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To Err is Human: Implementation in Quantal Response Equilibria

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To Err is Human: Implementation in Quantal Response Equilibria

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Abstract

We study the classical implementation problem when players are prone to make mistakes. To capture the idea of mistakes, Logit Quantal Response Equilibrium (LQRE) is used, and we consider a case in which players are almost rational, i.e., the sophistication level of players, λ, approaches infinity. We show that quasimonotonicity, a small variation of Maskin Monotonicity, and no worst alternative conditions are necessary for restricted Limiting LQRE (LLQRE) implementation. Moreover, these conditions are sufficient for both restricted and unrestricted LLQRE implementations if there are at least three players and each player's worst alternative set is constant over all states.

Keywords: implementation; mechanisms; bounded rationality; quantal response equilibria.

JEL classification: C72, D70, D78.

1 Introduction

Nowadays the fields of bounded rationality and mechanism design are increasingly attracting the attention of economists. This paper contributes to solving the classical implementation

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problem when the players are boundedly rational.

In implementation theory, we consider the cases in which there are many different states of the world but in each state what is optimal for a society (summarized by social choice rule (SCR)) is known. In the election problem, for example, states are defined by the society members' preferences, and the society agrees that the candidate who is preferred by the majority should be elected. In King Solomon's problem, states are defined by who the true mother of the baby is and the social optimum is to give the baby to the true mother. Even though the SCR is fixed, there are usually states in which the social optima contradict the individual optima for some agents. Therefore, it is unreasonable to expect the society to make a choice consistent with the SCR after a state is realized. However, a benevolent third party who is not aware of the realized state, or the society itself before the realization of a state, may be able to guarantee the socially optimal outcomes in each state by designing a mechanism (a set of rules that result in an outcome based on the information sent by the society members) that is played by the society members once the uncertainty is resolved. Implementation theory investigates whether any mechanism can deliver the socially optimal outcomes in each state.

Since Hurwicz's seminal works in 1960 and 1972, the implementation problem has been studied from many different perspectives.[1] However, a majority of papers assume that players are fully rational. But what if the players are not fully rational? That is the main concern of this paper.

In this paper, we model irrationality as simple mistakes that occur when the players evaluate their best responses. This means that the players try to be rational, but because of their imperfect calculating ability, they might play non-optimal strategies. If some probabilistic structure is imposed on the mistakes, then the players have probabilistic responses. Now, assuming that the players are aware that the others are mistake prone, one can define equilibrium as a fixed point of the players' responses. This equilibrium is the well known *Quantal Response Equilibrium* (QRE) from McKelvey and Palfrey (1995).

Logit QRE (LQRE) is a QRE when mistakes are distributed *iid* with an extreme value distribution parameterized by $\lambda \in \mathbb{R}_+$ which we here interpret as the sophistication level. Thanks to this specification of the mistakes, the logit quantal response function has two desirable features. First, the players are more likely to make a smaller mistake than a bigger one. Second, as the sophistication level approaches infinity, the probability of a player playing a strategy not in the true best response monotonically decreases to 0. Therefore, the higher the λ, the more rational players are. In addition, if $\lambda = \infty$, then the players are fully rational, hence, any *limit LQRE* (LLQRE) is a Nash equilibrium. This equilibrium has

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[1] For more information see Jackson (2001), Maskin and Sjostrom (2002) and Serrano (2004).

the following nice property: if the players are close enough to being fully rational, then any resulting LQRE is very close to one of the LLQREs.

In addition to its desirable theoretical features, LQRE seems to explain the experimental results better than the Nash Equilibrium does. The original paper of McKelvey and Palfrey (1995) demonstrates the predictive ability of LQRE on several well known experiments whose results systematically deviated from the ones Nash equilibrium predicts. Since then, LQRE has been used to explain many experimental results such as the ones in Anderson et al. (1998) (all-pay auctions), Goeree et al. (2002) (first price auction), Anderson et al. (2001) (coordination games), Capra et al. (1999), Goeree and Holt (2001) (the "traveler's dilemma"), and Goeree et al. (2007) (information cascade).

Given the theoretical and empirical plausibility of LQRE, we assume that games result in LQREs. This paper studies the implementation problem when the equilibrium concept is LLQRE (not LQRE) for the following reason. If the players' sophistication level is high enough, then the LQREs can be proxied by the corresponding LLQREs. Therefore, any mechanism which implements an SCR in LLQREs will implement the SCR in LQREs with high probability.

First we characterize the sufficient conditions for LLQRE implementation. In environments with at least three players, if each player's worst alternative set is constant over all states, then any SCR satisfying *quasimonotonicity* (a small variation of Maskin Monotonicity (Maskin (1999))) and *no worst alternative* (NWA) is LLQRE implementable. Quasimonotic SCRs must satisfy the following condition: if an alternative stops being an SCR alternative going from one state to another, then for some player, some other alternative must become weakly better than the original social alternative in the second state from being strictly worse in the first state. We say SCR satisfies NWA if it does not prescribe any player's worst alternative in any state.

In the proof of the sufficiency result, we construct a mechanism that delivers each SCR alternative in each state through some strict LLQRE of the corresponding state. We should remark that this does not mean that all the LLQREs have to be strict; there can be some non-strict LLQREs in any state as long as each one of them yields an alternative prescribed by the SCR in the corresponding state. We use the above mentioned restriction because non-strict LLQREs are sometimes not preserved[2] under monotonic transformations (including affine) of the utility functions of the players. This becomes a problem if the planner only has information about the players' preference relations in which case she needs to ensure that LLQRE implementation is robust to monotonic transformations of the utility functions of the players. Otherwise, some social choice alternatives might not be implemented for

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[2] See example 2.7.

certain utility representations which must be avoided. In this paper, we were able to show that strict LLQREs do not depend on the utility representations of the players' preferences. In addition, using several examples, we demonstrate some complexities of determining the conditions under which non-strict LLQREs are robust to monotonic transformations of the utility functions of the players. Consequently, for our sufficiency result, we look for a mechanism that delivers each SCR alternative in each state through some strict LLQRE of the corresponding state. If one concentrates on the LLQRE implementation in which each SCR alternative in each state is delivered by some strict LLQRE, then our next result shows that quasimonotonicity and NWA are also necessary conditions. In this sense, the paper (almost) fully characterizes LLQRE implementation under the restriction that each SCR alternative in each state is delivered by some strict LLQRE.

There are a handful of papers which consider the irrationality of players in implementation theory. Cabrales (1999) and Cabrales and Ponti (2000) consider implementation in existing mechanisms under learning dynamics. Cabrales and Serrano (2007) investigate the case in which the players adjust their strategies in the direction of better responses. Interestingly, quasimonotonicity, which is found to be crucial in our analysis, is also key for implementation in recurrent strategies of better response dynamics. These papers require dynamic settings, while the setting used for this paper is static. Sandholm (2005) studies simple pricing schemes used in implementing efficient SCRs in evolutionary setting. The idea that some players are completely unpredictable has been studied by Eliaz (2002). Even though in LQRE, players play in this fashion when the sophistication level approaches 0, there is a big difference between our paper and that of Eliaz. In his setup only some of the players make mistakes while the others are rational. In contrast, in this paper every player makes small mistakes.

This paper is organized as follows: Section 2 contains preliminaries. Section 3 defines LLQRE and restricted LLQRE implementations and discusses their sufficient and necessary conditions. Section 4 considers variants of LQRE implementation, and Section 5 concludes.

2 Preliminaries

2.1 Implementation

Let the set of players be $N = \{1, .., n\}$ and let $A = \{a_1, ..., a_k\}$ be the set of social alternatives. Finite set Θ is the set of states and we use θ for a typical state. Each player i has a utility function[3] $u_i : A \times \Theta \to \mathbb{R}$. Let the environment E be $E = \langle N, A, (u_i(\cdot, \theta))_{i \in N} \rangle$ and denote

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[3] Since QRE is defined using utilities, we are using utilities instead of preferences. However, the results in this paper are robust to monotonic transformations of the utilities, so we can easily translate utilities into preferences.

the set of possible environments by $\mathcal{E}$. We define social choice rule (SCR) as a mapping F which associates each state with a subset of A, i.e., $F : \Theta \to 2^A \backslash \emptyset$. The SCR, depending on the state, specifies the social alternatives desirable to the planner — someone who has authority to implement a social alternative for the society. Consequently, after a state is realized, the planner is willing to choose any of the SCR alternatives in the realized state, but this information is unavailable to her. The players, on the other hand, know the state.

The planner controls the design of a mechanism (game form) which is a pair $\Gamma = ((M_i)_{i\in N}, g)$ where M_i is player i's message (strategy) space, and $g : \prod_{i\in N} M_i \to A$ is the outcome function mapping message space to social alternatives. Each pair $\langle E, \Gamma\rangle$ is a game in which the set of players is N, the set of strategy profiles is $M = \prod M_i$, and the payoff function for each player i is $u_i(g(m), \theta)$ where $m = (m_i)_{i\in N}$ is a message profile. Let $\mathcal{S}$ be a solution concept of game theory. We say Γ implements SCR F via $\mathcal{S}$ if $g(\mathcal{S}(\Gamma, E)) = F(\theta)$ for any $E \in \mathcal{E}$. This says when the solution concept is $\mathcal{S}$, the outcomes of the game in a given state must coincide with the social alternatives in the SCR in that state. Moreover, SCR F is $\mathcal{S}$ implementable if there exists a mechanism which implements F via $\mathcal{S}$. This paper investigates the implementation problem when the game theoretic solution is the Limiting Logit Quantal Response Equilibrium (LLQRE) concept.

2.2 Logit Quantal Response Equilibrium

In this subsection we define LLQRE and investigate its properties.
Consider a game $\langle E, \Gamma\rangle$ or equivalently $\langle N, (M_i)_{i\in N}, (u_i(\cdot, \theta))_{i\in N}\rangle$. Let the set of strategy profiles M be finite and let each M_i consist of J_i pure strategies, i.e., $M_i = \{m_{i1}, ..., m_{iJ_i}\}$. Since the state is fixed throughout this section, we exclude state θ from the notation of the utility function. Moreover, we write $u_i(m)$ for $u_i(g(m))$. Also, for each strategy profile m, let $m_{-i} = (m_j)_{j\neq i}$. Sometimes we write (m_i, m_{-i}) for m.

Let $\Delta_i = \{p_i \in \mathbf{R}^{J_i} : \sum_{j=1,...,J_i} p_{ij} = 1\}$ be the set of mixed strategies for player i. Sometimes the notation $p_i(m_{ij})$ is used for p_{ij}. We write $\Delta = \prod \Delta_i$ and let a typical element of Δ be $p = (p_i)_{i\in N}$. Denote $p_{-i} = (p_l)_{l\neq i}$. We use the shorthand notation $p = (p_i, p_{-i})$. Slightly abusing the notation, we use m_{ij} for p_i with $p_{ij} = 1$. The players' utility functions are assumed to be of the expected utility form (von Neumann-Morgenstern), i.e., $u_i(p) = \sum_{m\in M} p(m)u_i(m)$ where $p(m) = \prod_{i\in N} p_i(m_i)$.

