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Centralized Allocation in Multiple Markets

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**Abstract**

The problem of allocating indivisible objects to different agents, where each individual is assigned at most one object, has been widely studied. Pápai (2000) shows that the set of strategy-proof, nonbossy, Pareto optimal and reallocation-proof rules are hierarchical exchange rules — generalizations of Gale's Top Trading Cycles mechanism. We study the centralized allocation that takes place in multiple markets. For example, the assignment of multiple types of indivisible objects; or the assignment of objects in successive periods. We show that the set of strategy-proof, Pareto efficient and nonbossy rules are sequential dictatorships, a special case of Pápai's hierarchical exchange rules.

JEL classification: C78, D61, D78, I20.

Keywords: Matching, Strategy-Proofness, Nonbossiness, Pareto efficiency.

# 1 Introduction

A central planner often faces the problem of designing a rule to assign (at most) one indivisible object to each agent. For example, municipalities assign public houses to families, education departments allocate students to public schools, and firms allocate projects among workers. This class of assignment problems has been widely studied from many different perspectives. Pápai (2000) characterizes the set of strategy-proof, Pareto optimal, nonbossy, and reallocation-proof rules. That is, if the desiderata is to implement a Pareto optimal allocation, then a way of implementing such an allocation with a nonbossy, strategy-proof and reallocation-proof rule (in fact, the *only way* when monetary transfers are not allowed)

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is by using a hierarchical exchange rule.[1] Not only is this result of theoretical importance, but it also provides an important guidance for practitioners and policy makers. For example, on April 16, 2012 it was announced that the New Orleans Recovery School District would utilize a version of the Top Trading Cycles as the algorithm for centralized enrollment of children in public schools (Vanacore, 2012).

In reality, some agents are typically involved in more than one assignment problem at once: people who participate in the allocation of public housing, for example, might also have their children enrolled in public schools. Moreover, many of these problems take place in multiple periods. An example is the allocation of new physicians in the United Kingdom, where each young doctor applies to two successive positions – a medical post and a surgical post (Roth, 1991; Irving, 1998). Another example of multiple market allocation is the assignment of young children to public daycares (Kennes et al., 2012). A simple, illustrative, example is the allocation of courses among the faculty of a department in which each professor teaches one course per semester.

We study this centralized allocation problem that takes place in multiple markets, where each market may be interpreted either as a different type of object or as a different period. There are $n$ agents and two (or more) markets and each agent must be assigned one object from each market. Agents have preferences over the different bundles, where a bundle is a vector consisting of one object per market. We mostly restrict our attention to the cases with additively separable preferences and in which markets are independent. By independent we mean that the set of objects available in a particular market is exogenous and not affected by other markets. This way, we keep our setting as close as possible to the setting of Pápai (2000).

In environments with multiple markets, if the allocation in each market is done separately, there might be scope for a mutually beneficial trade between agents even if the allocation is Pareto efficient within each market. In particular, if the hierarchical exchange rule is applied in each different market, the final allocation might not be efficient. This raises the following question: is it possible to characterize the set of rules –perhaps a subset of the hierarchical

[1]A hierarchical exchange rule is a generalization of Gale's Top Trading Cycles algorithm and can be described as follows. In the first stage, the planner distributes the objects to the agents; in particular, some agents might receive multiple objects while others might receive none. Then, the Top Trading Cycles is applied, with each agent pointing to her preferred object, while the object points to its owner. Once all endowed agents receive their objects, the agents who did not participate in this first stage inherit the left-over objects and the Top Trading Cycles is applied again. The procedure is repeated until all agents are assigned an object.

exchange rules– that can implement a Pareto efficient outcome?

We show that the set of rules that are strategy-proof and nonbossy and implement a Pareto efficient allocation are the sequential dictatorships. These rules generalize the serial dictatorship rule in that the order of agents who choose the objects might be a function of the choices made by agents that chose previously. The sequential dictatorship rules are a special case of Pápai's (2000) hierarchical exchange rules.[2]

Our result implies that if each allocation problem is considered in isolation then efficiency might fail to be satisfied. In particular, if one insists with Pareto efficiency within each market, then the Top Trading Cycles–or, more generally, a hierarchical exchange rule– might fail efficiency in the problem as whole, unless the rule is also a sequential dictatorship.

To the best of our knowledge, this is the first paper that provides a complete characterization of the centralized allocation in multiple markets without an endowment structure. Konishi et al. (2001)[3] considered the multi-type allocation problem, but in their work each agent is initially endowed with an object– as in the economy proposed by Shapley and Scarf (1974). They show that the core may be empty in these multi-type Shapley-Scarf economies and also that there are no Pareto efficient, individually rational and strategy-proof rules. Here, since we do not assume an initial endowment structure, we do not impose the individual rationality constraint, which plays a crucial role in the results obtained by Konishi et al. (2001).

This paper is organized as follows. In the following section we describe the model and state its main assumptions. In Section 3, we describe and define a mechanism and its main properties. We also describe in greater detail two mechanisms: the coordinatewise Top Trading Cycles and the sequential dictatorship. In Section 4, we prove the result for the special case of 2 goods in each market and 2 players. In Section 5 we prove our main result (theorem 3). Finally, in Section 6 we conclude the paper. The proof of Theorem 3 for the case in which the number of players is greater than 2 is left in the Appendix.

[2]Pápai (2001), Ehlers and Klaus (2003), and Hatfield (2009) study the problem of multi-unit allocation. They also show that the sequential dictatorship is the only rule that is strategy-proof, Pareto efficient and nonbossy. While our result has a similar flavor, the problem that we study here is substantially different from the multi-unit allocation objects.

[3]See Klaus (2008) for further reference.

## 2 Model

Let $N = \{1, \cdots, n\}$ where $n < \infty$ be the set of agents. There are two types of indivisible objects and $A$ and $B$ stand for the sets of type 1 and type 2 objects, respectively. We refer to a pair $(a, b) \in A \times B$ as a bundle. For convenience we assume that an artificial null object 0 is in both sets $A$ and $B$. Throughout the paper, we assume $|A \setminus \{0\}| \geq n$ and $|B \setminus \{0\}| \geq n$, i.e., there are enough objects of each type to distribute to the agents. An allocation $x = (x_1, \cdots, x_n)$ is a list of the assignments for the $n$ agents, where $x_i \in A \times B$. If $x_i = (a, b)$, then agent $i$ is assigned the bundle $(a, b)$. Often we write $x_i^A$ ($x_i^B$) to denote the type 1 (2) object player $i$ obtains under allocation $x$. An allocation $x$ is feasible if no object (except the 0 object) is assigned to more than 1 agent. Let $X$ stand for the set of all feasible allocations.

Each agent has a (weak) preference ordering $R_i$ over $A \times B$; where $R_i$ is a complete and transitive binary relation. We denote by $P_i$ the associated strict preference ordering to $R_i$. We make three assumptions on preferences:

**Assumption 1** (Strictness)**.** *For any $(a, b)$ and $(\hat{a}, \hat{b})$ in $A \times B$, $(a, b)R_i(\hat{a}, \hat{b})$ means that either $(a, b)P_i(\hat{a}, \hat{b})$ or $(a, b) = (\hat{a}, \hat{b})$.*

**Assumption 2** ((Additive) Separability)**.** *For each agent $i$, there exists $u_i : A \cup B \to \mathbb{R}$ such that*

$$(a, b)R(\hat{a}, \hat{b}) \text{ for some } a, \hat{a} \in A \; and b, \hat{b} \in B \quad \text{iff } u_i(a) + u_i(b) \geq u_i(\hat{a}) + u_i(\hat{b})$$

**Assumption 3** (Desirability)**.** *For any $(a, b) \neq (0, 0)$ and $i \in N$, $(a, b)R_i(0, 0)$.*

We remark that relaxing the separability and desirability assumptions do not affect our main characterization result. With these two assumptions, especially separability, our allocation problem in multiple markets remains as similar as possible to the one in a single market.

