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A parsimonious test of constancy of a positive definite correlation matrix in a multivariate time-varying GARCH model

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Abstract

We construct a parsimonious test of constancy of the correlation matrix in the multivariate conditional correlation GARCH model, where the GARCH equations are time-varying. The alternative to constancy is that the correlations change deterministically as a function of time. The alternative is a covariance matrix, not a correlation matrix, so the test may be viewed as a general test of stability of a constant correlation matrix. The size of the test in finite samples is studied by simulation. An empirical example is given.

JEL Classification Codes: C32; C52; C58

Keywords: Deterministically varying correlation; multiplicative time-varying GARCH; multivariate GARCH; nonstationary volatility; smooth transition GARCH.

1 Introduction

Successors of the constant conditional correlation (CCC-)GARCH model by Bollerslev (1990) have become quite popular in financial applications. For overviews of multivariate GARCH models, see Bauwens, Laurent and Rombouts (2006) and Silvennoinen and Teräsvirta (2009). The most popular time-varying conditional correlation GARCH model is the DCC-GARCH model by Engle (2002). Tse and Tsui (2002) independently developed a rather similar model called the Varying Correlation (VC-)GARCH model. Both nest the CCC-GARCH model. However, there do not exist tests for testing the CCC model against either one of them. The reason may be that when the data-generating process is the CCC-GARCH model, neither the DCC- nor the VC-GARCH model is identified. This causes problems in deriving an appropriate test.

Among multivariate regime-switching GARCH models, both the Markov-switching multivariate GARCH model (Pelletier 2006), and the smooth transition conditional correlation (STCC-)GARCH model (Berben and Jansen 2005, Silvennoinen and Teräsvirta 2005, 2015) nest the CCC-GARCH model. Neither of them is identified when data are generated from the smaller model. The latter authors circumvented the identification problem and developed a Lagrange multiplier type test of CCC-GARCH against STCC-GARCH.

In the meantime, GARCH equations of the CCC-GARCH model have been extended to accommodate potential nonstationarity in the series to be modelled. This has, to a large extent, been done through the so-called multiplicative decomposition of the variance of an individual series into the customary conditional variance and a deterministic component. Contributions include Feng (2004, 2018), van Bellegem and von Sachs (2004), Engle and Rangel (2008), Amado and Teräsvirta (2008, 2013, 2017), Brownlees and Gallo (2010) and Mazur and Pipień (2012). Amado and Teräsvirta (2014) incorporated this feature into CCC-, DCC- and VC-GARCH models. For a recent review, see Amado, Silvennoinen and Teräsvirta (2019). The problem for which multiplicative decomposition offers a solution is that many sufficiently long return series are nonstationary in the sense that the amplitude of volatility clusters that GARCH is designed to parameterise is not constant over time. The purpose of the deterministic component in the decomposition is to rescale the observations such that the rescaled series can be described by a standard weakly stationary GARCH model.

Silvennoinen and Teräsvirta (in press) retained the multiplicative decomposition of variances and, in addition, assumed that the correlations of their smooth transition correlation model were changing deterministically over time. As opposed to the DCC- and VC-GARCH, this allows systematic changes in correlations. For example, correlations may change from one level to another and remain there. Hall, Silvennoinen and Teräsvirta (2021) derived a test of CCC-GARCH against this Time-Varying Correlation (TVC-)GARCH model. A drawback of their test, called the HST-test for short, is that if the dimension of the model is large, the null hypothesis of the test will also be quite large. This limits the applicability of the HST-test in practical, large dimensional applications. In this paper we develop a parsimonious alternative to the HST-test. The main thrust is to use the spectral decomposition of the correlation matrix, thereby making the eigenvalues rather than individual correlation parameters the focal point of the test. As with the HST-test, while the statistic here has been derived using a linear time trend as a transition variable, it can be generalised to detect variation in correlations according to other variables of interest, see Silvennoinen and Teräsvirta (2015). As a consequence, both of these tests are designed to detect correlation movement as a function of the chosen transition variable, making them flexible in practical applications. The test presented in this paper does have a difference compared to the HST test: the alternative hypothesis is generally not a correlation matrix. The resulting test may therefore be viewed as a general misspecification test of the CCC-GARCH model when the correlations are allowed to change systematically over time.