In this model, the players try to be rational but they have an imperfect calculating ability. Specifically, given a strategy profile p, each player calculates the expected payoff for each of her strategies. However, she may make mistakes in the calculations and so could play non-optimal strategies.

For every player i and for each pure strategy m_{ij}, define the function $\bar{u}_{ij} : \Delta \to \mathbb{R}$ as $\bar{u}_{ij}(p) = u_i(m_{ij}, p_{-i})$. If player i has a perfect calculating ability, then $\bar{u}_{ij}(p)$ would be her evaluation of strategy m_{ij}. Let $\bar{u}_i(p) = (\bar{u}_{ij}(p))_{j=1,\cdots,J_i}$.

Now we introduce mistakes into the players' evaluations. For each player i, the mistake $\epsilon_i \in \mathbb{R}^{J_i}$ is distributed with a CDF F_i. Define player i's evaluation of the strategies with respect to mistake structure ϵ_i as:

$$\hat{u}_i(p) = \bar{u}_i(p) + \epsilon_i.$$

For a given realization of the mistakes, the players choose a strategy with the highest $\hat{u}_{ij}(p)$ (instead of the one with highest expected utility) against $p \in \Delta$. In other words, player i will play strategy m_{ij} if $\hat{u}_{ij}(p) \geq \hat{u}_{ik}(p)$ for all $k \neq j$ when the others are following p_{-i}. Because of the random structure of the mistakes, the players respond stochastically against others' strategies. Let us define player i's ij response set $R_{ij} \in \mathbb{R}^{J_i}$ for a given strategy profile p by

$$R_{ij}(p) = \{\epsilon_i \in \mathbb{R}^{J_i} : \bar{u}_{ij}(p) + \epsilon_{ij} \geq \bar{u}_{ik}(p) + \epsilon_{ik} \text{ for all } k \neq j\}.$$

Now we can find the probability of player i playing strategy m_{ij} for a given mistake structure ϵ_i

$$\sigma_{ij}(p) = \int_{R_{ij}(p)} dF_i(\epsilon_i).$$

The function $\sigma_i(p) = (\sigma_{ij}(p))_{j=1,\cdots,J_i}$ is called the quantal response function. Now assuming that each player knows that the others are mistake prone, we can define an equilibrium concept which is known as the quantal response equilibrium (QRE).

Definition 2.1. Let $G = \langle N, (M_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a game in normal form and the mistake for each player i be distributed with a CDF F_i. A *quantal response equilibrium* (QRE) is a mixed strategy $\pi \in \Delta$ such that for all $i \in N$, $1 \leq j \leq J_i$

$$\pi_{ij} = \sigma_{ij}(\pi).$$

We will put the following restriction on the structure of the mistakes, following McKelvey and Palfrey (1995).

Assumption 2.2. For each player i, ϵ_{ij}s are independently and identically distributed with CDF $F_{ij}(\epsilon_{ij}) = \exp\left(-\exp(-\lambda \epsilon_{ij} - \gamma)\right)$

This assumption says that each players' mistakes are independent of each other and follow the extreme value distribution.[4] Then the mean mistake is 0 and the variance is $\frac{1\pi^2}{6\lambda^2}$. Hence, as parameter λ increases, the mistakes will be more concentrated around 0. Therefore, we call λ the sophistication level. With the above specification of the mistakes, the logit quantal response function $\sigma_i(p)$ is given by

$$\sigma_{ij}(p, \lambda) = \frac{\exp(\lambda \bar{u}_{ij}(p))}{\sum_{k=1,\cdots,J_i} \exp(\lambda \bar{u}_{ik}(p))}.$$

The logit quantal response function has the following desirable properties. First, the strategies with a higher expected payoff is played more frequently for any level of sophistication $\lambda \neq 0$. Secondly, as the sophistication level increases, the players play the strategies not in the best response less frequently. Therefore, when λ converges to infinity, the logit quantal response function pointwise converges to the best response whenever it is single valued. If it is multi valued, all strategies in the best response are played with equal probabilities. Therefore, in the $\lambda = \infty$ case the players respond optimally to others' strategies, hence, they are rational.

The logit quantal response equilibrium (LQRE) is an QRE under assumption 2.2. We will denote the set of LQREs for a given game G and sophistication level λ by $L(\lambda, G)$. Now let us consider the Limiting LQRE which is the game theoretic solution concept for implementation in this paper.

Definition 2.3. Consider a finite game $G = \langle N, (M_i)_{i\in N}, (u_i)_{i\in N}\rangle$. $\pi^* \in \Delta$ is Limiting LQRE if there exists $\{\pi_t\} \to \pi^*$ where $\pi_t \in L(\lambda_t, G)$ for some $\{\lambda_t\} \to \infty$. Denote $L(G)$ as the set of LLQREs.

This equilibrium has an attractive property: if the players' sophistication level is high enough, then any resulting LQRE will be very close to one of the LLQREs. Therefore, if a mechanism implements an SCR in LLQREs, then the mechanism implements the SCR in LQREs with a high probability as long as the playerss are sophisticated enough.

Before we move on, we need to clarify the connection between the set of LLQREs and the one of Nash equilibria. It is well known that the former is a subset of the latter [5], but exactly what Nash equilibria are LLQREs is not clear. Since this information plays a crucial role for LLQRE implementation, we study when a pure Nash equilibrium is an LLQRE.

Lemma 2.4, which we will consider shortly, shows that all strict Nash equilibria are LLQREs. The reason is that in a close enough neighborhood of a strict Nash equilibrium, the logit quantal response must be arbitrarily close to the Nash equilibrium if the sophistication

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[4] See McKelvey and Palfrey (1995) for the justification of this specification.

[5] For more information see McKelvey and Palfrey (1995).

level of the players is high enough. Therefore, if the sophistication level of the players is high enough, then the logit quantal response function maps a small enough neighborhood of a strict Nash equilibrium to itself, implying that there is a fixed point or an LQRE thanks to the Brower's fixed point theorem. This intuition is formalized in lemma 2.4.

Lemma 2.4. *Let π^* be a pure strict Nash Equilibrium, i.e., $u_i(\pi^*) > u_i(\pi_i, \pi^*_{-i})$ for all $i \in N$ and $\pi_i \neq \pi^*_i$. Then π^* is an LLQRE.*

Proof. Define $B_\delta(\pi^*) :\equiv \{\pi \in \Delta : |\pi - \pi^*| \leq \delta\}$ and let $b_i(\pi)$ be the best response correspondence. Since π^* is a strict pure Nash equilibrium, there exists $\bar{\delta}$ such that $\pi^*_i = b_i(\pi)$ for any $\pi \in B_{\bar{\delta}}(\pi^*)$ and $i \in N$. Pick any $\{\lambda_t\} \to \infty$. We know the sequence of logit quantal response functions, $\{\sigma(\cdot, \lambda_t)\}$, pointwise converges to π^* on $B_{\bar{\delta}}(\pi^*)$. Since $B_{\bar{\delta}}(\pi^*)$ is compact and each $\sigma(\cdot, \lambda_t)$ is continuous, we can find $\tau_t = \max_{\pi \in B_{\bar{\delta}}(\pi^*)} |\sigma(\pi, \lambda_t) - \pi^*|$. Clearly, the sequence $\{\tau_t\}$ converges to 0. Therefore, there must exist $\bar{t}$ such that $\tau_t \leq \bar{\delta}$ for $t > \bar{t}$. This implies that if $t > \bar{t}$, then $\sigma(\pi, \lambda_t) \in B_{\bar{\delta}}(\pi^*)$ for any $\pi \in B_{\bar{\delta}}(\pi^*)$. Consider any $t > \bar{t}$. Since $B_{\bar{\delta}}(\pi^*)$ is convex and compact, and $\sigma(\pi, \lambda_t)$ is continuous and maps $B_{\bar{\delta}}(\pi^*)$ to itself, there must exist a fixed point by Brower's fixed point theorem. Denote this fixed point by π_t. Since $\{\sigma(\pi, \lambda_t)\} \to \pi^*$ for any $\pi \in B_{\bar{\delta}}(\pi^*)$, $\{\pi_t\}$ must also converge to π^*. This proves the lemma. ☐

Thanks to lemma 2.4, it is clear that the set of strict LLQREs coincides with the set of strict Nash equilibria. It is also well known that strict Nash equilibria are preserved under monotonic transformations of the players' utilities.[6] Consequently, strict LLQREs are preserved under monotonic transformations of the players' utilities.

Now let us turn our attention to the relation between non-strict Nash equilibria and LLQREs. First, in the next lemma, we show that any pure Nash equilibrium for which exactly $n - 1$ players have a single valued best response cannot be an LLQRE. To see the intuition, consider a Nash equilibrium to which exactly $n - 1$ players have a unique best response. We know that the odd players' logit quantal response to this Nash equilibrium is to play all strategies in her best response with equal probability. Therefore, in order for the odd player to play the strategy that is part of the Nash equilibrium, the others should not play strategies that are "too close" to the Nash equilibrium. However, it happens that the quantal response function converges at an infinitely faster rate to the best response function for the players whose best response correspondence is single valued, compared to the one whose best response correspondence is not single valued. Therefore, the $n - 1$ players who have a unique best response play exactly those strategies that are "too close" to the Nash equilibrium, hence, we obtain a contradiction.

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[6] This is also true for non-strict Nash equilibria.

Lemma 2.5. *Let G be a game in which each player has at least two pure strategies. If π^* is a pure LLQRE in G, then*

(a) *there cannot exist exactly $n-1$ players whose best response set to π^* is single valued.*

(b) *for any player i and her strategy m_{ij_i} with $\pi^*_{ij_i} = 1$, there cannot exist any other strategy $m_{ij} \neq m_{ij_i}$ such that $u_i(m_{ij}, m_{-i}) \geq u_i(m_{ij_i}, m_{-i})$ for all m_{-i}.*

Proof. See Appendix. □

As a result of lemmas 2.4 and 2.5, we know that if any pure Nash equilibrium is an LLQRE, then either this Nash equilibrium is strict or there are at least two players who have a multi-valued best response to this equilibrium. We must remark that lemmas 2.4 and 2.5 do not completely specify the relation between pure Nash equilibria and LLQREs, which one would like to determine. However, the examples we will consider next demonstrate that such a hope is, perhaps, too optimistic.