We will use the notation $\mathcal{R} = \Pi_{i \in N} \mathcal{R}_i$ where $\mathcal{R}_i$ stands for the set of separable preferences for player $i$. With slight abuse of notation we write $aR_i^A a'$ if $(a, b) \, R_i \, (a', b)$, for $\forall b \in B$. Observe here that $R_i^A$ is a preference relation on $A$, due to the separability assumption. We use $R_i^B$ in a similar manner.

**Definition 1** (Pareto Dominance)**.** *An allocation $x$ (weak) Pareto dominates $y$ if $x_i R_i y_i$ for* *all $i$ and $x_j P_j y_j$ for at least one $j$. An allocation $x$ is (strongly) Pareto efficient if there exists no feasible $y$ which Pareto dominates $x$.*

# 3 Mechanism and Its Properties

A mechanism $\varphi$ is a mapping from the set of separable preferences $\mathcal{R}$ to the set of feasible allocations. The notation $\varphi_i^A(R)$ ($\varphi_i^B(R)$) denotes the type 1 (type 2) object that mechanism $\varphi$ assigns to agent $i$ when the reported preference profile is $R$. In the subsequent sections, we will characterize the set of mechanisms that are Pareto efficient, strategy-proof, and nonbossy. We formally define these concepts below.

First, we say a mechanism is efficient if it returns an efficient allocation for each preference profile.

**Definition 2** (Pareto Efficiency)**.** *A mechanism $\varphi$ is* Pareto efficient*if for all $R$, the allocation $\varphi(R)$ is Pareto efficient under $R$.*

A mechanism is strategy-proof if reporting one's true preferences is a weakly dominant strategy for every agent.

**Definition 3** (Strategy-Proofness)**.** *A mechanism $\varphi$ is* strategy-proof *if for all $i \in I$, all $R_i$, all $\hat{R}_i$, and all $\hat{R}_{-i}$,*

$$\varphi_i\left(R_i, \hat{R}_{-i}\right) R_i \varphi_i\left(\hat{R}_i, \hat{R}_{-i}\right)$$

*where $R_i$ is $i$'s true preferences while $\hat{R}_i$ and $\hat{R}_{-i}$ are the reported preferences of $i$ and the others.*

Finally, a mechanism is nonbossy if no player can change the others' allocations without changing her own allocation.

**Definition 4** (Nonbossiness)**.** *A mechanism $\varphi$ is nonbossy if for all $R$, $i \in N$ and $\hat{R}_i$,*

$$\varphi_i\left(R_i, R_{-i}\right) = \varphi_i\left(\hat{R}_i, R_{-i}\right) \Longrightarrow \varphi\left(R_i, R_{-i}\right) = \varphi\left(\hat{R}_i, R_{-i}\right).$$

Below we present examples of mechanisms and how they work in a problem of centralized allocation in multiple markets.

## 3.1 Coordinatewise Top Trading Cycles

The top trading cycles mechanism was studied, among others, by Shapley and Scarf (1974), Roth and Postlewaite (1977), Pápai (2000) and Abdulkadiroğlu and Sönmez (2003). Let us define the commodity-wise top trading cycle algorithm (CTTC) under the assumption that $|A \setminus \{0\}| = |B \setminus \{0\}| = n$, to avoid lengthy technical discussions.[4] The CTTC allocates objects as follows:

Fix any allocation $x$ in which each agent is assigned exactly one object of each type. Then we allocate type 1 objects as follows:

*Round 1*: Each agent $i$ points to her favorite type 1 object under $R_i^A$ and each object points to its owner under $x$. Then we look for a cycle, which is an alternating sequence of agents and objects, $\{i_1, \bar{a}_1, i_2, \bar{a}_2, \cdots, i_k, \bar{a}_k\}$, such that $\bar{a}_j$ is agent $i_j$'s favorite type 1 object, whereas agent $i_l$ is the owner of $\bar{a}_{l-1}$, for $l = 2, ..., k$; and agent $i_1$ is the owner of $\bar{a}_k$ under allocation $x$. There must exist at least one cycle and any agent or type 1 object can be a part of only one cycle. Then each agent who is a part of a cycle obtains the type 1 object she points to, i.e., her top choice.

In general, at:

*Round k*: All agents who obtain type 1 object in rounds $1, \cdots, k-1$ do not participate in step $k$. Each remaining agent $i$ points to her favorite type 1 object under $R_i^A$ among the unassigned type 1 objects. Each pointed object points to its owner under $x$. Again, each agent who is a part of a cycle obtains the object she points to.

The process continues until all agents are allocated a type 1 object. After assigning all the type 1 objects, type 2 objects are allocated in a similar manner.

Given that the agents' preferences are separable, the CTTC mechanisms are strategy-proof and nonbossy.[5] On the other hand, it turns out that CTTC is not necessarily efficient which we demonstrate with the following example.

**Example 1** (CTTC: Failure of Pareto Efficiency)**.** *Let* $n = 2$ *and* $A = \{a_1, a_2\}$ *and* $B = \{b_1, b_2\}$*. Agent 1 owns* $(a_1, b_1)$ *while agent 2 owns* $(a_2, b_2)$*.*

[4] See Pápai (2000) for a more generalized version of the top trading cycles algorithm.
[5] See Pápai (2000).

*The preferences of the agents are given as follows:* $a_1 R_i^A a_2$ *and* $b_2 R_i^B b_1$ *for both* $i = 1, 2$. *However,* $(a_2, b_2) R_1 (a_1, b_1)$ *and* $(a_1, b_1) R_2 (a_2, b_2)$.

*The CTTC allocates* $(a_1, b_1)$ *to agent 1 and* $(a_2, b_2)$ *to agent 2. Clearly, this allocation is Pareto dominated by the one in which* $(a_2, b_2)$ *and* $(a_1, b_1)$ *are allocated to players* 1 *and* 2*, respectively.*

In Example 1, CTTC yields an allocation that both players cannot improve if they trade their allocations in one market. However, the agents will improve if they trade their allocations in both markets. This is the reason why CTTC is not necessarily efficient. Consequently, when going from the allocation problem in a single market to the one in multiple markets, the set of strategy-proof, nonbossy and Pareto efficient mechanisms narrows, as Pápai (2000) shows that the hierarchical exchange mechanisms — a generalized version of TTC — are the *only* strategy-proof, Pareto efficient, nonbossy and reallocation proof mechanisms in single market settings.

We have already mentioned that CTTC is both strategy-proof and nonbossy. In fact, because the agents have separable preferences in our setting, running strategy-proof and nonbossy mechanisms in both markets would be strategy-proof and nonbossy. In this sense, achieving strategy-proofness and nonbossiness in multiple market settings is no more difficult than achieving them in single market settings. On the other hand, efficiency is much harder to achieve in a multiple market setting than in a single market setting as we already noted for the case of CTTC. Therefore, we conclude that efficiency is the driving force why the set of strategy-proof, nonbossy and Pareto efficient mechanisms narrow in multiple market settings.

## 3.2 Sequential Dictatorship

In this subsection, we define the sequential dictatorship mechanism which is strategy-proof, nonbossy and Pareto efficient. In this mechanism, the first player who makes her choice is exogenously given and is free to choose any bundle. The first player's choice determines the second player to make a choice and this player is free to choose any available bundle out of the bundles left after the first player has made her choice. Then the second player's choice determines the third player to choose and she is free to choose any available bundle left after the first two players' choices. The process continues until all players have made their choices. Below we define the sequential dictatorship algorithm formally.