The plan of the paper is as follows. Section 2 contains an overview of previous tests of constant GARCH equations and correlations. The model and the null hypothesis to be tested are also presented there. The log-likelihood, score and the information matrix can be found in Sections 3 and 4 and the test statistic in Section 5. In Section 6, the performance of the test in finite samples is examined by simulation, including a few cases in which the GARCH equations are misspecified. Section 7 contains a real-world application. Conclusions can be found in Section 8. Proofs and further simulation evidence are relegated to an appendix.

2 Previous literature and the Time-Varying Smooth Transition Correlation GARCH Model

Before considering our Time-Varying Smooth Transition Correlation (TV-STC-GARCH) model, we take a quick look at the literature on tests of constancy of the error covariance matrix of a possibly nonlinear vector model. This literature is not very large, and rather few tests actually focus on the correlation matrix. There exist tests against conditional heteroskedasticity. Lütkepohl (2004, pp. 130–131) constructed a test of no multivariate ARCH against multivariate ARCH of order q. This Lagrange multiplier test works best when q and N, the dimension of the model, are small. The test statistic has an asymptotic χ^2-distribution with $qN^2(N+1)^2/4$ degrees of freedom when the null hypothesis of no ARCH holds.

Eklund and Teräsvirta (2007) designed a test in which the covariance matrix $\boldsymbol{\Sigma}_t$ is decomposed as in Bollerslev (1990) such that $\boldsymbol{\Sigma}_t = \boldsymbol{D}_t \boldsymbol{P} \boldsymbol{D}_t$ where $\boldsymbol{D}_t = \mathrm{diag}(d_{1t}, \ldots, d_{Nt})$ is a time-varying matrix with positive diagonal elements and $\boldsymbol{P}$ is a positive definite correlation matrix. The null hypothesis is that $\boldsymbol{D}_t = \boldsymbol{D} = \mathrm{diag}(d_1, \ldots, d_N)$ where $d_i > 0$, $i = 1, \ldots, N$. The alternative is $d_{it} \neq d_i$ at least for one i. Typically $d_{it} = d_i + d(\boldsymbol{x}_t)$, where both the (parametric) function $d(\cdot)$ and the argument $\boldsymbol{x}_t$ can be defined in various ways. The restriction that $\boldsymbol{P}$ is constant saves degrees of freedom but in some situations has a negative effect on the power of the test.

A similar decomposition is employed by Catani, Teräsvirta and Yin (2017), but the purpose of their test is more limited. The decomposition has the form $\boldsymbol{\Sigma}_t = \boldsymbol{D}_t \boldsymbol{S}_t \boldsymbol{P} \boldsymbol{S}_t \boldsymbol{D}_t$, where $\boldsymbol{D}_t = \mathrm{diag}(h_{1t}^{1/2}, \ldots, h_{Nt}^{1/2})$ such that h_{it}, $i = 1, \ldots, N$, are ARCH- or GARCH-type conditional variances. For example,

$$h_{it} = \alpha_{i0} + \alpha_{i1} \frac{\varepsilon_{i,t-1}^2}{g_{i,t-1}} + \kappa_{i1} \frac{\varepsilon_{i,t-1}^2}{g_{i,t-1}} I(\varepsilon_{i,t-1} < 0) + \beta_{i1} h_{i,t-1}, \tag{1}$$