Example 2.6. Consider the following two player game:

		$P2$	
		m_1	m_2
P1	m_1	(a, b)	(c, b)
	m_2	(a, d)	$(0, 0)$

Suppose a, b, c, and d are strictly positive, then unlike the conditions in lemma 2.5, neither players' best response to (m_1, m_1) is unique. In this game, (m_1, m_1) is an LLQRE.
To show this, let p_{ij} be the probability that player i plays strategy j. Then the following condition must be satisfied for any $p \in L(\lambda)$.

$$d(1 - p_{11}) \ln \frac{p_{11}}{1 - p_{11}} = c(1 - p_{21}) \ln \frac{p_{21}}{1 - p_{21}} \tag{1}$$

From the proof of lemma 2.5, we know that both sides of equation 1 would go to 0, if $p_{i1} \to 1$ where $i = 1, 2$. This implies that there exists a small enough $\epsilon > 0$ and $\delta > 0$, such that for any $p_{11} \in [1 - \epsilon, 1)$, there exists $p_{21} \in [1 - \delta, 1)$ satisfying equation 1. This means there is a function $p_{21} : [1 - \epsilon, 1) \to [1 - \delta, 1)$. From $\frac{p_{11}}{1 - p_{11}} = \exp\left(\lambda c(1 - p_{21}(p_{11}))\right)$ we can find $\lambda(p_{11}) = \frac{\ln \frac{p_{11}}{1 - p_{11}}}{c(1 - p_{21}(p_{11}))}$. Observe that $\lambda(p_{11}) \to_{p_{11} \to 1} \infty$ from 1. Now pick any strictly increasing sequence of p_{11}s converging to 1, then find the corresponding sequence of $\lambda(p_{11})$s. For this sequence of $\lambda(p_{11})$s, p_{11} and $p_{21}(p_{11})$ are LQREs and they converge to 1. Hence, (m_1, m_1) is an LLQRE. ◇

Example 2.6 illustrates that sometimes a Nash equilibrium to which at least 2 players' best response is multi-valued is an LLQRE. Based on this example, one might expect similar results to emerge in games with more than 2 players. The following example shows that is not necessarily the case.

Example 2.7. Let $n = 3$ and the strategies and the payoffs are given in the following table.

P3

m_1

P2

		m_1	m_2
P1	m_1	$(1, 0.5, 1)$	$(1, 0.5, 1)$
	m_2	$(1, 0.5, 1)$	$(0, 0, 0)$

m_2

P2

		m_1	m_2
$P1$	m_1	$(1, 0.5, 1)$	$(0, 0, 0)$
	m_2	$(0, 0, 0)$	$(0, 0, 0)$

For this game $(m_1)_{i=1,2,3}$ is not an LLQRE.
To show this, suppose $(m_1)_{i=1,2,3}$ is an LLQRE; this means that there exists a sequence $\{p_t\} \to (m_1)_{i=1,2,3}$ such that $p_t \in L(\lambda_t)$ for some $\{\lambda_t\} \to \infty$. For any $p \in \{p_t\}$, the following conditions must be satisfied.

$$(p_{11}p_{32}0.5 + p_{12}p_{31}0.5) \ln \frac{p_{11}}{p_{12}} = (p_{21}p_{32} + p_{22}p_{31}) \ln \frac{p_{21}}{p_{22}}$$

$$(p_{11}p_{32}0.5 + p_{12}p_{31}0.5) \ln \frac{p_{31}}{p_{32}} = (p_{11}p_{22} + p_{12}p_{21}) \ln \frac{p_{21}}{p_{22}}$$

By symmetry $p_{11} = p_{31}$ at LQRE since p_{11} and p_{31} are around 1, so we obtain:

$$p_{11}p_{12} \ln \frac{p_{11}}{p_{12}} = (p_{21}p_{12} + p_{22}p_{11}) \ln \frac{p_{21}}{p_{22}}$$

For this equation to hold the following two inequalities must be satisfied: $p_{11}p_{12} \ln \frac{p_{11}}{p_{12}} > p_{21}p_{12} \ln \frac{p_{21}}{p_{22}}$ and $p_{11}p_{12} \ln \frac{p_{11}}{p_{12}} > p_{22}p_{11} \ln \frac{p_{21}}{p_{22}}$. The first one yields $p_{11} > p_{21}$ and the second one gives $p_{11} < p_{21}$ if p_{21} is close to 1. These inequalities contradict each other, so $(m_1)_{i=1,2,3}$ is not an LLQRE. ◇

The above example shows the difficulties of supporting a non-strict Nash equilibrium as an LLQRE, and this example is quite robust to perturbations of payoffs.

Another equally important feature of non-strict LLQRE is that it is not preserved under monotonic transformation (including affine) of the utilities of players. To see this, suppose that we doubled player 2's utilities in example 2.7. Now we can easily show that $(m_1)_{i=1,2,3}$ is an LLQRE. This complicates implementation in LLQREs if the planner has information only about the preferences of players, as discussed in the next section.

Equipped with definitions of LQRE and lemmas 2.4 and 2.5, we can start investigating the LLQRE implementation in the next section.

3 LLQRE Implementation

In this section, we introduce the concept of LLQRE implementation and identify the sufficient and necessary conditions that characterize the family of SCRs which are LLQRE implementable. Before we start, let us remark that we concentrate only on implementation in pure LLQREs, which is somewhat restrictive.

For a given mechanism $\Gamma = ((M_i)_{i \in N}, g)$, let $L(\lambda, \Gamma, \theta)$ be the set of LQREs of $\langle E, \Gamma \rangle$ when the sophistication level of the players are λ. In addition, $L(\Gamma, \theta)$ be the set of *pure* LLQREs of $\langle E, \Gamma \rangle$. Now we are ready to define the LLQRE implementation.

Definition 3.1. Mechanism $\Gamma = ((M_i)_{i \in N}, g)$ implements[7] SCR F via LLQREs if $g(L(\Gamma, \theta)) = F(\theta)$ for each $E \in \mathcal{E}$. We say SCR F is LLQRE implementable if there exists a mechanism that implements F via LLQREs.

In the above definition, if a mechanism implements a given SCR then it must satisfy two requirements: (1) in each state, each SCR alternative must be reached via some LLQRE of the mechanism and (2) in each state, each LLQRE of the mechanism must deliver some SCR alternative.

3.1 Sufficient Conditions

Now we present the sufficient conditions that characterize the family of SCRs that can implemented in LLQREs.

Definition 3.2. SCR F is *quisimonotonic* if whenever an alternative $a \in F(\theta)$ and $a \notin F(\theta')$ for some θ and θ', there exist player $i \in N$ and $a_i \in A$ such that

$$u_i(a_i, \theta) < u_i(a, \theta) \text{ and } u_i(a_i, \theta') \geq u_i(a, \theta') \tag{2}$$

Player i and corresponding alternative a_i that satisfy condition 2 are called test player and test alternative for player i with respect to triplet (a, θ, θ'); we use notation $I(\theta, \theta', a)$ for the set of test players with respect to (a, θ, θ') and $A_i(\theta, \theta', a)$ for the set of test alternatives for test player i with respect to (a, θ, θ'). Furthermore, we decompose $I(a, \theta, \theta')$ into $I^1(\theta, \theta', a)$

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[7] The message space must be finite, otherwise the probability of playing any specific strategy is 0 at any LQRE regardless of λ, so this probability remains 0 at LLQRE.

and $I^2(\theta, \theta', a)$, and $A_i(a, \theta, \theta')$ into $A_i^1(\theta, \theta', a)$, and $A_i^2(\theta, \theta', a)$. For any player $i \in I(a, \theta, \theta')$, social alternative $a_i \in A_i(a, \theta, \theta')$ that satisfies the second inequality of condition 2 with strict inequality/equality belongs to $A_i^1(a, \theta, \theta')/A_i^2(a, \theta, \theta')$. Player $i \in I(a, \theta, \theta')$ whose $A_i^1(a, \theta, \theta') \neq \emptyset / A_i^1(a, \theta, \theta') = \emptyset$ belongs to $I^1(\theta, \theta', a)/I^2(\theta, \theta', a)$.

Quasimonotonicity is closely related to Maskin Monotonicity which is known to be the necessary and almost sufficient condition for implementation in Nash equilibria. To make the differences between the two conditions clear, let us give the formal definition of Maskin Monotonicity.

Definition 3.3. SCR F is *Maskin monotonic* if whenever an alternative $a \in F(\theta)$ and $a \notin F(\theta')$ for some θ and θ', there exist player $i \in N$ and $a_i \in A$ such that

$$u_i(a_i, \theta) \leq u_i(a, \theta) \text{ and } u_i(a_i, \theta') > u_i(a, \theta') \tag{3}$$

If a Maskin monotonic SCR prescribes social alternative a in state θ and if every player's lower contour set of a weakly expands from state θ to state θ', then the SCR must prescribe a in state θ'. On the other hand, if a quasimonotonic SCR prescribes social alternative a in state θ and if every player's *strict* lower contour set of a weakly expands from state θ to state θ', then the SCR must prescribe a in state θ'. Logically, quasimonotonicity is neither a weaker nor stronger condition than Maskin monotonicity. However, these conditions coincide if the preferences of players are strict or continuous in every state.

Even though quasimonotonicity is very similar to Maskin monotonicity, there are some interesting SCRs that are quasimonotonic but not monotonic or vise versa. For example, weak pareto correspondence is well known to be monotonic, yet is not necessarily quasimonotonic. To illustrate this point let us consider the following example:

Example 3.4. There are two players $\{1, 2\}$, two alternatives $\{a, b\}$, and two states $\{\theta, \theta'\}$. Suppose the utilities and SCR are given as follows:

	θ		θ'	
	P1	P2	P1	P2
a	5	5	5	5
b	5	0	0	0
	$F(\theta) = \{a, b\}$		$F(\theta') = \{a\}$	

One can easily check that SCR F is Maskin monotonic, however it is not quasimonotonic.$\diamond$

On the other hand, strong pareto correspondence is well known to be not necessarily Maskin monotonic, however, it is quasimonotonic with a slight restriction on the preferences of the players.

Example 3.5. Consider environments in which at least one player has strict preferences in all states. Then the strong pareto correspondence F is quasimonotonic.
Suppose that F is not quasimonotonic. Then there must exist a, θ and θ' such that $a \in F(\theta)$, $a \notin F(\theta')$ and for any i and $a_i \in A$, $u_i(a_i, \theta) < u_i(a, \theta)$ implies that $u_i(a_i, \theta') < u_i(a, \theta')$. However, since F is a strong pareto correspondence, there must exist a' which weakly pareto dominates a in state θ' but not in state θ. Let i^* be the player whose preferences are strict in all states. First, let us eliminate the $u_{i^*}(a', \theta) < u_{i^*}(a, \theta)$ case. If $u_{i^*}(a', \theta) < u_{i^*}(a, \theta)$, then $u_{i^*}(a', \theta') > u_{i^*}(a, \theta')$ thanks to i^* having a strict preference and a' weakly dominating a in state θ'. This contradicts that for any i and $a_i \in A$, $u_i(a_i, \theta) < u_i(a, \theta)$ implies $u_i(a_i, \theta') < u_i(a, \theta')$. Hence, $u_{i^*}(a', \theta) > u_{i^*}(a, \theta)$. Then there must exist $i' \neq i^*$ such that $u_{i'}(a', \theta) < u_{i'}(a, \theta)$. Otherwise, $u_i(a', \theta) \geq u_i(a, \theta)$ for all $i \neq i^*$ and $u_{i^*}(a', \theta) > u_{i^*}(a, \theta)$ for i^*. Consequently, a' weakly pareto dominates a in state θ which contradicts that $a \in F(\theta)$. Now observe that for i', $u_{i'}(a', \theta) < u_{i'}(a, \theta)$ and $u_{i'}(a', \theta') \geq u_{i'}(a, \theta')$. This contradicts that for all i and $a_i \in A$, $u_i(a_i, \theta) < u_i(a, \theta)$ implies $u_i(a_i, \theta') < u_i(a, \theta')$. As a result, F is quasimonotonic. ◇

Now that we have discussed the differences between quasimonotonicity and Maskin monotonicity, let us now present the final condition required for LLQRE implementation.