Let $\pi : N \times \mathcal{R} \to \{1, \cdots, n\}$ be a permutation of $N$ that depends on the preference profiles of the players. In addition, let $i_j(R, \pi)$ be the player for whom $\pi(i_j(R, \pi), R) = j$, that is, if the reported preference profile is $R$, then $i_j(R, \pi)$ is the individual who will make the $j$th move under the specific permutation map $\pi(\cdot, R)$.

When the order of choice is given according to $\pi(\cdot, R)$, agent $i_1(R, \pi)$ chooses her favorite bundle from $A \times B$ and then agent $i_2(R, \pi)$ chooses her favorite bundle from the remaining set of bundles, and so on. To formalize this process, we define the favorite bundle of each agent $i_j(R, \pi)$: the favorite bundle of player for agent $i_1(R, \pi)$, $f(i_1(R, \pi)) = (f^A(i_1(R, \pi), f^B(i_1(R, \pi))$, is the most preferred bundle of agent $i_1(R, \pi)$ in $A \times B$. That is, if $a = f^A(i_1(R, \pi))$ then $aR^A_{i_1(R,\pi)}a'$ for $\forall a' \in A$, and similarly for $f^B(i_1(R, \pi))$. We then define $f(i_j(R, \pi))$ successively as follows: $f(i_j(R, \pi))$, for $j = 2, 3, ..., n$, is the most preferred bundle of $i_j(R, \pi)$ in the set:

$$A \setminus \left\{\cup_{k=1}^{j-1} f^A(i_k(R, \pi))\right\} \times B \setminus \left\{\cup_{k=1}^{j-1} f^B(i_k(R, \pi))\right\}$$

With the above notations, the choice of player $i_j(R, \pi)$ is $f(i_j(R, \pi))$ when the order of the agents is given by $\pi(\cdot, R)$. Now we are ready to define the sequential dictatorship mechanism.

**Definition 5** (Sequential Dictatorship)**.** *A mechanism $\varphi$ is a sequential dictatorship mechanism if there is a permutation $\pi : N \times \mathcal{R} \to \{1, \cdots, n\}$ such that*

1. *for all $R \in \mathcal{R}$ and $j = 1, \cdots, n$, $\varphi_{i_j(R,\pi)}(R) = f(i_j(R, \pi))$.*

2. *whenever $i = i_1(R, \pi)$ for some $R$, then $i = i_1(R', \pi)$ for all $R' \in \mathcal{R}$.*

3. *if $i = i_j(R, \pi)$ for some $R$ and $j \le n$, then $i = i_j(R', \pi)$ for all $R'$ in which $f(i_k(R, \pi)) = f(i_k(R', \pi))$ for all $k \le j - 1$.*

The first item in the definition of the sequential mechanism means that each player must choose her favorite bundle among the available bundles. The second item requires that there is only one player who makes the first choice. The third item requires that if the first $j - 1$ players make the same choices, then the $j$th player who makes a choice must be the same agent.

The standard serial dictatorship mechanism is the one in which $\pi(\cdot, R)$ is constant for all $R \in \mathcal{R}$. That is, the order in which the agents make their choices is the same regardless of the reported preference profile.

**Example 2** (Example 1 revisited.). *Under the sequential dictatorship mechanism, player 1 obtains* $(a_1, b_2)$ *if she is the one who makes the first choice and* $(a_2, b_1)$ *if she is the second one to choose. Clearly, in both cases the final allocation is efficient.*

**Remark 1.** *In the example above, observe that the sequential dictatorship mechanism never yields the allocation* $x_1 = (a_2, b_2)$ *and* $x_2 = (a_1, b_1)$ *which is also Pareto efficient. This result contrasts with the result in the allocation problem in single markets, in which all Pareto efficient allocations are reached through some serial dictatorship (Abdulkadiroğlu and Sönmez, 1999).*

# 4 The 2x2 Cases and the Gibbard-Satterthwaite Theorem

In this section, we present a preview of our main result. Here, we focus on the specific case of two objects in each market and two players only. That is, $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$ and n=2. A nice feature of this proof is that it makes use of the Gibbard-Satterthwaite theorem.[6] The key aspect here is that the allocation of one player fully determines the allocation of the other player. For example, when the allocation of player 1 is $x_1 = (a_1, b_1)$, the allocation of player 2 must be $x_2 = (a_2, b_2)$ and so on.[7] The strict preference ordering of each agent over the set of her own final allocations induces a strict ordering over the set of player 1's allocations. In this newly interpreted setting, a mechanism maps the players' reported preferences to player 1's allocations. Perhaps the most important observation here is that in the reinterpreted setting, a mechanism is a social choice function as used in the implementation literature. Now, using the Gibbard-Satterthwaite theorem (Gibbard, 1973; Satterthwaite, 1975), one obtains that if the mechanism is strategy-proof and efficient, then it must be a dictatorship, or in our setting a sequential dictatorship mechanism (which is also a serial dictatorship as there are only 2 objects of each type and 2 players).

**Theorem 1.** *Any nonbossy, strategy-proof, Pareto efficient mechanism for the* $|N| = |A| = |B| = 2$ *case is a sequential dictatorship.*

[6] The result can also be proved using an alternative method, which we do in the following section, for the general case of any number of players ($n \geq 2$) and any number of objects per market.

[7] The same is not true if there are more than 2 goods even when there are only 2 players or if there (strictly) more than 2 agents.

*Proof.* Fix an efficient and strategy-proof mechanism $\varphi$, where, recall, $\varphi : \mathcal{R}_1 \times \mathcal{R}_2 \to X$.

Let us use the following notations: $t_1 = (a_1, b_1)$, $t_2 = (a_1, b_2)$, $t_3 = (a_2, b_1)$, $t_4 = (a_2, b_2)$ and let $\mathcal{T} = \{t_1, t_2, t_3, t_4\}$. First let us show that $\varphi_1$ is an onto function. Fix any $t = (a, b) \in A \times B$. Consider $R_1 \in \mathcal{R}_1$ and $R_2 \in \mathcal{R}_2$ such that $(a, b)$ is player 1's favorite bundle in $A \times B$ while the remaining pair in $A \times B$ is player 2's top choice. Because $\varphi$ is efficient, $\varphi_1(R) = (a, b)$. This means that $\varphi_1$ is an onto function. Now we will show that $\varphi_1 : \mathcal{R} \to \mathcal{T}$ must be dictatorial.

We will view $\varphi_1 : \mathcal{R} \to \mathcal{T}$ as a social choice function that assigns player 1 some object $t$. Specifically, $t \in \mathcal{T}$ stands for the objects that player 1 obtains. On the other hand, if player 1 is assigned $t_1/t_2/t_3/t_4$ then player 2 is assigned $t_4/t_3/t_2/t_1$ by feasibility. Player 1's preferences rank alternatives assuming that these are the alternatives she would obtain, while player 2's preferences rank alternatives based on what is left after player 1 is allocated some alternative. With this relabeling, one can view $\varphi_1 : \mathcal{R} \to \mathcal{T}$ as a social choice function. Then the Gibbard-Satterthwaite theorem[8] yields the desired result (Gibbard, 1973; Satterthwaite, 1975) $\square$

# 5 Efficiency and Strategy-Proofness

In this section we characterize the nonbossy, strategy-proof and Pareto efficient mechanisms. First let us note that any sequential dictatorship mechanism is nonbossy, strategy-proof and Pareto efficient.