where $I(\cdot)$ is an indicator variable, with $\alpha_{i0} > 0$, $\alpha_{i1} \geq 0$, $\alpha_{i1}+\kappa_{i1} > 0$, and $\beta_{i1} \geq 0$, so (1) has a GJR-GARCH structure, see Glosten, Jagannathan and Runkle (1993). Furthermore $\boldsymbol{S}_t = \mathrm{diag}(g_{1t}^{1/2}, \ldots, g_{Nt}^{1/2})$ where $g_{it} = 1+\sum_{j=1}^{q_i} \delta_{ij} z_{i,t-j}^2$, and $\boldsymbol{z}_t = (z_{1t}, \ldots, z_{Nt})' \sim \mathrm{iid}(\boldsymbol{0}, \boldsymbol{P})$. The null hypothesis H_0: $\boldsymbol{S}_t = \boldsymbol{I}_N$, or $g_{it} = 1$, $i = 1, \ldots, N$, which means that after estimating the CCC-GARCH model, there is no structure unmodelled in conditional variances. When $\boldsymbol{D}_t = \boldsymbol{D}$, this test may be viewed as a parsimonious version of Lütkepohl's test of no multivariate ARCH. The authors point out that their test can also be interpreted as a generalisation of the more parsimonious test by Ling and Li (1997).

The aforementioned tests are tests of $\boldsymbol{\Sigma}_t$ such that in the decomposition $\boldsymbol{\Sigma}_t = \boldsymbol{D}_t \boldsymbol{P}_t \boldsymbol{D}_t$ or $\boldsymbol{\Sigma}_t = \boldsymbol{D}_t \boldsymbol{S}_t \boldsymbol{P}_t \boldsymbol{S}_t \boldsymbol{D}_t$, it is assumed $\boldsymbol{P}_t = \boldsymbol{P}$, and the hypothesis to be tested has been $\boldsymbol{D}_t = \boldsymbol{D}$. In this work the focus is on testing H_0: $\boldsymbol{P}_t = \boldsymbol{P}$. Assuming $\boldsymbol{S}_t = \boldsymbol{I}$, Tse (2000) derived a portmanteau type constancy test of this hypothesis and found that it has reasonable power against the alternatives he was interested in. Péguin-Feissolle and Sanhaji (2016) proposed two portmanteau tests that are in fact extensions to Tse's test. The authors showed by simulation that the power of their tests is superior to that of Tse. A common feature of these tests is that the alternative is not a correlation matrix.

The TV-STC-GARCH model is a multivariate GARCH model with time-varying GARCH equations and correlations

$$\boldsymbol{\varepsilon}_t = \boldsymbol{H}_t^{1/2} \boldsymbol{\zeta}_t = \boldsymbol{D}_t \boldsymbol{S}_t \boldsymbol{P}_t^{1/2} \boldsymbol{\zeta}_t, \tag{2}$$

where $\boldsymbol{\varepsilon}_t = (\boldsymbol{\varepsilon}_{1t}, \ldots, \boldsymbol{\varepsilon}_{Nt})'$ is a stochastic $N \times 1$ vector and $\boldsymbol{H}_t = \boldsymbol{D}_t \boldsymbol{S}_t \boldsymbol{P}_t \boldsymbol{S}_t \boldsymbol{D}_t$ is an $N \times N$ conditional covariance matrix of $\boldsymbol{\varepsilon}_t$, typically the vector of returns in applications. The diagonal matrix $\boldsymbol{S}_t = \mathrm{diag}(g_{1t}^{1/2}(\boldsymbol{\theta}_{g1}), \ldots, g_{Nt}^{1/2}(\boldsymbol{\theta}_{gN}))$ is a matrix of square roots of positive-valued deterministic components to be defined below and $\boldsymbol{D}_t = \mathrm{diag}(h_{1t}^{1/2}(\boldsymbol{\theta}_{g1}, \boldsymbol{\theta}_{h1}), \ldots, h_{Nt}^{1/2}(\boldsymbol{\theta}_{gN}, \boldsymbol{\theta}_{hN}))$ contains the conditional standard deviations of $\boldsymbol{\phi}_t = \boldsymbol{S}_t^{-1} \boldsymbol{\varepsilon}_t = (\phi_{1t}, \ldots, \phi_{Nt})'$, where $\phi_{it} = \boldsymbol{\varepsilon}_{it}/g_{it}^{1/2}$, $i = 1, \ldots, N$. In what follows it is assumed that the elements of $\boldsymbol{\phi}_t$ have a first-order GJR-GARCH representation, see Glosten et al. (1993):

$$h_{it}(\boldsymbol{\theta}_{gi}, \boldsymbol{\theta}_{hi}) = \alpha_{i0} + \alpha_{i1} \phi_{i,t-1}^2 + \kappa_{i1} I(\phi_{i,t-1} < 0) \phi_{i,t-1}^2 + \beta_{i1} h_{i,t-1},$$