Definition 3.6. SCR F satisfies no worst alternative (NWA) property if $a \in F(\theta)$ implies that for any i there exists an alternative b such that $u_i(b, \theta) < u_i(a, \theta)$.
We use the notation $A_i(a, \theta)$ to denote the set $\{b : u_i(b, \theta) < u_i(a, \theta)\}$

This property says that in any given state, SCR must not prescribe any player's worst alternative which could differ from one state to another. In many situations NWA property is satisfied naturally. For example, in an exchange economy setting consider Paretian SCR F which prescribes some consumption greater than the subsistence level of consumption, $\epsilon > 0$, to every player in every state. This SCR F satisfies NWA.

We derive the sufficiency conditions for LLQRE implementation for the environments satisfying the following assumption.

Assumption 3.7. For every player $i \in N$ there exists an alternative set $W_i \subset A$ such that

1. $u_i(w_i, \theta) < u_i(a, \theta)$ for all $\theta \in \Theta$, $w_i \in W_i$, and $a \in A \backslash W_i$
2. $u_i(w_i, \theta) = u_i(w_j, \theta)$ for all $\theta \in \Theta$, and $w_i, w_j \in W_i$ if $|W_i| > 1$.

This assumption says that for each player there must exist a social alternative set whose elements are the worst for the player in each state. We call W_i the worst alternative set player i. The worst alternative set for different players can vary. Even though this assumption is somewhat restrictive, many natural situations do satisfy it. For example, in an exchange economy setting with strictly increasing preferences, 0 consumption is the worst alternative for everyone. If an environment satisfies assumption 3.7, then NWA reduces to condition $W_i \cap F(\theta) = \emptyset$ for all θ.

Now we are ready to present the sufficiency result for LLQRE implementation.

Theorem 3.8. *Suppose $n \geq 3$ and the environment satisfies assumption 3.7. If SCR F satisfies quasimonotonicity and NWA then F is LLQRE implementable.*

Sketch of the proof. Consider a mechanism $\Gamma = ((M_i)_{i \in N}, g)$, such that $M_i = A \times \Theta \times \Theta \times \{0, 1\}$. Let a typical message m_i of player i be of the form $(a_i, \theta_i^1, \theta_i^2, \nu_i)$. The outcome function g is as follows:

1. Every player sends $m_i = (a, \theta, \theta, 0)$ where $a \in F(\theta)$. Then $g(m) = a$
2. Every player $i \neq j$ sends $m_i = (a, \theta, \theta, 0)$ where $a \in F(\theta)$ and player j sends message $m_j = (a_j, \theta_j^1, \theta_j^2, \nu_j) \neq (a, \theta, \theta, 0)$. Then
    (a) $g(m) = a_j$ if $m_j = (a_j, \theta, \theta_j^2, \nu_j)$, $j \in I(a, \theta, \theta_j^2)$, and $a_j \in A_j(a, \theta, \theta_j^2)$
    (b) $g(m) = w_j$ where $w_j \in W_j$ if m_j violates one of the conditions in $2a$
3. Some players send $m_i = (a, \theta, \theta, \nu_i)$ where $a \in F(\theta)$ while at least 2 players send $m_j = (a_j, \theta, \theta_j^2, \nu_j)$. If $j \in I(a, \theta, \theta_j^2)$, and $a_j \in A_j(a, \theta, \theta_j^2)$, then $g(m) = a_l$ where l is the lowest indexed player among those whose message contains differing states. (If all players send $m_j = (a_j, \theta, \theta_j^2, \nu_j)$, we check whether there exists $a \in F(\theta)$ such that all $j \in I(\theta, \theta_j^2, a)$ and $a_j \in A_j(\theta, \theta_j^2, a)$. If such a exists, then a_1 is implemented.)
4. In all other cases,
    (a) $g(m) = a_1$ if $\theta_1^1 \neq \theta_1^2$
    (b) $g(m) = w_1$ if $\theta_1^1 = \theta_1^2$

The above mechanism LLQRE implements F. We delegate the formal proof to the Appendix. Here, let us demonstrate the key reasons why the canonical mechanism implements F in LLQREs using an example. Suppose there are 3 players, 2 states $\{\theta, \theta'\}$, and 4 alternatives $\{a, b, c, w\}$. Let the preference ranking of the players be given in the following way.

θ			θ'		
P1	P2	P3	P1	P2	P3
a	$a \sim b$	a	a	b	a
$b \sim c$	c	b	b	a	b
w	w	c	c	c	c
		w	w	w	w

Let $F(\theta) = \{a\}$ and $F(\theta') = \{a, b\}$

Obviously this SCR does not satisfy Maskin Monotonicity, hence, Nash implementation is impossible. However, this SCR is LLQRE implementable.

The hardest part to prove is that the message profile $(b, \theta', \theta', 0)_{i=1,2,3}$, which would implement b, is not an LLQRE in state θ. Other cases can be handled easily.

Now let us consider message profile $(b, \theta', \theta', 0)_{i=1,2,3}$ in state θ. Clearly, player 3 cannot unilaterally deviate without strictly hurting herself, but thanks to rule 2, players 1 and 2 can do so by deviating to $(c, \theta', \theta, \nu_1)$ and $(a, \theta', \theta, \nu_2)$, respectively. Furthermore, if players 1 and 2 deviate at the same time then the outcome would be c. This is shown in the following table in which we assume player 3 sends $(b, \theta', \theta', 0)$.

		$P2$	
		$(b, \theta', \theta', 0)$	$(a, \theta', \theta, \nu_2)$
$P1$	$(b, \theta', \theta', 0)$	b	a
	$(c, \theta', \theta, \nu_1)$	c	c

We can see clearly that if player 3 sends $(b, \theta', \theta', 0)$, then player 2 is indifferent between $(b, \theta', \theta', 0)$ and $(a, \theta', \theta, \nu_2)$ as long as player 1 plays $(b, \theta', \theta', 0)$ or $(c, \theta', \theta, \nu_1)$. This feature allows us to prove that $(b, \theta', \theta', 0)_{i=1,2,3}$ is not an LLQRE in state θ. $\square$

In the proof of theorem 3.8, we construct a mechanism that LLQRE implements any given SCR F satisfying NWA and quasimonotonicity in environments with at least three players if the players' worst alternative set is constant. As mentioned earlier, the mechanism must satisfy the following two requirements: (1) in each state, each alternative in the SCR must be reached via some LLQRE and (2) in each state, each LLQRE of the mechanism must yield an SCR alternative. In fact, each SCR alternative in any state is reached via some *strict* LLQRE in our proposed mechanism. Hence, this mechanism satisfies an even stronger requirement than (1). To be specific, in our proposed mechanism, each player sends a message consisting of four components: a social alternative, two states, and one of 0 or 1. If every player reports the true state, coordinates on an SCR in the true state, and sends 0 as the last component of her message, then the outcome function "implements" the alternative every one selects according to rule 1. If anyone unilaterally deviates from this message profile, then the outcome function must follow rule 2 which is constructed so that it punishes the deviator. We are able to do this thanks to NWA. Therefore, the message profile in which every player reports the true state, the same social social choice alternative in the true state, and 0 as the last component of one's message is a strict Nash Equilibrium. This profile is also a strict LLQRE thanks to lemma 2.4.

We need rules 2, 3, and 4 to satisfy requirement (2). These rules are designed so that any message profile in which someone's message contains different states cannot be an LLQRE strategy. This is because the outcome function never follows rule 1 whenever a player sends different states in her message. Then this player must be indifferent between this message and the one in which she changes only the fourth component of her original message. Consequently, if some message profile is an LLQRE in some state, then everyone's message must contain two identical states. This means that any message profile for which the outcome function follows rules 2a, 3 or 4a is not an LLQRE. For any message profile for which the outcome function follows rule 2b, the deviator's worst alternative is implemented. Then deviator can unilaterally change her message and induce the case in which the outcome function follows rule 1. Hence, for any LLQRE strategy profile, the outcome function must not follow rule 2b. Also, for any message profile for which the outcome function follows rule 4b, player 1's worst alternative is implemented. But by appropriately changing the first and second components of her message, player 1 can induce the case in which the outcome function follows rule 4a and in which her top choice is implemented. Hence, for any LLQRE strategy profile, the outcome function must not follow rule 4b. Consequently, if a message profile is an LLQRE in some state, then for this profile, the outcome function must follow rule 1. Consider such a profile. If this profile yields an SCR alternative of the realized state, then we are done. If not, then quasimonotonocity and rule 2a guarantee the existence of an player who can unilaterally deviate from the original profile without hurting her. If this player strictly improves by a unilateral deviation, then the original profile is not LLQRE. Consequently, there is no player who improves strictly by unilaterally deviating from the original message profile. However, there is at least one player who has multiple best responses to the original profile. If there is only one such player, then lemma 2.5 yields that the original message profile is not an LLQRE. Now suppose there are multiple players who have multiple best responses to the original profile. Then rule 3 ensures that the player with the highest index among those who have multiple best responses to the original profile is indifferent between her best responses as long as every other player plays some best response strategy to the original profile. This property enables us to prove that the original message profile is not an LLQRE which we show in the formal proof.

We also must remark that the mechanism in the proof of theorem 3.8 does not use a modulo game or integer game. However, for the proof of the sufficiency result, we restrict our attention only to pure LLQREs, which enables us to dispose of the undesired LLQREs. Jackson (1992) points out the shortcomings of not considering mixed equilibria in the context of Nash implementation. This criticism is not easily addressed in our setting. First, by considering mixed LLQREs, one complicates the analysis significantly. An even bigger issue of considering mixed LLQREs is that they are not robust to monotonic transformations of the utility functions. We discuss this complication in detail when we study the necessary conditions for LLQRE implementation.[8]

Implementation in LLQREs vs. Implementation in Strict Nash Equilibria

Cabrales and Serrano (2007) find that quasimonotonicity and NWA are necessary conditions for implementation in *strict* Nash equilibria. Furthermore, they show that these conditions are sufficient if there are at least 3 players. As we have seen in this section, theorem 3.8 shows that if the players' worst alternative set is constant over all states, quasimonotonicity and NWA are sufficient conditions for implementation in LLQREs in environments with at least 3 players. Consequently, we obtain the following corollary.

Corollary 3.9. *Suppose $n \geq 3$ and the environment satisfies assumption 3.7. Then any strict Nash implementable SCR F is LLQRE implementable.*

The connection between the sufficient conditions for implementation in LLQREs and the ones for implementation in strict Nash equilibria can be explained easily. Let us fix some SCR and consider the mechanism used in the proof of theorem 3.8. This mechanism implements the SCR in strict Nash equilibria. As we mentioned earlier, this mechanism delivers each SCR alternative in any state via some strict LLQRE of that state. But we know that any strict LLQRE is a strict Nash equilibrium thanks to lemma 2.4. Hence, the mechanism indeed delivers each SCR alternative in any state via some strict Nash equilibrium. Now all we have to do is to show that in any state the mechanism has no "bad" strict Nash equilibrium in the sense that it delivers some non-SCR alternative in that state. But if a "bad" strict Nash equilibrium existed in some state, it would be a strict LLQRE in the same state thanks to lemma 2.4. Hence, the mechanism would fail in LLQRE implementing the SCR. As a result, the mechanism considered in the proof of theorem 3.8 implements a given SCR in both LLQREs and strict Nash equilibria.