**Theorem 2.** *The sequential dictatorship mechanisms are nonbossy, strategy-proof and Pareto efficient.*

Now we turn our attention to the main result of the paper: only the sequential dictatorship mechanisms satisfy nonbossiness, strategy-proofness and Pareto efficiency. In the body of the text, we prove the result for $n = 2$, and we leave the proof of the case in which $n > 2$ for the Appendix.

Now let us give the definition of monotonicity which is closely related to the notion of Maskin Monotonicity used in the implementation literature.

[8]The precise statement of the Gibbard-Satterthwaite theorem is the following: In any environments with at least three social alternatives, any strategy-proof and onto social choice function is a dictatorship (The proof is well-known and can be found, for example, in Mas-Colell et al. (1995), Proposition 23.C.3).

**Definition 6** (Monotonicity). *A preference profile $R^1$ is a monotonic change of $R$ with respect to mechanism $\varphi$ if for each agent $i$, the relative ranking of the allocation $\varphi_i(R)$ weakly improves under $R^1$, i.e.,*

$$\{(a,b) \in A \times B : \varphi_i(R) R_i (a,b)\} \subseteq \left\{(a,b) \in A \times B : \varphi_i(R) R_i^1 (a,b)\right\}.$$

*A mechanism $\varphi$ is monotonic if $\varphi(R) = \varphi(R^1)$ for any $R$ and $R^1$ where $R^1$ is a monotonic change of $R$ with respect to $\varphi$.*

In words, a mechanism is monotonic if whenever each agent's lower contour set of $\varphi(R)$ expands weakly going from preference profile $R$ to $R^1$, the allocations prescribed by the mechanism under $R$ and $R^1$ must be the same. The next lemma which establishes that all the nonbossy and strategy-proof mechanisms are also monotonic is from Svensson (1999).

**Lemma 1** (Lemma 1 of Svensson (1999)). *If a mechanism $\varphi$ is nonbossy and strategy-proof, then $\varphi$ is monotonic.*

We are now ready for our main result in the paper. Here, we present the proof for the $n = 2$ case, and we leave the general proof of $n > 2$ for the Appendix.

**Theorem 3.** *Any nonbossy, strategy-proof, and Pareto efficient mechanism is a sequential dictatorship.*

*Proof.* Assume that $n = 2$ and fix an efficient and strategy-proof mechanism $\varphi$.
*Claim 1. For any $a \in A$ and $b \in B$, there exists $j \in N$ such that whenever player $j$ reports $P_j$ in which $(a,b)$ is her most preferred bundle, $\varphi_j(P_j, P_{-j}) = (a,b)$ for any $P_{-j}$.*
*Proof of Claim 1.* Without loss of generality (WLOG) let $a = a_1$ and $b = b_1$. Consider the preference profile $(P_1^1, P_2^1)$ in which the most preferred 4 choices of the players are shown in the table below (ordered from top to bottom):

| $P_1^1$ | $P_2^1$ |
|---|---|
| $(a_1, b_1)$ | $(a_1, b_2)$ |
| $(a_2, b_1)$ | $(a_2, b_2)$ |
| $(a_1, b_2)$ | $(a_1, b_1)$ |
| $(a_2, b_2)$ | $(a_2, b_1)$ |
| ... | ... |

In this case, efficiency yields that players 1 and 2 must obtain either (1) $(a_1, b_1)$ and $(a_2, b_2)$, respectively, or (2) $(a_2, b_1)$ and $(a_1, b_2)$.

*Case (1).* Now we will show that player 1 obtains $(a_1, b_1)$ under $\varphi$ if she reports $(a_1, b_1)$ as her most preferred bundle regardless of player 2's report.

Let us consider 2 more preference profiles $P_1^2$ for player 1 and $P_2^2$ for player 2.

| $P_1^2$ | $P_2^2$ |
|---|---|
| $(a_1, b_1)$ | $(a_1, b_2)$ |
| $(a_1, b_2)$ | $(a_1, b_1)$ |
| $(a_2, b_1)$ | $(a_2, b_2)$ |
| $(a_2, b_2)$ | $(a_2, b_1)$ |
| ... | ... |

As $(P_1^2, P_2^1)$ is a monotonic change of $(P_1^1, P_2^1)$ with respect to $\varphi$, $\varphi(P_1^2, P_2^1) = \varphi(P_1^1, P_2^1)$ by Lemma 1. Now let us show that $\varphi(P_1^1, P_2^2) = \varphi(P_1^1, P_2^1)$. The strategy-proofness of $\varphi$ implies that $\varphi_2(P_1^1, P_2^2)$ is either $(a_2, b_2)$ or $(a_1, b_1)$. If $\varphi_2(P_1^1, P_2^2) = (a_1, b_1)$, then $\varphi_1(P_1^1, P_2^2) = (a_2, b_2)$ by efficiency. But this allocation is Pareto dominated by the allocation which assigns $(a_2, b_1)$ to player 1 and $(a_1, b_2)$ to player 2. This means that $\varphi_2(P_1^1, P_2^2) = (a_2, b_2)$ which along with efficiency implies $\varphi_1(P_1^1, P_2^2) = (a_1, b_1)$. Finally, let us show $\varphi(P_1^2, P_2^2) = \varphi(P_1^1, P_2^2)$. This easily follows from Lemma 1 as $(P_1^2, P_2^2)$ is a monotonic change of $(P_1^1, P_2^2)$ with respect to $\varphi$.

Now consider preferences for player 2, $P_2^3$ in which player 2's top 4 choices are:

| $P_2^3$ |
|---|
| $(a_1, b_1)$ |
| $(a_1, b_2)$ |
| $(a_2, b_1)$ |
| $(a_2, b_2)$ |
| ... |

Now let us show that $\varphi(P_1^1, P_2^3) = \varphi(P_1^1, P_2^2)$. The strategy-proofness of $\varphi$ implies that $\varphi_2(P_1^1, P_2^3)$ is either $(a_2, b_2)$ or $(a_2, b_1)$. If $\varphi_2(P_1^1, P_2^3) = (a_2, b_1)$, then $\varphi_1(P_1^1, P_2^3) = (a_1, b_2)$ by efficiency. But this allocation is Pareto dominated by the one in which players 1 and 2 obtain $(a_2, b_1)$ and $(a_1, b_2)$, respectively. Hence, $\varphi_2(P_1^1, P_2^3) = (a_2, b_2)$. Then efficiency implies that $\varphi_1(P_1^1, P_2^3) = (a_1, b_1)$. This shows that $\varphi(P_1^1, P_2^3) = \varphi(P_1^1, P_2^2)$. Furthermore, recall that $(a_1, b_1)$ is player 1's most preferred bundle in both $P_1^1$ and $P_1^2$, thus, as $(P_1^2, P_2^3)$ is a monotonic change of $(P_1^1, P_2^3)$, we must have that $\varphi(P_1^2, P_2^3) = \varphi(P_1^1, P_2^3)$ by Lemma 1. In addition, the strategy-proofness of $\varphi$ implies that player 2 can never obtain any of $(a_1, b_1)$, $(a_2, b_1)$, and $(a_1, b_2)$ as long as player 1 reports $P_1^1$ or $P_1^2$. Combining this with Lemma 1, we obtain that player 2 can never obtain any of $(a_1, b_1)$, $(a_2, b_1)$, and $(a_1, b_2)$ as long as player 1 reports preferences in which $(a_1, b_1)$ is her most preferred bundle. Note that this completes the proof for the case in which $|A| = |B| = 2$.