$i = 1, \ldots, N$. Furthermore, in $\boldsymbol{S}_t$,

$$g_{it} = \delta_{i0} + \sum_{j=1}^{r_i} \delta_{ij} G_{ij}(t/T, \gamma_{ij}, \boldsymbol{c}_{ij}) \tag{3}$$

with

$$G_{ij}(t/T) = G_{ij}(t/T, \gamma_{ij}, \boldsymbol{c}_{ij}) = (1 + \exp\{-\gamma_{ij} \prod_{k=1}^{K_{ij}} (t/T - c_{ijk})\})^{-1}, \tag{4}$$

$i = 1, \ldots, N$, where $\gamma_{ij} > 0$ and $\boldsymbol{c}_{ij} = (c_{ij1}, \ldots, c_{ijK_{ij}})'$ such that $c_{ij1} \leq \ldots \leq c_{ijK_{ij}}$. Note that $\delta_{i0} > 0$ is assumed known to solve the identification problem arising from both h_{it} and g_{it} having an intercept. It is often convenient to set $\delta_{i0} = 1$, but any positive constant will do. Finally, $\boldsymbol{P}_t$ is a positive definite deterministically varying covariance matrix of $\boldsymbol{z}_t = \boldsymbol{D}_t^{-1} \boldsymbol{S}_t^{-1} \boldsymbol{\varepsilon}_t$, and $\boldsymbol{\zeta}_t \sim \mathrm{iid}(\boldsymbol{0}, \boldsymbol{I}_N)$. For the purposes of this paper it is assumed that

$\boldsymbol{P}_t$ is rotation invariant: $\boldsymbol{P}_t = \boldsymbol{Q}\boldsymbol{\Lambda}_t\boldsymbol{Q}'$, where the matrix $\boldsymbol{Q} = (\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n)'$ holds the time-invariant eigenvectors as its columns, and the time-varying eigenvalues are

$$\begin{aligned} \boldsymbol{\Lambda}_t &= (1 - G(t/T))(\boldsymbol{\Lambda} - \boldsymbol{\Lambda}^*) + G(t/T)(\boldsymbol{\Lambda} + \boldsymbol{\Lambda}^*) \\ &= \boldsymbol{\Lambda} + \{2G(t/T) - 1\}\boldsymbol{\Lambda}^* \end{aligned} \tag{5}$$

with

$$G(t/T) = (1 + \exp\{-\gamma(t/T - c_1)(t/T - c_2)\})^{-1}, \ \gamma > 0. \tag{6}$$

If changes in the elements of $\boldsymbol{\Lambda}_t = \mathrm{diag}(\lambda_{1t}, \ldots, \lambda_{Nt})$ are assumed monotonic, the exponent of order one in (6) is sufficient. If nonmonotonicity is allowed, a second-order exponent is necessary. Further note that these elements are required to be positive and sum up to N. It is assumed that the elements of the diagonal matrix $\boldsymbol{\Lambda}$ satisfy the same conditions, and the elements of the diagonal matrix $\boldsymbol{\Lambda}^*$ sum up to zero. When $\boldsymbol{P}_t = \boldsymbol{P}$, it is assumed that $\boldsymbol{P}$ is a positive definite correlation matrix, in which case $\boldsymbol{H}_t$ is a slightly generalised version of the decomposition of the conditional covariance matrix Bollerslev (1990) suggested.

Hall et al. (2021) derived a constancy test in a more general situation in which

$$\boldsymbol{P}_t = \{1 - G(t/T)\}\boldsymbol{P}_{(1)} + G(t/T)\boldsymbol{P}_{(2)},$$

where $G(t/T)$ is defined as in (6) and $\boldsymbol{P}_{(1)}$ and $\boldsymbol{P}_{(2)}$ are two positive definite correlation matrices. In that set-up, as a convex combination of these two matrices $\boldsymbol{P}_t$ is always a positive definite correlation matrix . Their Lagrange multiplier test statistic of the null hypothesis $\gamma = 0$, i.e. $\boldsymbol{P}_t = \boldsymbol{P}$, is asymptotically χ^2-distributed with $N(N-1)$ degrees of freedom when the null hypothesis holds.