However, one should point out that not all mechanisms that implement a given SCR in

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[8] The discussion for mixed LLQREs is the same as the one for non-strict LLQREs.

strict Nash equilibria will implement the same SCR in LLQREs. To see this consider some mechanism that implements a given SCR F in strict Nash equilibria. Then the definition of strict Nash implementation and lemma 2.4 yield that each SCR alternative in each state is delivered by some strict LLQRE of the mechanism in the same state. However, the definition of strict Nash implementation does not rule out the possibility of the mechanism having a "bad" non-strict Nash equilibrium in some state that delivers a non-SCR alternative in that state. If this "bad" non-strict Nash equilibrium is an LLQRE, then the mechanism cannot implement F in LLQREs. We demonstrate this point in example 3.10. First we need one more definition: let $SNE(\Gamma, \theta)$ denote the set of strict Nash equilibria of game $\langle E, \Gamma \rangle$.

Example 3.10. Consider the following three player $\{1, 2, 3\}$, two state $\{\theta, \theta'\}$ and four alternative example $\{a, b, c, w\}$. The utility function of player $i = 1, 2, 3$ in state $\hat{\theta} = \theta, \theta'$ is given as follows:

	θ			θ'		
	P1	P2	P3	P1	P2	P3
a	2	2	2	2	2	2
b	2	2	2	1	1	1
c	1	1	1	1	1	1
w	0	0	0	0	0	0

Suppose $F(\theta) = \{a, b\}$ and $F(\theta') = \{a\}$. Consider the following mechanism $\Gamma = (M, g)$: For each player i, $M_i = \{m_{i1}, m_{i2}, m_{i3}\}$. The outcome function g is as follows. If each player i sends message m_{i1}, then b is implemented. If each player $i \neq j$ sends m_{i1} while player $j = 1, 2, 3$ alone sends m_{j2}, then c is implemented. If each player i sends m_{i3}, then a is implemented. In all other cases w is implemented.

Claim: Mechanism Γ implements F in strict Nash equilibria but not in LLQREs.

Proof of the Claim **Step 1.** Γ implements F in strict Nash equilibria.

First let us consider state θ. Then the players play the game in which each player's payoffs are described as follows:

P3

		m_1		
		P2		
		m_1	m_2	m_3
P1	m_1	2	1	0
	m_2	1	0	0
	m_3	0	0	0

		m_2		
		P2		
		m_1	m_2	m_3
P1	m_1	1	0	0
	m_2	0	0	0
	m_3	0	0	0

		m_3		
		P2		
		m_1	m_2	m_3
P1	m_1	0	0	0
	m_2	0	0	0
	m_3	0	0	2

It is easy to see that $(m_{i1})_{i=1,2,3}$ and $(m_{i3})_{i=1,2,3}$ are strict Nash equilibria in state θ. Clearly, $(m_{i1})_{i=1,2,3}$ results in b and $(m_{i3})_{i=1,2,3}$ in a. In addition, these two message profiles are the only Nash equilibria. To see this consider any message profile m that results in w according to g. Because w is the worst alternative for each player, by unilaterally deviating, each player cannot get worse. Hence, m is not a strict Nash equilibrium. Now consider a message profile m' that results in c according to g. Then two players must send message 1, while one sends message 2. However, if the odd player sends her message 1, b is implemented. Since, the odd player prefers b to c in state θ, she has a profitable deviation. Hence, m' is not a strict Nash equilibrium. Consequently, $F(\theta) = SNE(\Gamma, \theta)$.

Now let us consider state θ'. Then the players play the game in which each player's payoffs are described as follows:

P3

m_1

P2

		m_1	m_2	m_3
	m_1	1	1	0
P1	m_2	1	0	0
	m_3	0	0	0

m_2

P2

		m_1	m_2	m_3
	m_1	1	0	0
P1	m_2	0	0	0
	m_3	0	0	0

m_3

P2

		m_1	m_2	m_3
	m_1	0	0	0
P1	m_2	0	0	0
	m_3	0	0	2

Now we show that $(m_{i3})_{i=1,2,3}$ which results in a is the unique strict Nash equilibrium in state θ'. It is easy to see that $(m_{i3})_{i=1,2,3}$ is a strict Nash equilibrium. Now we need to show that there is no other strict Nash equilibrium strategy profile. To see this consider any message profile m that results in w according to g. Because w is the worst alternative for each player, by unilaterally deviating, each player cannot get worse. Hence, m is not a strict Nash equilibrium. Now consider a message profile m' that results in c according to g. Then two players must send message 1, while one sends message 2. However, if the odd player sends her message 1, b is implemented. Since the odd player is indifferent between b and c in state θ', m' is not a Nash equilibrium. Lastly, consider m'' that results in b. This means every player sends message 1. However, if any of the players unilaterally deviate to message 2, then c is implemented. Since each player is indifferent between b and c in state θ', m'' is not a strict Nash equilibrium. Hence, $F(\theta') = SNE(\Gamma, \theta')$.

Step 2. Γ does not implement F in LLQREs.

We show that $(m_{i1})_{i=1,2,3}$ which results in b is an LLQRE in state θ'. This means we must show that there exists a sequence of $\{p_t\} \to (m_{i1})_{i=1,2,3}$ such that $p_t \in L(\lambda_t, \Gamma, \theta')$ for some

$\{\lambda_t\} \to \infty$. For any $p \in \{p_t\}$, then the following conditions must be satisfied.

$$\ln\frac{p_{11}}{p_{12}} = \lambda\left(p_{21}p_{32} + p_{22}p_{31}\right)$$
$$\ln\frac{p_{11}}{p_{13}} = \lambda\left(p_{21}p_{31} + p_{21}p_{32} + p_{22}p_{31} - 2p_{23}p_{33}\right)$$

Using the symmetry between the players' payoffs, we assume that $p_{11} = p_{21} = p_{31} = \pi_1$, $p_{12} = p_{22} = p_{32} = \pi_2$ and $p_{13} = p_{23} = p_{33} = \pi_3$. As a result, the above 2 equations reduce to:

$$\ln\frac{\pi_1}{\pi_2} = \lambda\left(2\pi_1\pi_2\right)$$
$$\ln\frac{\pi_1}{\pi_3} = \lambda\left(\pi_1^2 + 2\pi_1\pi_2 - 2\pi_3^2\right)$$

Equating by λ, the above 2 equations yield

$$2\pi_1\pi_2\ln\pi_3 + \left(\pi_1^2 - 2\pi_3^2\right)\ln\pi_1 - \left(\pi_1^2 + 2\pi_1\pi_2 - 2\pi_3^2\right)\ln\pi_2 = 0. \tag{4}$$

Now fix sufficiently small $\pi_3 > 0$. We proceed to show that the above equation has a solution. By definition, $\pi_2 = 1 - \pi_1 - \pi_3$. Clearly, the left hand side of equation 4 is a continuous function. When $\pi_1 \to 0.5$, $\pi_2 \to 0.5 - \pi_3$. Since π_3 is small enough, the term $2\pi_1\pi_2\ln\pi_3$ dominates the left hand side of equation 4 as this term approaches $-\infty$ as $\pi_3 \to 0$ while the other terms are bounded. On the other hand, if $\pi_1 \to 1 - \pi_3$, $\pi_2 \to 0$. When $\pi_2 < \pi_3$, from the proof of lemma 2.5, the left hand side of equation 4 is approximately $-\left(\pi_1^2 - 2\pi_3^2\right)\ln\pi_2$ which converges to $+\infty$. Now using the intermediate value theorem we obtain that for any small enough π_3, there must exist $\pi_1(\pi_3)$ and $\pi_2(\pi_3)$ that satisfy equation 4. Now let us show that $\pi_1(\pi_3) \to 1$ when $\pi_3 \to 0$. Suppose otherwise. The the term $2\pi_1\pi_2\ln\pi_3$ dominates the left hand side of equation 4 as this term approaches $-\infty$. Hence, equation 4 is never satisfied. Thus $\pi_1(\pi_3) \to 1$ when $\pi_3 \to 0$. Now set

$$\lambda(\pi_3) = \frac{\ln\frac{\pi_1(\pi_3)}{\pi_3}}{\left(\pi_1(\pi_3)\right)^2 + 2\pi_1(\pi_3)\pi_2(\pi_3) - 2\pi_3^2}$$

Clearly, when $\pi_3 \to 0$, $\lambda(\pi_3) \to \infty$. To complete the proof, consider any sequence of π_3s converging to 0. Now find the corresponding $\lambda(\pi_3)$s. As argued earlier, the sequence of $\lambda(\pi_3)$s converge to ∞. Now for each $\lambda(\pi_3)$, everyone playing a strategy in which $p_{11} = p_{21} = p_{31} = \pi_1(\pi_3)$, $p_{12} = p_{22} = p_{32} = \pi_2(\pi_3)$ and $p_{13} = p_{23} = p_{33} = \pi_3$ is an LQRE. As argued earlier, this sequence of LQREs must converge to $(m_{i1})_{i=1,2,3}$. Hence, Γ does not LLQRE implement F. ◇

Based on example 3.10, one can say that the paper contributes to the literature by designing a mechanism that implements any given SCR in both LLQREs and strict Nash equilibria.

3.2 Necessary Conditions and Restricted LLQRE Implementation

As we mentioned before, the mechanism used in the proof of theorem 3.8 delivers each SCR alternative in any state through a strict LLQRE of the same state. This naturally leads to the question of why one wants each SCR alternative in any state to be delivered via a strict LLQRE but not via a non-strict LLQRE. To answer this question, let us suppose that the planner has information only about the players' preferences. In other words, the planner does not know the utility representations of the players' preferences. In this case, the mechanism must be designed so that for all utility representations of the players' preferences, it delivers the SCR alternatives in each state via LLQREs. In addition, it is well known that if a utility function represents the underlying preference, a monotonic transformation of the utility function must also represent the underlying preference. Consequently, one needs to make sure that implementation in LLQREs is robust under monotonic transformation of utility functions. But as we discussed in the previous section, non-strict LLQREs are not preserved under monotonic or even affine transformation of the players' utilities in some cases. Hence, even if a mechanism delivers an SCR alternative in some state through an LLQRE for some utility representations, the same mechanism might not deliver the same SCR alternative in the same state through any LLQRE for some other ulity representations. We illustrate this point in the following example.