From now on, let at least one of the markets have more than two objects, i.e. $|A| > 2$ and $|B| > 2$ is satisfied. To complete the proof we must show that player 2 can never obtain any of $(a, b_1)$ where $a \neq a_1, a_2$ or $(a_1, b)$ where $b \neq b_1, b_2$ for as long as player 1 reports preferences in which $(a_1, b_1)$ is her most preferred bundle. Consider $P_2^4$ in which the most preferred 5 bundles of player 2 are:

| $P_2^4$ |
|---|
| $(a_1, b_1)$ |
| $(a_1, b_2)$ |
| $(a_2, b_1)$ |
| $(a, b_1)$ |
| $(a_2, b_2)$ |
| ... |

The strategy-proofness of $\varphi$ implies that $\varphi_2(P_1^1, P_2^4)$ is either $(a_2, b_2)$ or $(a, b_1)$. If $\varphi_2(P_1^1, P_2^4) = (a, b_1)$ then efficiency yields that $\varphi_1(P_1^1, P_2^4) = (a_1, b_2)$. But this allocation is Pareto dominated by the allocation in which players 1 and 2 obtain $(a_2, b_1)$ and $(a_1, b_2)$ respectively. Therefore, $\varphi_2(P_1^1, P_2^4) = (a_2, b_2)$ and then efficiency yields that $\varphi(P_1^1, P_2^4) = \varphi(P_1^1, P_2^1)$. This along with the strategy-proofness of $\varphi$ and monotonicity yields that player 2 can never obtain any of $(a, b_1)$ where $a \neq a_1, a_2$ as long as $(a_1, b_1)$ is the most preferred bundle in player 1's reported preferences. Finally, consider $P_2^5$ in which the most preferred 5 bundles of player 2 are

| $P_2^5$ |
|---|
| $(a_1, b_1)$ |
| $(a_1, b_2)$ |
| $(a_2, b_1)$ |
| $(a_1, b)$ |
| $(a_2, b_2)$ |
| ... |

The strategy-proofness of $\varphi$ implies that $\varphi_2(P_1^2, P_2^5)$ is either $(a_2, b_2)$ or $(a_1, b)$. If $\varphi_2(P_1^2, P_2^5) = (a_1, b)$ then efficiency yields that $\varphi_1(P_1^2, P_2^5) = (a_2, b_1)$. But this allocation is Pareto dominated by the allocation in which players 1 and 2 obtain $(a_1, b_2)$ and $(a_2, b_1)$ respectively. This along with the strategy-proofness of $\varphi$ (and using monotonicity) yields that player 2 can never obtain any of $(a_1, b)$ where $b \neq b_1, b_2$ as long as $(a_1, b_1)$ is the most preferred bundle in player 1's reported preferences. This completes the proof of case (1).

*Case (2).* Now we will show that player 2 obtains $(a_1, b_1)$ under $\varphi$ if she reports $(a_1, b_1)$ as her most preferred bundle regardless of player 1's report.

The similar proof used in Case (1) yields that player 2 obtains $(a_1, b_2)$ under $\varphi$ if she reports $(a_1, b_2)$ as her most preferred bundle regardless of player 1's report. Now consider the preference profile $(Q_1^1, Q_2^1)$ in which the most preferred 4 bundles of the two players are

| $Q_1^1$ | $Q_2^1$ |
|---|---|
| $(a_1, b_2)$ | $(a_1, b_1)$ |
| $(a_2, b_2)$ | $(a_2, b_1)$ |
| $(a_1, b_1)$ | $(a_1, b_2)$ |
| $(a_2, b_1)$ | $(a_2, b_2)$ |
| ... | ... |

In this case, efficiency yields that players 1 and 2 obtain either $(i)$ $(a_1, b_2)$ and $(a_2, b_1)$ respectively or $(ii)$ $(a_2, b_2)$ and $(a_1, b_1)$. In case $(ii)$, using a similar logic as in Case (1) we know that player 2 always obtains $(a_1, b_1)$ as long as she reports $(a_1, b_1)$ as her top choice. This completes the proof. In case (i), again using the logic in case (1), we know that player 1 always obtains $(a_1, b_2)$ as long as she reports $(a_1, b_2)$ as her most preferred bundle. Then suppose both players report $(a_1, b_2)$ as their top choice. But both players cannot obtain $(a_1, b_2)$ which means case (i) cannot occur.

To complete the proof, we need to show that there is a player who obtains her most preferred reported bundle regardless of the other player's report. It suffices to show that for any $(a, b)$ and $(\hat{a}, \hat{b})$, there is only 1 one player who obtains $(a, b)$ whenever she reports $(a, b)$ as her most preferred bundle and $(\hat{a}, \hat{b})$ whenever she reports $(\hat{a}, \hat{b})$ as her most preferred bundle choice. By Claim 1, there must be a player who obtains $(a, b)$ when she reports $(a, b)$ as her most preferred choice. WLOG assume that this is player 1. Now let us argue that player 1 obtains $(a, \hat{b})$ when she reports $(a, \hat{b})$ as her most preferred choice. Otherwise, player

2 must obtain $(a, \hat{b})$ (by Claim 1) when she reports it as her most preferred choice. But then when player 1 reports $(a, b)$ as her most preferred choice and while player 2 reports $(a, \hat{b})$, $a$ has to be assigned to both players which is a contradiction. Hence, player 1 obtains $(a, \hat{b})$ when she reports $(a, \hat{b})$ as her most preferred choice. Iterating this argument one more time, we conclude that player 1 obtains $(\hat{a}, \hat{b})$ when she reports $(\hat{a}, \hat{b})$ as her most preferred choice.

□

# 6 Conclusion

We have studied the problem of centralized assignment in multiple markets, which includes the class of dynamic matching problems. In our main result, we showed that the set of rules that are strategy-proof, nonbossy and implement a Pareto efficient allocation is the set of sequential dictatorship rules. This result sharply contrasts with the centralized allocation in a single market, and with the single object allocation in static environments. In those problems, the Top Trading Cycles–and its generalizations denoted hierarchical exchange rules– satisfy the above mentioned criteria.

Our result provides further support for the use of sequential dictatorships in applications of dynamic matching problems. While these rules have the shortcoming that some agents might have a larger choice set than others, in some applications this shortcoming is less severe. For example, Kennes et al. (2012) suggest the use of a mechanism which is a variation of the sequential dictatorship for the dynamic problem of allocating children to public daycares. There, they proved that there is no algorithm that is both strategy-proof and stable and they also argued that, due to the dynamic nature of the problem and the fact that all agents have a known exit rate from daycares, the shortcoming described above is much less severe. The result of the current paper provides an additional –and perhaps stronger– justification for the use of the sequential dictatorship mechanism in the daycare problem.

We conclude by suggesting three practical approaches for this important class of market design problems. One approach is to use the sequential dictatorship algorithm, which is both Pareto efficient and strategy-proof. As we argued in the previous paragraph, in some contexts the shortcomings of this mechanism might be less severe. The other approach is to search for an algorithm that may not be strategy-proof but delivers a Pareto efficient matching. The difficulty here is that whenever revealing the true preferences is not a dominant strategy, coordination failure is likely to occur – unless the algorithm yields a unique Nash equilibrium that is simple enough such that all agents can fully understand the strategic nature of the game. Finally, one might want to weaken the solution concept and search for a strategy-proof algorithm that yields a matching that, although not necessarily Pareto efficient, achieves some appropriately defined welfare criterion.

One possible direction for future research is to identify classes of problems (appropriate restrictions on the preference profiles) within the multiple matching framework presented herein, for which Pareto efficiency and strategy-proofness are not incompatible with rules other than the sequential dictatorships.

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# 7 Appendix

Here, we prove Theorem 3 for the $n \geq 3$ case. One of the challenges of the main proof is that one has to use separable preferences only. Specifically, if one changes the relative ranking of $(a, b)$ and $(a, \hat{b})$ then the ranking of $(\bar{a}, b)$ and $(\bar{a}, \hat{b})$ must also be changed. Therefore, we do not have the luxury of isolating the relative ranking change of only two bundles. However, it turns out there is a set of specific preferences that are easy to work with. Below we define these preferences formally.

