(r) = 1$.

Case 1: $0 < k < 1$. Since $G_s(r) = 1$ and all naive types of the seller ask at most r, for any value type of the strategic buyer, the bid of r is strictly better than any bid greater than r. Moreover, since the probability that the strategic buyer bids less than r is 0, it must be that almost all value types of the strategic buyer bid equal to r. Similarly, almost all cost types of the strategic seller ask equal to r.

Let $\phi(c)$ be the expected payoff of the strategic seller of cost c if she asks equal to r. Then $\phi(c) = (1-\epsilon)(r-c) + \epsilon \int_r^{\bar{v}} \big[kv + (1-k)r - c\big] f_b(v)dv$, which is a continuous function of c. Thus, $U_s(c, 1) = \phi(c)$ for almost all c.

We show that $\exists z_s > r$ such that the strategic seller of value r gets a higher expected payoff by asking z_s instead of r. Consider the difference in these payoffs,

$$\begin{aligned}
&\epsilon\left(\int_{z_s}^{\bar{v}} \big[kv + (1-k)z_s - r\big] f_b(v)dv - \int_r^{\bar{v}} \big[kv + (1-k)r - r\big] f_b(v)dv\right) \\
=&\epsilon\left((1-k)\int_{z_s}^{\bar{v}} (z_s - r) f_b(v)dv - k\int_r^{z_s} (v-r) f_b(v)dv\right) \\
\geq&\epsilon(z_s - r)\left((1-k)\int_{z_s}^{\bar{v}} f_b(v)dv - k\int_r^{z_s} f_b(v)dv\right).
\end{aligned}$$

There exists a $z_s > r$ such that the last term is positive. If not, then $\epsilon > 0$ implies that $\lim_{z_s \searrow r}\left((1-k)\int_{z_s}^{\bar{v}} f_b(v)dv - k\int_r^{z_s} f_b(v)dv\right) = (1-k) \leq 0$, a contradiction.

Therefore, $U_s(r, 1) > \phi(r)$. Since $\phi(c)$ is continuous, it must be that for all c close enough to r, we have $U_s(r, 1) > \phi(c)$. However, the third condition in Lemma 2.4 implies that $U_s(c, 1) \geq U_s(r, 1), \forall c$. This contradicts $U_s(c, 1) = \phi(c)$ for almost all c.

Case 2: $k = 1$. Like in the previous case, we can argue that almost all value types of the strategic buyer bid equal to r. We now show that there exists a positive measure of value types of the strategic buyer who would prefer to bid less than r, which is a contradiction. If the strategic buyer with value v bids r, then her expected payoff is $v - r$, whereas if she were to bid $\frac{v+\underline{c}}{2}$, then her expected payoff is $\frac{v-\underline{c}}{2}\tilde{G}_s\left(\frac{v-\underline{c}}{2}\right)$, where $\tilde{G}_s$ is the distribution of the seller's bid. However, $\frac{v-\underline{c}}{2}\tilde{G}_s\left(\frac{v-\underline{c}}{2}\right) > v - r$ for v close enough to r since $F_s\left(\frac{v-\underline{c}}{2}\right) > 0$ and $\epsilon > 0$.

Case 3: $k = 0$. This case can be argued like the case when $k = 1$. □

References

I. Alger I, R. Renault, Screening ethics when honest agents care about fairness, Int. Econ. Rev. 47 (2006), 59-85.

I. Alger, R. Renault, Screening ethics when honest agents keep their word, Econ. Theory 30 (2007), 291-311.

K. Chatterjee, W. Samuelson, Bargaining under incomplete information, Operations Research 31 (1983), 835-851.

O. Compte, P. Jehiel, Veto constraint in mechanism design: Inefficiency with correlated types, American Economic Journal: Microeconomics 1 (2009), 182-206.

J. Erard , B. Feinstein, Honesty and evasion in the tax compliance game, RAND J. Econ. 25 (1994), 1-19.

J. Farrell, R. Gibbons, Cheap talk can matter in bargaining, J. Econ. Theory 48 (1989), 221-237.

F. Forges, Ex post individually rational trading mechanisms, in: A. Alkan, C. D. Aliprantis, N. C. Yannelis (Eds.), Current Trends in Economics: Theory and Applications, Springer, New York, 1999, 157-175.

K. M. Hagerty, W. P. Rogerson, Robust trading mechanisms, J. Econ. Theory 42 (1987), 94-107.

T. A. Gresik, Ex ante efficient, ex post individually rational trade, J. Econ. Theory 53 (1991a), 131-145.

T. A. Gresik, The efficiency of linear equilibria of sealed-bid double auctions, J. Econ. Theory 53 (1991b), 173-184.

B. Holmström, R. Myerson, Efficient and durable decision rules with incomplete information, Econometrica 51 (1983), 1799-1819.

W. Leininger, P. Linhart, R. Radner, Equilibria of the sealed-bid mechanism for bargaining with incomplete information, J. Econ. Theory 48 (1989), 63-106.

S. Mathews, A. Postlewaite, Pre-play communication in two-person sealed-bid double auctions, J. Econ. Theory 48 (1989), 238-263.

K. McGinn, L. Thompson, M. Bazerman, Dyadic processes of disclosure and reciprocity in bargaining with communication, Journal of Behavioral Decision Making 16 (2003), 17-34.

R. Myerson, M. Satterthwaite, Efficient mechanisms for bilateral trading, J. Econ. Theory 29 (1983), 265-281.

R. Radner, A. Schotter, The sealed-bid mechanism: An experimental study, J. Econ. Theory 48 (1989), 179-220.

R. Saran, How communication improves efficiency in bargaining: Reconciling theory with evidence, METEOR Research Memoranda RM/09/22.

M. Satterthwaite, S. Williams, Bilateral trade with the sealed bid k-double auction: Existence and efficiency, J. Econ. Theory 48 (1989), 107-133.

S. Severinov, R. Deneckere, Screening when not all agents are strategic: Does a monopoly need to exclude?, RAND J. Econ. 37 (2006), 816-841.

S. Spiekermann, J. Grossklags, B. Berendt, E-privacy in 2nd generation e-commerce: Privacy preferences versus actual behavior, in: Proceedings of the 3rd ACM Conference on Electronic Commerce, ACM, New York, 2001, pp. 38-47.

K. Valley, L. Thompson, R. Gibbons, M. Bazerman, How communication improves efficiency in bargaining games, Games Econ. Behav. 38 (2002), 127-155.

S. Williams, Efficient performance in two agent bargaining, J. Econ. Theory 41 (1987), 154-172.