Example 3.11. Consider the following three player $\{1,2,3\}$, two state $\{\theta,\theta'\}$ and five alternative example $\{a,b,c,d,w\}$. The utility function of player $i=1,2,3$ in state $\hat{\theta}=\theta,\theta'$ is given as follows:

	θ			θ'		
	P1	P2	P3	P1	P2	P3
a	0	0	0	2	2	2
b	2	2	2	1	1	1
c	0	0	0	1	1	1
d	0	0	0	0	0	0
w	-1	-1	-1	-1	-1	-1

Suppose $F(\theta)=\{b\}$ and $F(\theta')=\{a,b\}$. Consider the following mechanism $\Gamma=(M,g)$: For each player i, $M_i=\{m_{i1},m_{i2},m_{i3}\}$. The outcome function g is as follows. If each player

i sends message m_{i1}, then b is implemented. If each player $i \neq j$ sends m_{i1} while player $j = 1, 2, 3$ alone sends m_{j2}, then c is implemented. If each player i sends m_{i3}, then a is implemented. In all other cases d is implemented.

Claim: Mechanism Γ implements F in LLQREs. However, mechanism Γ does not implement F in LLQREs if player 2's payoffs are scaled down by two.

Proof of the Claim. **Step 1.** Γ implements F in LLQREs.

Consider state θ. Then the players play the game in which each player's payoffs are described as follows:

P3

m_1

P2

		m_1	m_2	m_3
	m_1	2	0	0
P1	m_2	0	0	0
	m_3	0	0	0

m_2

P2

		m_1	m_2	m_3
	m_1	0	0	0
P1	m_2	0	0	0
	m_3	0	0	0

m_3

P2

		m_1	m_2	m_3
	m_1	0	0	0
P1	m_2	0	0	0
	m_3	0	0	0

It is easy to see that $(m_{i1})_{i=1,2,3}$ is the only LLQRE in state θ.

Now let us consider state θ'. Then the players play the game in which each player's payoffs are described as follows:

P3

m_1

P2

		m_1	m_2	m_3
	m_1	1	1	0
P1	m_2	1	0	0
	m_3	0	0	0

m_2

P2

		m_1	m_2	m_3
	m_1	1	0	0
P1	m_2	0	0	0
	m_3	0	0	0

m_3

P2

		m_1	m_2	m_3
	m_1	0	0	0
P1	m_2	0	0	0
	m_3	0	0	2

Clearly, every player i sending m_{i3} is a strict Nash equilibrium, hence is an LLQRE. In addition, from example 4 we know that every player i sending m_{i1} is an LLQRE in state θ'. Therefore, Γ implements F in LLQREs.

Step 2. Γ does not implement F in LLQREs if player 2's payoffs are scaled down by two.

We need to show that every player i sending m_{i1} is not an LLQRE in state θ'. On the contrary, suppose $(m_{i1})_{i=1,2,3}$ is an LLQRE; this means that there exists a sequence $\{p_t\} \to (m_{i1})_{i=1,2,3}$ such that $p_t \in L(\lambda_t, \Gamma, \theta')$ for some $\{\lambda_t\} \to \infty$. For any $p \in \{p_t\}$, the following conditions must be satisfied.

$$(p_{11}p_{32}0.5 + p_{12}p_{31}0.5) \ln \frac{p_{11}}{p_{12}} = (p_{21}p_{32} + p_{22}p_{31}) \ln \frac{p_{21}}{p_{22}}$$

$$(p_{11}p_{32}0.5 + p_{12}p_{31}0.5) \ln \frac{p_{31}}{p_{32}} = (p_{11}p_{22} + p_{12}p_{21}) \ln \frac{p_{21}}{p_{22}}$$

By symmetry $p_{11} = p_{31}$ at LQRE since p_{11} and p_{31} are around 1. Thus we obtain:

$$p_{11}p_{12} \ln \frac{p_{11}}{p_{12}} = (p_{21}p_{12} + p_{22}p_{11}) \ln \frac{p_{21}}{p_{22}}$$

For this equation to hold the following two inequalities must be satisfied: $p_{11}p_{12} \ln \frac{p_{11}}{p_{12}} > p_{21}p_{12} \ln \frac{p_{21}}{p_{22}}$ and $p_{11}p_{12} \ln \frac{p_{11}}{p_{12}} > p_{22}p_{11} \ln \frac{p_{21}}{p_{22}}$. The first one yields $p_{11} > p_{21}$ and the second one gives $p_{11} < p_{21}$ if p_{21} is close to 1. These inequalities contradict each other, so $(m_{i1})_{i=1,2,3}$ is not an LLQRE in state θ'. $\diamond$

As example 5.4 demonstrates, whether a mechanism succeeds in implementing a given SCR in LLQREs could depend on the utility representations of the players' preferences if a mechanism is designed so that some SCR alternative in some state is delivered by a non-strict LLQRE in that state. One must avoid this problem. Hence, one must determine the conditions required for non-strict LLQREs to be robust to utility representations of the players' preferences. However, this seems to be a rather complicated problem given examples 2.6, 2.7 and lemma 2.5.

Another simple way to ensure that LLQRE implementation is robust under different utility representations of the players' preferences is to use the fact that strict LLQREs are preserved under monotonic transformation. In other words, when designing a mechanism one can make sure that (1) any social choice alternative in any state to be delivered by a *strict* LLQRE in that state and (2) in any state, any LLQRE (strict or non-strict) must yield an SCR alternative. Requirement (1) ensures that each SCR alternative in each state is delivered by some LLQRE without depending on utility representations. For some utility representations, there might be non-strict LLQREs but they should not cause any harm as requirement (2) ensures that each LLQRE in any state delivers some SCR alternative of the same state. This is exactly what our mechanism in the proof of theorem 3.8 accomplishes. Now let us formalize the above discussed requirements (1) and (2) and define a slightly stronger version of LLQRE implementation. We call this restricted LLQRE implementation.

Definition 3.12. Social choice rule F is restricted LLQRE implementable if there is a mechanism $\Gamma = ((M_i)_{i \in N}, g)$, such that

1. for every environment $E \in \mathcal{E}$ and for any $a \in F(\theta)$, there exists $m^* \in L(\Gamma, \theta)$ such that $g(m^*) = a$ and every player's best response to m^* is single valued
2. if $m^* \in L(\Gamma, t)$, then $g(m^*) \in F(\theta)$

Now let us present the following theorem which characterizes the necessary conditions for restricted LLQRE implementation.

Theorem 3.13. *If SCR F is restricted LLQRE implementable, then*

1. *F is quasimonotonic*

2. *F satisfies NWA.*

Proof. Let $\Gamma = ((M_i)_{i=1}^n, g)$ restricted LLQRE implement SCR F. Now let us prove F is quasimonotonic. Suppose otherwise. Then there exists a mechanism $\Gamma = ((M_i)_{i=1}^n, g)$ which restricted LLQRE implements a non quasimonotonic SCR F. This means there exist states t and θ', and a social alternative a such that $a \in F(\theta)$ but $a \notin F(\theta')$, while all $b \in \{c \in A : u_i(c, \theta) < u_i(a, \theta)\}$ remain in $\{c \in A : u_i(c, \theta') < u_i(a, \theta')\}$ for any player $i \in N$. Let m^* be a strict LLQRE in environment E that delivers a. Since no one's relative ranking of a to some alternative b, which is strictly less preferred to a in state t, worsened in state θ', m^* stays as a strict Nash equilibrium in state θ'. Now by lemma 2.4, m^* is an LLQRE in environment E'. This contradicts Γ restricted LLQRE implements SCR F.
Proof of $a \in F(\theta)$ not being the worst alternative for anyone in state θ directly follows from lemma 2.4. □

As discussed earlier, thanks to theorem 3.8 the sufficiency result for restricted LLQRE implementation remains the same as the one for LLQRE implementation.

Corollary 3.14. *Suppose $n \geq 3$ and assumption 3.7 is satisfied. If SCR F satisfies quasimonotonicity and NWA, then F is restricted LLQRE implementable.*

Thanks to theorem 3.13 and corollary 3.14 we fully identify the necessary and sufficient conditions for restricted LLQRE implementation if the are at least 3 players and if each player's worst alternative set is constant over all states.

4 Other Variants of LQRE Implementation

This section discusses possible variants of LLQRE implementation. Specifically, we are ask the question of what would happen if the players' sophistication level does not approach infinity. Then the first observation to make is that there is no hope for the exact LQRE implementation, given that an LQRE is always in the interior of probability simplex when λ is finite. In other words, in an LQRE all strategies are played with positive probabilities and as a result, any non constant SCR is not exact LQRE implementable. Therefore, instead of the exact implementation we have to consider the LQRE implementation with almost certainty[9].

We could think of many possible LQRE implementation concepts depending on the planner's knowledge of the sophistication levels of the players, but we consider only the extreme cases.

1. The planner does not know the players' sophistication level. In this case the planner's goal would be to design a mechanism which LQRE implements a given SCR with high probability for all values of λ. Not surprisingly, this task is not feasible because when λ approaches 0, players choose their strategies in a completely random fashion. This eliminates any hope of LQRE implementation with probability more than 1/2 for all values of λ.

2. The planner knows the exact value of the players' sophistication level. In this case it is impossible to talk about LQRE implementation in a general setting because LQRE is very dependent on a utility representation. In addition, it is hard to imagine the case in which the authority designs a mechanism for the specific value of players' sophistication level. Even if one manages this task, it would have very little value since the mechanism is unlikely to be robust.

5 Conclusion

This paper has studied LLQRE implementation where the players are assumed to be susceptible to making mistakes. In particular, we have identified necessary and sufficient conditions for restricted LLQRE implementation. Quasimonotonicity, a slight variation of Maskin Monotonicity, and NWA emerge as critical conditions to our exercise.

Appendix

Proof of Lemma 2.5. **(a)** Let us prove this lemma by contradiction. Suppose that there exists a sequence $\{\pi_t\} \to \pi^*$ where $\pi_t \in L(\lambda_t, G)$ for some $\{\lambda_t\} \to \infty$ and exactly $n-1$ players have a unique best response strategy to π^*. Since π^* is a pure equilibrium, denote the strategy that player i plays with probability 1 by m_{ij_i}, i.e., $\pi^*_{ij_i} = 1$. Therefore, $\bar{u}_{ij_i}(\pi^*) = u_i(\pi^*)$ for every $i \in N$. Let player i^* be the player whose best response set to π^* is not single valued, i.e., there exists a strategy m_{i^*j} for player i^* such that $\bar{u}_{i^*j}(\pi^*) = \bar{u}_{i^*j_{i^*}}(\pi^*)$. Also, by condition of the lemma for every player $i \neq i^*$, π^* satisfies $\bar{u}_{ij_i}(\pi^*) > \bar{u}_{ik}(\pi^*)$ for all $k \neq j_i$.

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[9] As in virtual implementation considered in Matsushima (1988) and Abreu and Sen (1991).