**Definition 7** (Generalized lexicographical preference)**.** *Consider a bijective function $\eta : A \cup B \to \{1, \cdots, |A| + |B|\}$. The preference relation of agent $i$, $R_i$, is a (generalized) lexicographical preference of the order $\eta$ if for any $(a, b), (\bar{a}, \bar{b}) \in A \times B$, $(a, b)P_i(\bar{a}, \bar{b})$ implies that one of the following conditions is satisfied:*

$$\min\{\eta(a), \eta(b)\} < \min\{\eta(\bar{a}), \eta(\bar{b})\}$$

$$\min\{\eta(a), \eta(b)\} = \min\{\eta(\bar{a}), \eta(\bar{b})\} \,\&\, \max\{\eta(a), \eta(b)\} < \max\{\eta(\bar{a}), \eta(\bar{b})\}.$$

Before we move on, let us consider an example of lexicographical preferences. Let $A = \{a_1, a_2 \cdots, a_m\}$, $B = \{b_1, b_2, \cdots, b_m\}$, $\eta(a_i) = 2i - 1$ and $\eta(b_i) = 2i$. Then any agent with the lexicographical preferences of the order $\eta$ would rank alternatives as follows:

$$\begin{gathered}(a_1, b_1)P_i(a_1, b_2)P_i \cdots P_i(a_1, b_m)P_i \\ (a_2, b_1)P_i(a_3, b_1)P_i \cdots P_i(a_m, b_1)P_i \\ (a_2, b_2)P_i(a_2, b_3)P_i \cdots P_i(a_2, b_m)P_i \\ (a_3, b_2)P_i(a_4, b_2)P_i \cdots P_i(a_m, b_2)P_i \\ \vdots\end{gathered}$$

We often use the following notation: $A^\eta(c) = \{a \in A : \eta(a) \geq \eta(c)\}$ and $B^\eta(c) = \{b \in B : \eta(b) \geq \eta(c)\}$. Let a set $D$ be the totally ordered set of $(A \cup B)$ in which the order of its members is determined by $\eta$. We find it convenient to say that $R_i$ is a lexicographical preference of order $D$ instead of saying that $R_i$ is a lexicographical preference of order $\eta$. Also, we use $A^D(c)$ and $B^D(c)$ instead of $A^\eta(c)$ and $B^\eta(c)$, respectively.

We will use the lexicographical preferences extensively in the proof of our main theorem because they are separable and possess several nice properties. First, let us show the separability of the lexicographical preferences in the following lemma.

**Lemma 2** (Separability)**.** *Any lexicographical preference $R_i$ of order $\eta$ is separable.*

*Proof.* For all $c \in A \cup B$, set $u_i(c) = 2^{-\eta(c)}$. Consider any $a, \hat{a} \in A$ and $b, \hat{b} \in B$. One can easily check that $(a, b) R_i (\hat{a}, \hat{b})$ if and only if $u(a) + u(b) \geq u(\hat{a}, \hat{b})$. □

Below let us state three additional properties of the lexicographical preferences which will be used later.

**Lemma 3.** *Let $\bar{a}$ and $\bar{b}$ be some neighboring members of $D$, i.e. $\left|\eta\left(\bar{a}\right) - \eta\left(\bar{b}\right)\right| = 1$. Let $R_i$ be player $i$'s lexicographical preference of order $D$.*

1. *Then $(\bar{a}, \bar{b})$ is the most preferred bundle of player $i$ in the set $A^D(\bar{a}) \times B^D(\bar{b})$ under $R_i$.*

2. *Let $D'$ be the ordered set which is obtained from $D$ by only reversing the orders of $\bar{a}$ and $\bar{b}$. Let $R'_i$ be player $i$'s lexicographical preference of order $D'$. Then $A^{D'}(\bar{a}) \times B^{D'}(\bar{b}) = A^D(\bar{a}) \times B^D(\bar{b})$, and $(\bar{a}, \bar{b})$ is the favorite bundle of player $i$ in the set $A^{D'}(\bar{a}) \times B^{D'}(\bar{b})$ under both $R_i$ and $R'_i$.*

3. *Let $D''$ be another ordered set in which any alternative's relative orders of $\bar{a}$ and $\bar{b}$ to the other alternatives are the same as under $D$. Suppose $R''$ is a lexicographical preference of order $D''$. Then $A^{D''}(\bar{a}) \times B^{D''}(\bar{b}) = A^D(\bar{a}) \times B^D(\bar{b})$ and $(\bar{a}, \bar{b})$ is the top choice of player $i$ within the set $A^{D''}(\bar{a}) \times B^{D''}(\bar{b})$ under both $R_i$ and $R''_i$.*

*Proof.* This lemma follows directly from the definition of lexicographical preferences. □

*Proof of Theorem 3.* Let $n \geq 3$, $|A| \geq n$ and $|B| \geq n$. To prove the theorem for this case we will use an induction argument.

**Induction Assumption** For all $n = 2, \cdots, m-1$ where $m - 1 \geq 2$, $|A| \geq m-1$, and $|B| \geq m-1$, there is a player to whom $\varphi$ always assigns her favorite reported bundle. We already know that this is true for the $n = 2$ case.

Now we will show that there exists player $j$ such that $\varphi$ always assigns player $j$ her favorite reported bundle for any case in which $n = m$, $|A| \geq m$, and $|B| \geq m$. For this proof we will need several steps.

*Claim 2. Let $|A| \geq m \geq 2$ and $|B| \geq m \geq 2$. Consider any set of agents $S \subset N$ and $P \in \mathcal{P}$. Let $A_S = \{a \in A : \varphi_i^A(P) = a \text{ for some } i \in S\}$. Similarly, define $B_S$. Then there must be player $i \notin S$ such that $\varphi_i(P)$ is the favorite bundle of $i$ in the set $A \setminus A_S \times B \setminus B_S$.*

*Proof of Claim 2.* Suppose not. Let $P^1$ be a monotonic change of $P$ with respect to $\varphi$ satisfying the following conditions:

1. if $j \in S$, then $\varphi_j(P)$ is the favorite bundle of $j$ in $A \times B$
2. if $i \notin S$, then $i$'s preferences satisfy that
   (a) whenever $(a,b) \in A \setminus A_S \times B \setminus B_S$ and $(\bar{a},\bar{b}) \notin A \setminus A_S \times B \setminus B_S$, then $(a,b)P_i^1(\bar{a},\bar{b})$
   (b) whenever $(a,b) \in A \setminus A_S \times B \setminus B_S$ and $(\bar{a},\bar{b}) \in A \setminus A_S \times B \setminus B_S$, then $(a,b)P_i^1(\bar{a},\bar{b})$ if and only if $(a,b)P_i(\bar{a},\bar{b})$

By Lemma 1, $\varphi(P^1) = \varphi(P)$. We will reach a desired contradiction once we show that there is a player $i \notin S$ whose favorite bundle in $A \setminus A_S \times B \setminus B_S$ is $\varphi_i(P^1)$.

To show this consider the class of preferences $\mathcal{P}^S$ such that $P' \in \mathcal{P}_S$ satisfies the following conditions:

1. if $j \in S$, then $P'_j = P^1_j$
2. if $i \notin S$, then $i$'s preferences satisfy that
   (a) whenever $(a,b) \in A \setminus A_S \times B \setminus B_S$ and $(\bar{a},\bar{b}) \notin A \setminus A_S \times B \setminus B_S$, then $(a,b)P'_i(\bar{a},\bar{b})$

Observe that $P^1 \in \mathcal{P}$. For this class of preferences, each player $i \in S$ must obtain $\varphi_i(P)$ as $\varphi$ is efficient. Consequently, for $\mathcal{P}^S$, we can treat $\varphi$ as the allocation rule that allocates $A \setminus A_S \times B \setminus B_S$ among the players not in $S$. Now by the induction assumption, we know that $\varphi$ must assign some player $j \notin S$ her reported favorite bundle in $A \setminus A_S \times B \setminus B_S$. This means that $\varphi_j(P^1)$ is the favorite bundle of player $j$ in $A \setminus A_S \times B \setminus B_S$ reaching a contradiction.

In fact, we can strengthen Claim 2 as follows:

*Claim 3. Suppose that whenever a set of players $S \subset N$ reports $P^S = (P_i)_{i \in S}$ then $\varphi$ assigns the same allocation to each $i \in S$ regardless of the other players' reports. Then there must be a player $j \notin S$ who obtains her favorite reported bundle in $A \setminus A_S \times B \setminus B_S$. In addition,* *if $\varphi_i(P) = \varphi_i(\bar{P})$ for all $i \in S \subset N$, there must be a player $j \notin S$ who obtains her favorite reported bundle in $A \setminus A_S \times B \setminus B_S$ for both reported preferences.*

*Proof of Claim 3.* This immediately follows from the previous claim and the induction assumption.

In the next 3 claims (4-6), we prove that for any $(a, b) \in A \times B$, there exists player $j$ who obtains $(a, b)$ whenever she reports $(a, b)$ as her favorite bundle. WLOG, let us set $a = a_1$ and $b = b_1$.

*Claim 4. Let us consider a lexicographical preference profile $L$ in which the order of the objects is $(a_1, b_1, a_2, b_2, \cdots)$.*[9] *Then each $(a_k, b_k)$ where $k \leq n$ must be allocated to some player under $L$.*

*Proof of Claim 4.* Since $n \geq 3$ there must be a player who does not obtain neither $a_1$ nor $b_1$. Now using Claim 2, we obtain there must be a player $i$ who obtains $(a_1, b_1)$. Given that player $i$ gets $(a_1, b_1)$ under $L$, by Claim 2 there must be a player $j \neq i$ who obtains $(a_2, b_2)$. We complete the proof by applying Claim 2 repeatedly.

WLOG, let us assume $\varphi_i(L) = (a_i, b_i)$. Now we show that $\varphi$ assigns player 1 $(a_1, b_1)$ if it is her favorite reported bundle.

*Claim 5.* Consider any lexicographical preference profile in which all players' order of the objects is the same and starts with $(a_1, b_1)$ and then alternates the remaining elements of $A$ and $B$. Then player 1 must obtain $(a_1, b_1)$.

*Proof of Claim 5.* We will first prove the following statement: Consider another lexicographical preference profile $\hat{L}$ which is obtained from $L$ by changing the order of the objects as follows:

$$(a_1, b_1, a_2, b_2, \cdots, a_{j-1}, b_{j-1}, a_j, \mathbf{b_{j+1}}, a_{j+1}, \mathbf{b_j}, a_{j+2}, b_{j+2}, a_{j+3}, b_{j+3}, \cdots)$$

where $j \geq 2$. Then for each $i < j$, $\varphi_i(\hat{L}) = \varphi_i(L)$ and $\varphi_j(\hat{L}) = (a_j, b_{j+1})$.

Observe that the proof of the statement above yields Claim 5 as any lexicographical preference profile specified in Claim 5 can be reached through a sequence of changes which starts from $L$ and in each change in the sequence, the order of only two neighboring objects of the same type is reversed.

If $j > n$, then $\hat{L}$ is a monotonic change of $L$ with respect to $\varphi$. Hence, Lemma 1 yields

[9] If $|A| > |B|$ (or $|A| < |B|$), then $a_j$ where $j > |B|$ is listed at the end.

the statement above. Let $j \leq n$. Consider a lexicographical preference profile $L^1$ in which

**(i)** for player $i < j$ the order of the objects is $(a_1, b_1, \cdots, a_i, b_i, a_j, b_{j+1}, \cdots)$

**(ii)** for player $j$ the order of the objects is $(a_1, b_1, \cdots, a_j, b_j, b_{j+1}, a_{j+1}, \cdots)$

**(iii)** for player $i > j$, $(a_j, b_1, a_1, b_2, \cdots, a_{j-1}, b_j, a_{j+1}, b_{j+1}, \cdots, a_i, b_i, \cdots)$.

Clearly $L^1$ is a monotonic change of $L$ with respect to $\varphi$. Hence, $\varphi(L) = \varphi(L^1)$. Now let $L^2$ be a lexicographical preference obtained from $L^1$ by changing only player $j$'s order of the objects as follows: $(a_1, b_1, \cdots, a_j, b_{j+1}, b_j, a_{j+1}, \cdots)$. Going from $L^1$ to $L^2$ only the relative ranking of $(a_j, b_{j+1})$ improves with respect to $(a_j, b_j)$ for player $j$. As $\varphi$ is strategy-proof, $\varphi_j(L^2)$ is either $(a_j, b_j)$ or $(a_j, b_{j+1})$. In the first case, thanks to nonbossiness, $\varphi(L^2) = \varphi(L^1)$. Then by Claim 3, player $j$ should obtain her favorite bundle among $A \setminus \{a_1, \cdots, a_{j-1}\} \times B \setminus \{b_1, \cdots, b_{j-1}\}$ which is $(a_j, b_{j+1})$ under $L^2$. This is a contradiction. Hence, $\varphi_j(L^2) = (a_j, b_{j+1})$. Because $(a_1, b_1)$ is the favorite bundle of every player in the set $\{A \setminus a_j \times B \setminus b_{j+1}\}$, someone other than $j$ must obtain $(a_1, b_1)$ by Claim 2. In addition, when $j > 2$, $(a_2, b_2)$ is the favorite bundle of every player in the set $\{A \setminus \{a_j, a_1\} \times B \setminus \{b_{j+1}, b_1\}$, someone must obtain $(a_2, b_2)$ by Claim 2. A similar logic yields that each of the $\{(a_1, b_1), \cdots, (a_{j-1}, b_{j-1})\}$ must be allocated to some player. However, observe that $(a_1, b_1)$ cannot be allocated to any player $i > j$. Otherwise, there is a Pareto improvement by swapping the allocations of players $j$ and $i$. Similarly, we obtain that none of the $\{(a_1, b_1), \cdots, (a_{j-1}, b_{j-1})\}$ are allocated to players $\{j+1, \cdots, n\}$. Now let us show that player 1 obtains $(a_1, b_1)$. Otherwise, she obtains one of the $\{(a_2, b_2), \cdots, (a_{j-1}, b_{j-1})\}$. But then players 1 and $j$ can swap their allocations and Pareto improve. Then player 2 must earn $(a_2, b_2)$; otherwise players 2 and $j$ can swap their allocations and Pareto improve. A similar logic yields that all players $i \leq j - 1$, $\varphi_i(L^2) = (a_i, b_i)$ and $\varphi_j(L^2) = (a_i, b_{j+1})$.

Now we need to change every player's preferences to $\hat{L}$ and show that this change does not alter the allocation of the players $\{1, \cdots, j\}$ under $L^2$. To prove this, we need some extra steps. First, observe that Claim 2 yields that there must be a player who obtains $(a_{j+1}, b_j)$ under $L^2$. Call this player $k > j$. Now consider a lexicographical preference $L^3$ under which the preferences of players $N \setminus k$ is the same as the ones under $\hat{L}$ but $k$'s is the same as the one under $L^2$. Observe that this is a monotonic change of $L^2$, hence $\varphi(L^3) = \varphi(L^2)$.