We know any $\pi \in \{\pi_t\}$ must satisfy:

1. $\frac{\pi_{ij_i}}{\pi_{ik}} = \frac{\exp(\lambda \bar{u}_{ij_i}(\pi))}{\exp(\lambda \bar{u}_{ik}(\pi))}$ for all $i \neq i^*$ and $k \neq j_i$
2. $\frac{\pi_{i^*j_{i^*}}}{\pi_{i^*j}} = \frac{\exp(\lambda \bar{u}_{i^*j_{i^*}}(\pi))}{\exp(\lambda \bar{u}_{i^*j}(\pi))}$

Combining the above equations we obtain

$$(\bar{u}_{i^*j_{i^*}}(\pi) - \bar{u}_{i^*j}(\pi)) \ln \frac{\pi_{ij_i}}{\pi_{ik}} = (\bar{u}_{ij_i}(\pi) - \bar{u}_{ik}(\pi)) \ln \frac{\pi_{i^*j_{i^*}}}{\pi_{i^*j}} \text{ for all } i \neq i^* \text{ and } k \neq j_i. \tag{5}$$

By supposition, we know that $\{\pi_t\} \to \pi^*$. For this proof we will use the abusive notation $\pi \to \pi^*$ to denote $\{\pi_t\} \to \pi^*$. Now let us evaluate the left and right hand sides of equation 5 when $\pi \to \pi^*$. If the LHS and RHS do not converge to the same value for some i and $k \neq j_i$ then we obtain the desired contradiction.

Right Hand Side: Since $\bar{u}_{ij_i}(\pi^*) > \bar{u}_{ik}(\pi^*)$ for any player $i \neq i^*$ and $k \neq j_i$ if π is close enough to π^*, then $\bar{u}_{ij_i}(\pi) > \bar{u}_{ik}(\pi)$. Combining this condition with $\pi^*_{i^*j_{i^*}} = 1$ guarantees that the RHS approaches ∞ for all $k \neq j_i$ when $\pi \to \pi^*$.

Left Hand Side: Now we will show for some $k \neq j_i$, the LHS converges to 0 as $\pi \to \pi^*$. This is true if the absolute value of the LHS converges to 0.

$$\begin{aligned} |\bar{u}_{i^*j_{i^*}}(\pi) - \bar{u}_{i^*j}(\pi)| &= \left| \sum_{m_{-i^*} \in M_{-i^*}} (u_{i^*}(m_{i^*j_{i^*}}, m_{-i^*}) - u_{i^*}(m_{i^*j}, m_{-i^*})) prob(m_{-i^*}) \right| \\ &\leq \max_{m_{-i^*} \in M_{-i^*}} |u_{i^*}(m_{i^*j_{i^*}}, m_{-i^*}) - u_{i^*}(m_{i^*j}, m_{-i^*})| \left(1 - \prod_{l \neq i^*} \pi_{lj_l}\right). \end{aligned}$$

To see this, observe that $\left(1 - \prod_{l \neq i^*} \pi_{lj_l}\right)$ is the probability of at least one player $l \neq i^*$ not playing strategy m_{lj_l}. Define $o :\equiv \max_{m_{-i^*} \in M_{-i^*}} |u_{i^*}(m_{i^*j_{i^*}}, m_{-i^*}) - u_{i^*}(m_{i^*j}, m_{-i^*})|$, then $|LHS| \leq o \left(1 - \prod_{l \neq i^*} \pi_{lj_l}\right) \ln \frac{\pi_{ij_i}}{\pi_{ik}}$ for all $i \neq i^*$. Clearly, $\lim_{\pi \to \pi^*} \left(1 - \prod_{l \neq i^*} \pi_{lj_l}\right) \ln \pi_{ij_i} = 0$. Therefore, we only need to show that there exists a player $i \neq i^*$ and strategy $k \neq j_i$ such that $\lim_{\pi \to \pi^*} \left(- \left(1 - \prod_{l \neq i^*} \pi_{lj_l}\right) \ln \pi_{ik}\right) \to 0$. Clearly there exists at least one strategy k satisfying $\pi_{ik} \geq \frac{1-\pi_{ij_i}}{|J_i|-1}$ for any π and i, and accordingly $-\left(1 - \prod_{l \neq i^*} \pi_{lj_l}\right) \ln \pi_{ik} \leq -\left(1 - \prod_{l \neq i^*} \pi_{lj_l}\right) \ln \frac{1-\pi_{ij_i}}{|J_i|-1}$. Let $Z_i(\pi) = \left\{k : \pi_{ik} \geq \frac{1-\pi_{ij_i}}{|J_i|-1}\right\}$. Consider $\bar{Z}_i :\equiv \limsup Z_i(\pi_t)$. This set is finite and not empty because cardinality of strategy space is finite. Fix any $k \in \bar{Z}_i$. Consider any subsequence $\{\pi_{ts}\}$ such that $k \in Z_i(\pi_{ts})$. Then $-\left(1 - \prod_{l \neq i^*} \pi_{lj_l}\right) \ln \pi_{ik} \leq -\left(1 - \prod_{l \neq i^*} \pi_{lj_l}\right) \ln \frac{1-\pi_{ij_i}}{|J_i|-1}$ for any $\pi \in \{\pi_{ts}\}$. Without loss of generality, assume that player $\bar{i} \neq i^*$ is the player with the smallest π_{ij_i}. For player $\bar{i}$, we know $-\left(1 - \prod_{l \neq i^*} \pi_{lj_l}\right) \ln \frac{1-\pi_{\bar{i}j_{\bar{i}}}}{|J_{\bar{i}}|-1} \leq$

$-\left(1-\pi_{\bar{i}j_{\bar{i}}}^{|N|-1}\right)\ln\frac{1-\pi_{\bar{i}j_{\bar{i}}}}{|J_{\bar{i}}|-1}$. Now applying L'Hospital's rule and simplifying the terms, we obtain

$$-\left(1-\pi_{\bar{i}j_{\bar{i}}}^{|N|-1}\right)\ln\frac{1-\pi_{\bar{i}j_{\bar{i}}}}{|J_{\bar{i}}|-1} \to_{\pi_{\bar{i}j_{\bar{i}}}\to 1} 0. \tag{6}$$

Expression 6 shows that equation 5 is not satisfied in the limit of some subsequence for some strategy k of player $\bar{i}$. This means that for $\pi \in \{\pi_t\}$, which is close enough to π^*, equation 5 is violated, contradicting the supposition π^* is an LLQRE.

(b) Suppose for some player i and her strategy m_{ij_i} with $\pi^*_{ij_i} = 1$, there exists strategy m_{ij} such that $u_i(m_{ij}, m_{-i}) \geq u_i(m_{ij_i}, m_{-i})$ for all m_{-i}. This implies that $\bar{u}_{ij}(\pi) \geq \bar{u}_{ij_i}(\pi)$ for all $\pi \in \Delta$. Now let us show that π^* is not an LLQRE. Suppose otherwise. Then there must exist sequences $\{\pi_t\} \to \pi^*$ and $\{\lambda_t\} \to \infty$ such that $\pi_t \in L(\lambda_t, G)$. Consider any $\pi \in \{\pi_t\}$ and $\lambda \in \{\lambda_t\}$ such that $\pi \in L(\lambda, G)$. Then by the definition of LQRE,

$$\frac{\pi_{ij}}{\pi_{ij_i}} = \exp\left(\lambda(\bar{u}_{ij}(\pi) - \bar{u}_{ij_i}(\pi))\right).$$

Because $\bar{u}_{ij}(\pi) - \bar{u}_{ij_i}(\pi) \geq 0$, the above equation yields that $\pi_{ij} \geq \pi_{ij_i}$. Since this is true for all $\pi \in \{\pi_t\}$, $\{\pi_t\} \nrightarrow \pi^*$. $\square$

Proof of Theorem 3.8. Consider the mechanism Γ defined in the sketch of the proof.

Step 1. For any $a \in F(\theta)$ and $\theta \in \Theta$, there exists an LLQRE $m^* \in L(\Gamma, t)$ with $g(m^*) = a$.

Proof of step 1. Observe that a is implemented if the players send message profile $m^* = (a, \theta, \theta, 0)_{i\in N}$. In order to prove $m^* \in L(\Gamma, t)$, we need to show that m^* is a strict Nash equilibrium in $\langle N, \Gamma, (u(\cdot, \theta))_{i\in N}\rangle$ thanks to lemma 2.4. If any player unilaterally deviates from m^*, then the outcome function must follow rule 2. Obviously, the deviator will be strictly worse off. Therefore, m^* is a strict Nash equilibrium which yields the desired result.

Step 2. If $m' \in L(\Gamma, \theta)$ then $g(m') \in F(\theta)$.

Proof of step 2. Suppose $m' \in L(\Gamma, \theta)$. First, assume that $g(m')$ falls under rule 1. Let $m' = (a', \theta', \theta', 0)_{i\in N}$. To simplify the notations let $s' = (a', \theta', \theta', 0)$. As $g(m')$ falls under rule 1, $a' \in F(\theta')$. If $a' \in F(\theta)$, then we are done. Suppose that $a' \notin F(\theta)$. Then, by quasimonotonicity, there exists a player i and a social alternative a_i such that $u_i(a_i, \theta') < u_i(a', \theta')$, but $u_i(a_i, \theta) \geq u_i(a', \theta)$. This means that player i can unilaterally deviate from m' without worsening herself. This could be consistent with 3 possible cases:

1. $I^1(a', \theta', \theta) \neq \emptyset$

2. If not 1

    (a) $|I^2(a', \theta', \theta)| = 1$

(b) $|I^2(a', \theta', \theta)| \geq 2$

Suppose that case 1 occurs. If player $i \in I^1(a', \theta', \theta)$ sends message $(a_i, \theta', \theta, \nu_i)$ where $a_i \in A_i^1(a', \theta', \theta)$ then a_i would be implemented by rule 2a. As player i strictly prefers a_i to a' in state θ, $(a_i, \theta', \theta, \nu_i)$ is a profitable deviation. Therefore, m' is not a Nash equilibrium. Hence, $m' \notin L(\Gamma, \theta)$, which is a contradiction.

Suppose that case 2 occurs. Observe that for any player $j \notin I^2(a', \theta', \theta)$, s' must be the unique best response to m'. Let us consider case 2a. Player $i \in I^2(a', \theta', \theta)$ has more than one best response strategies to m'. For example, s' and $(a_i, \theta', \theta, \nu_i)$ where $a_i \in A_i^2(a', \theta', \theta)$ are best responses to m' thanks to rules 1 and 2a. Because $|I^2(a', \theta', \theta)| = 1$, $m' \notin L(\Gamma, \theta)$ by lemma 2.5. This is a contradiction.

Suppose that case 2b occurs. Obviously, s' must be one of the best response strategies to m' for everyone. Consider player $i \in I^2(a', \theta', \theta)$. First let us show that any player i's best response strategy to m' must be of the form $(a_i, \theta', \theta_i^2, \nu_i)$ where $a_i \in A_i(a', \theta', \theta_i^2)$. Suppose otherwise. Let player i play a such strategy $\bar{m}_i$ in response to m'. Then thanks to rule 2b, $w_i \in W_i$ is implemented. But because NWA is satisfied, $F(\theta') \cap W_i = \emptyset$. Hence, $u_i(w_i, \theta) < u_i(a', \theta)$. Consequently, $\bar{m}_i$ is not a best response strategy to m', meaning that player i's best response strategy to m' is of the form $(a_i, \theta', \theta_i^2, \nu_i)$ where $a_i \in A_i(a', \theta', \theta_i^2)$.