Let us consider $L^4$ in which

**(i)** for player $i < j$ the order of the objects is $(a_1, b_1, \cdots, a_i, b_i, a_{j+1}, b_{j+1}, a_{i+1}, b_{i+1}, \cdots)$

**(ii)** for player $j$ the order of the objects is $(a_1, b_1, \cdots, a_j, b_{j+1}, a_{j+1}, b_j, a_{j+2}, b_{j+2}, \cdots)$

**(iii)** for player $k$ the order of the objects is $(a_1, b_1, \cdots, a_j, b_j, b_{j+1}, a_{j+1}, \cdots)$ and

**(iv)** for player $i > j$ and $i \neq k$, $(a_{j+1}, b_1, a_1, b_2, \cdots, a_{j-1}, b_j, a_j, b_{j+1}, \cdots, a_i, b_i, \cdots)$.

Clearly $L^4$ is a monotonic change of $L^3$ with respect to $\varphi$. Hence, $\varphi(L^4) = \varphi(L^3)$. Now let $L^5$ be a lexicographical preference obtained from $L^4$ by changing player $k$'s order of the objects as follows: $(a_1, b_1, \cdots, a_j, b_{j+1}, a_{j+1}, b_j, \cdots)$. Going from $L^4$ to $L^5$ only the relative ranking of $(a_{j+1}, b_{j+1})$ improves with respect to $\varphi_k(L^4) = (a_{j+1}, b_j)$ for player $k$. As $\varphi$ is strategy-proof, $\varphi_j(L^5)$ is either $(a_{j+1}, b_j)$ or $(a_{j+1}, b_{j+1})$. Now we rule out the latter case. Suppose the latter case occurs. Using the same steps as we used to prove that $\{(a_1, b_1), \cdots, (a_{j-1}, b_{j-1})\}$ is allocated among the players $\{1, \cdots, j-1\}$ under $L^2$, we obtain that $\{(a_1, b_1), \cdots, (a_j, b_j)\}$ is allocated among the players $\{1, \cdots, j\}$. If player 1 does not obtain $(a_1, b_1)$, by swapping the allocations of 1 and $k$, we can Pareto improve. Similarly, players $i \leq j-1$, must obtain $(a_i, b_i)$. Therefore, player $j$ obtains $(a_j, b_j)$. But this is a contradiction with Claim 3 as $j$ is not obtaining her favorite bundle in $A \backslash \{a_1, \cdots, a_{j-1}\} \times B \backslash \{b_1, \cdots, b_{j-1}\})$ under $L^5$. Hence, $\varphi_k(L^5) = \varphi_k(L^4) = (a_{j+1}, b_j)$. Now nonbossiness gives that $\varphi(L^5) = \varphi(L^4)$. Now change everyone's preferences to $\hat{L}$ which is a monotonic change of $L^5$ with respect to $\varphi$. Hence, $\varphi(\hat{L}) = \varphi(L^5)$. This completes the proof of Claim 5.

*Claim 6. For any preference profile in which player 1 ranks $(a_1, b_1)$ as her favorite bundle, player 1 obtains $(a_1, b_1)$ regardless of what others report.*

*Proof of Claim 6.* Pick any preference profile $P$ in which player 1 ranks $(a_1, b_1)$ as her favorite bundle. Now let us construct a lexicographical preference $L^n$ in $n$ iterative rounds.

*Round 1.* Set $i_1 = 1$. Pick any lexicographical preference $L^1$ in which everyone's order of the objects is the same and starts with $(a_1, b_1)$ and alternates the members of $A_1$ and $B_1$. Set $I_1 = \{i_1\}$ and $A_1 = A \setminus a_1$ and $B_1 = B \setminus b_1$. Observe that $\varphi_{i_1}(L^1) = (a_1, b_1)$ by Claim 5.

*Round 2.* Pick the player who is in $N \setminus I_1$ and who obtains her favorite bundle under $L^1$ in $A_1 \times B_1$. This is always feasible thanks to Claim 3. (In fact, this player is the second player.) Call her $i_2$. Set $I_2 = I_1 \setminus i_2$. Pick the highest ranked alternative of player $i_2$ among $A_1 \times B_1$ under preference profile $P$. Let it be $(\hat{a}_2, \hat{b}_2)$. Set $A_2 = A_1 \setminus \hat{a}_2$ and $B_2 = B_1 \setminus \hat{b}_2$.

Pick a lexicographical preference $L^2$ in which the order of the objects is the same for everyone, starts with $(a_1, b_1, \hat{a}_2, \hat{b}_2)$ and alternates the members of $A_2$ and $B_2$. Observe that $\varphi_{i_1}(L^2) = (a_1, b_1)$ and $\varphi_{i_2}(L^2) = (\hat{a}_2, \hat{b}_2)$.

*Round k.* Pick the player who is in $N \setminus I_{k-1}$ and who obtains her favorite bundle under $L^{k-1}$ in the set $A_{k-1} \times B_{k-1}$. This is always feasible thanks to Claim 3. Call her $i_k$ and set $I_k = I_k \setminus I_{k-1}$. Pick the most preferred bundle of player $i_k$ among $A_{k-1} \times B_{k-1}$ under preference profile $P$. Let it be $(\hat{a}_k, \hat{b}_k)$. Set $A_k = A_{k-1} \setminus \hat{a}_k$ and $B_k = B_{k-1} \setminus \hat{b}_k$. Pick a lexicographical preference $L^k$ in which the order of the objects is the same, starts with $(a_1, b_1, \hat{a}_2, \hat{b}_2, \cdots, \hat{a}^k, \hat{b}^k)$ and then alternates the members of $A_k$ and $B_k$. Observe that $\varphi_{i_1}(L^k) = (a_1, b_1)$ and $\varphi_{i_j}(L^k) = (\hat{a}_j, \hat{b}_j)$ where $j \leq k$.

Now consider $L^n$. Now observe that $L^n$ is a monotonic change of $P$ with respect to $\varphi$. Therefore, by Lemma 1, player 1 must obtain $(a_1, b_1)$. This completes the proof of Claim 6.

*Claim 7. There exists a player who obtains her favorite reported bundle.*
*Proof of Claim 7.* Claims 2-6 show that for any $(a, b)$ there is a player who obtains her top choice whenever she reports $(a, b)$ as her top choice. This claim is proved once we show that for any other $(\hat{a}, \hat{b})$ the same player obtains $(\hat{a}, \hat{b})$ whenever she reports $(\hat{a}, \hat{b})$ as her top choice. Suppose otherwise. This means that there exists a player $i$ who obtains $(a, b)$ if she reports $(a, b)$ as her favorite bundle. Then there must be a player who obtains $(\hat{a}, b)$ whenever she reports $(\hat{a}, b)$ as her favorite bundle. This player must be $i$: otherwise, when this player reports $(\hat{a}, b)$ as her favorite bundle and player $i$ reports $(a, b)$ as hers, both players must obtain $b$ which is a contradiction. Iterating this argument one more time we obtain Claim 7.

*Claim 8. Any strategy-proof, Pareto efficient mechanism is a sequential serial dictatorship.*
*Proof of Claim 8.* From Claim 7, we know that there is a player who obtains her favorite reported bundle. Let this player be $i_1$. Now consider any two profiles $P \neq P'$ in which $i_1$'s favorite bundles are the same under both profiles. Then by Claim 3, there must be a player $i_2$ who obtains her favorite bundle after player $i_1$ chooses her favorite alternative. Iterating this argument we obtain Claim 8. $\square$

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