It is clear that for player $i \in I^2(a', \theta', \theta)$, $(a_i, \theta', \theta, \nu_i)$ where $a_i \in A_i^2(a', \theta', \theta)$ is a best response strategy to message profile m'. In addition to s' and $(a_i, \theta', \theta, \nu_i)$ where $a_i \in A_i^2(a', \theta', \theta)$, player i could have other best response strategies to m'. To see this, suppose there is a state $\theta_i^2 \neq \theta$ and an alternative $\bar{a}_i$ such that $u_i(\bar{a}_i, \theta') < u_i(a', \theta')$ and $u_i(\bar{a}_i, \theta_i^2) \geq u_i(a', \theta_i^2)$. If player i sends $(\bar{a}_i, \theta', \theta_i^2, \nu_i)$, then $\bar{a}_i$ is implemented by rule 2a. The case $u_i(\bar{a}_i, \theta) > u_i(a', \theta)$ cannot occur. Otherwise, this contradicts $I_i^1(a', \theta', \theta) = \emptyset$. However, one cannot rule out the case $u_i(\bar{a}_i, \theta) = u_i(a', \theta)$. Indeed, if $u_i(\bar{a}_i, \theta) = u_i(a', \theta)$, then $(\bar{a}_i, \theta', \theta_i^2, \nu_i)$ is a best response strategy for player i to m'. In this case, one should note that $a_i \in A_i^2(a', \theta', \theta)$.

For each player i let us define S_i to be player i's best response correspondence strategies[10] to message profile m'. Observe that for player $i \in I^2(a', \theta', \theta)$, S_i consists of strategies of the form $(a_i, \theta', \theta_i^2, \nu_i)$ where $a_i \in A_i^2(a', \theta', \theta)$ and $a_i \in A_i^2(a', \theta', \theta_i^2)$. If each player i sends a message $s_i \in S_i$, then the outcome function follows one of rules 1, 2a and 3. Let $\bar{i}$ be the highest indexed player in set $I^2(a', \theta', \theta)$. Player $\bar{i}$ is indifferent between strategies in $S_{\bar{i}}$ as long as each player $i \in N$ sends a message in S_i thanks to rules 1, 2a and 3.

Now we show that $m' \notin L(\Gamma, \theta)$. Let π be a mixed strategy profile and let notation π^* denote the strategy profile in which every player plays strategy s' with probability 1.

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[10] Observe S_i is defined for all players.

Suppose $\pi^* \in L(\Gamma, \theta)$. Then there exist sequences $\{\pi_t\} \to \pi^*$ and $\{\lambda_t\} \to \infty$, such that $\pi_n \in L(\lambda_t, \Gamma, \theta)$. Since the proof is done for state θ, we omit θ from the notations for the remainder of this proof, whenever the omission does not cause confusion.

We will use s_i and l_i to denote the typical strategy in S_i and in $M_i \backslash S_i$, respectively. Clearly, S_i is not empty. In addition, $M_i \backslash S_i$ is not empty as we argued that $u_i(w_i, \theta) < u_i(a', \theta)$. Let π_{im_i} be the probability that player i plays strategy m_i. If $\pi \in \{\pi_t\}$, the following conditions must be satisfied.

1. $\frac{\pi_{is'}}{\pi_{il_i}} = \frac{\exp(\lambda \bar{u}_{is'}(\pi))}{\exp(\lambda \bar{u}_{il_i}(\pi))}$ for all $l_i \notin S_i$ and for all $i \neq \bar{i}$
2. $\frac{\pi_{\bar{i}s'}}{\pi_{\bar{i}s_{\bar{i}}}} = \frac{\exp(\lambda \bar{u}_{\bar{i}s'}(\pi))}{\exp(\lambda \bar{u}_{\bar{i}s_{\bar{i}}}(\pi))}$ where $s_{\bar{i}} \in S_{\bar{i}}$ and $s_{\bar{i}} \neq s'$

Combining the above equations we obtain

$$\left(\bar{u}_{\bar{i}s'}(\pi) - \bar{u}_{\bar{i}s_{\bar{i}}}(\pi)\right) \ln \frac{\pi_{is'}}{\pi_{il_i}} = \left(\bar{u}_{is'}(\pi) - \bar{u}_{il_i}(\pi)\right) \ln \frac{\pi_{\bar{i}s'}}{\pi_{\bar{i}s_{\bar{i}}}} \text{ for all } i \neq \bar{i},\ l_i \notin S_i \text{ and } s_{\bar{i}} \in S_{\bar{i}}. \quad (7)$$

By supposition, we know that $\{\pi_n\} \to \pi^*$. We simplify the notations by using $\pi \to \pi^*$ to denote $\{\pi_n\} \to \pi^*$. Now let us evaluate the left and right hand sides of equation 7 when $\pi \to \pi^*$. If the LHS and RHS do not converge to the same value for some i and $l_i \notin S_i$, then we get the desired contradiction.

Right Hand Side: Since $l_i \notin S_i$, $\bar{u}_{is'}(\pi^*) > \bar{u}_{il_i}(\pi^*)$. Combining this condition with $\pi_{\bar{i}s'} \to 1$ guarantees that the RHS approaches infinity.

Left Hand Side: Now we show that for some player $i \neq \bar{i}$ and $l_i \notin S_i$, the LHS converges to 0 as $\pi \to \pi^*$. This is true if the absolute value of the LHS converges to 0.

$$\begin{aligned}\left|\bar{u}_{\bar{i}s'}(\pi) - \bar{u}_{\bar{i}s_{\bar{i}}}(\pi)\right| &= \left| \sum_{m_{-\bar{i}} \in M_{-\bar{i}}} (u_{\bar{i}}(s', m_{-\bar{i}}) - u_{\bar{i}}(s_{\bar{i}}, m_{-\bar{i}})) prob(m_{-\bar{i}}) \right| \\ &\leq \max_{m_{-\bar{i}} \in M_{\bar{i}}} |u_{\bar{i}}(s', m_{-\bar{i}}) - u_{\bar{i}}(s_{\bar{i}}, m_{-\bar{i}})| \left(1 - \prod_{j \neq \bar{i}} \left(\sum_{s_j \in S_j} \pi_{js_j} \right)\right)\end{aligned}$$

This inequality is due to the fact that player $\bar{i}$ is indifferent as long as all players play strategies from their best responses to m'. Define $o :\equiv \max_{m_{-\bar{i}} \in M_{\bar{i}}} |u_{\bar{i}}(s', m_{-\bar{i}}) - u_{\bar{i}}(s_{\bar{i}}, m_{-\bar{i}})|$. Then it must be that $|LHS| \leq o \left(1 - \prod_{j \neq \bar{i}} \left(\sum_{s_j \in S_j} \pi_{js_j}\right)\right) \ln \frac{\pi_{is'}}{\pi_{il_i}}$ for all $i \neq \bar{i}$ and $l_i \notin S_i$. Since $\lim_{\pi \to \pi^*} \left(1 - \prod_{j \neq \bar{i}} \left(\sum_{s_j \in S_j} \pi_{js_j}\right)\right) \ln \pi_{is'} = 0$, we are left to evaluate

$$- \lim_{\pi \to \pi^*} \left(1 - \prod_{j \neq \bar{i}} \left(\sum_{s_j \in S_j} \pi_{js_j} \right)\right) \ln \pi_{il_i}$$

for all $l_i \notin S_i$ and for all $i \neq \bar{i}$.

Let $|M_i|$ be the cardinality of player i's strategy set. We know that for any player $i \neq \bar{i}$, there exists at least one strategy l_i such that $\pi_{il_i} \geq \frac{1-(\sum_{s_i \in S_i} \pi_{is_i})}{|M_i|-|S_i|}$ for any $\pi \in \{\pi_n\}$. The remaining part of the proof that the LHS converges to infinity is the exact replica of the proof of lemma 2.5, hence, we omit the rest of the proof. This concludes the proof that $m' \notin L(\Gamma, \theta)$ if $g(m') \notin F(\theta)$ and falls under rule 1.

Suppose that $g(m')$ falls under rule 2. Let player i be the dissident player who sends $m'_i = (a_i, \theta_i^1, \theta_i^2, \nu_i)$ while others send $(a', \theta', \theta', 0)$ where $a' \in F(t')$. If $g(m')$ falls under rule 2a, then $\theta_i^1 \neq \theta_i^2$. Observe that as long as player i sends $(a_i, \theta_i^1, \theta_i^2, \nu_i)$ where $\theta_i^1 \neq \theta_i^2$, no matter what messages others send, the outcome function never falls under rule 1. Then by construction of the outcome function, whatever messages the other players send, the implemented alternative when i sends $(a_i, \theta_i^1, \theta_i^2, 0)$ does not change if i sends $(a_i, \theta_i^1, \theta_i^2, 1)$. Hence, at any LLQRE, player i plays $(a_i, \theta_i^1, \theta_i^2, 0)$ and $(a_i, \theta_i^1, \theta_i^2, 1)$ with equal probability thanks to lemma 2.5b. This contradicts that $m' \in L(\Gamma, \theta)$. Suppose $g(m')$ falls under rule 2b. Then w_i is implemented. But $w_i \in W_i$, hence, player i strictly improves by sending message $(a', \theta', \theta', 0)$. Therefore, $m' \notin L(\Gamma, \theta)$ if $g(m')$ falls under rule 2.

Suppose that $g(m')$ falls under rule 3. Then for some player i, $\theta_i^1 \neq \theta_i^2$. Observe that as long as player i sends $(a_i, \theta_i^1, \theta_i^2, \nu_i)$ where $\theta_i^1 \neq \theta_i^2$, no matter what messages others send, the outcome function never falls under rule 1. Then, by construction of the outcome function, whatever messages the other players send, the implemented alternative when i sends $(a_i, \theta_i^1, \theta_i^2, 0)$ does not change if i sends $(a_i, \theta_i^1, \theta_i^2, 1)$. Hence, at any LLQRE, player i plays $(a_i, \theta_i^1, \theta_i^2, 0)$ and $(a_i, \theta_i^1, \theta_i^2, 1)$ with equal probability thanks to lemma 2.5b. This contradicts that $m' \in L(\Gamma, \theta)$. Therefore, $m' \notin L(\Gamma, \theta)$ if $g(m')$ falls under rule 3.

Suppose $g(m')$ falls under rule 4. If $\theta_1^1 = \theta_1^2$, then w_1 is implemented. But player 1 can obtain her top choice by changing only her first and second messages. This means $\theta_1^1 \neq \theta_1^2$. Observe that as long as player 1 sends $(a_1, \theta_1^1, \theta_1^2, \nu_1)$ where $\theta_1^1 \neq \theta_1^2$, no matter what messages others send, the outcome function never falls under rule 1. Then, by construction of the outcome function, whatever messages the other players send, the implemented alternative when player 1 sends $(a_1, \theta_1^1, \theta_1^2, 0)$ does not change if player 1 sends $(a_i, \theta_i^1, \theta_i^2, 1)$. Hence, at any LLQRE, player 1 plays $(a_1, \theta_1^1, \theta_1^2, 0)$ and $(a_1, \theta_1^1, \theta_1^2, 1)$ with equal probability thanks to lemma 2.5b. This contradicts that $m' \in L(\Gamma, \theta)$. Therefore, $m' \notin L(\Gamma, \theta)$ if $g(m')$ falls under rule 4. This completes the proof. □

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