Even here, the focus is on testing $\boldsymbol{P}_t = \boldsymbol{P}$ against the alternative that the matrix varies deterministically with time. As already indicated, $\boldsymbol{P}_t$ is not a correlation matrix when $\boldsymbol{P}_t \neq \boldsymbol{P}$. Testing constancy of $\boldsymbol{P}_t$ in this framework is motivated by the fact that the test of the null hypothesis H_0: $\gamma = 0$ in (6) involves fewer parameters than the test of Hall et al. (2021) when $N > 2$. It may be viewed as a parsimonious version of their test, which is an advantage when N becomes large. When H_0 holds, $G(t/T) = 1/2$, and $\boldsymbol{\Lambda}_t \equiv \boldsymbol{\Lambda}$. It is seen from (5) and (6) that in that situation the covariance matrix (5) is not identified. Both $\boldsymbol{\Lambda}^*$, c_1 and c_2 are unidentified nuisance parameters.

In order to derive a test of this null hypothesis, we circumvent the identification problem as in Luukkonen, Saikkonen and Teräsvirta (1988) and develop $G(t/T)$ into a Taylor series around the null hypothesis. After reparameterising, (5) becomes

$$\boldsymbol{\Psi}_t = \boldsymbol{\Psi}_{(0)} + \boldsymbol{\Psi}_{(1)}t/T + \boldsymbol{\Psi}_{(2)}(t/T)^2 + \boldsymbol{\Psi}_{(R)}, \tag{7}$$

where $\boldsymbol{\Psi}_{(R)}$ is a residual matrix and $\boldsymbol{\Psi}_{(j)} = \mathrm{diag}(\psi_{j1}, \ldots, \psi_{jN})$, $j = 0, 1, 2$. Requiring the diagonal elements of $\boldsymbol{\Psi}_t$ to sum up to N implies that $\psi_{0N} = N - \sum_{i=1}^{N-1} \psi_{0i}$ and $\psi_{jN} = -\sum_{i=1}^{N-1} \psi_{ji}$ for $j = 1, 2$. Under H_0, $\boldsymbol{\Psi}_{(0)} = \boldsymbol{\Lambda}$. The elements ψ_{ji}, $i = 1, \ldots, N$; $j = 1, 2$, in (7) are of the form $\psi_{ji} = \gamma^j \widetilde{\psi}_{ji}$, $\widetilde{\psi}_{ji} \neq 0$, $j = 1, 2$, so the new $2(N-1)$-dimensional null hypothesis is H_0': $\boldsymbol{\Psi}_{(1)} = \boldsymbol{\Psi}_{(2)} = \boldsymbol{0}$.

Since we shall construct a Lagrange multiplier test that only requires estimating the model under the null hypothesis we can ignore the residual vector $\boldsymbol{\psi}_R$ because it is a null vector when H_0 (or H_0') holds. It does contribute to the power of the test when the alternative is true. This leads to the following auxiliary covariance matrix:

$$\boldsymbol{P}_t^A = \boldsymbol{Q}(\boldsymbol{\Psi}_{(0)} + \boldsymbol{\Psi}_{(1)}t/T + \boldsymbol{\Psi}_{(2)}(t/T)^2)\boldsymbol{Q}'. \tag{8}$$

Matrix (8) is a correlation matrix only under H_0', and its purpose is to function as a basis for a test of constant correlations. We call the model (2) in which (5) is replaced by (8), the *auxiliary* time-varying correlation GARCH model. It is a device constructed to derive the test and not a data-generating process. Its log-likelihood and score are considered in the next section.

The test we propose is similar to the one by Yang (2014) in that both make use of the spectral decomposition of $\boldsymbol{\Sigma}_t$. It should be noted, however, that Yang (2014) did not decompose the covariance matrix further into conditional variances and correlations. He constructed instead a test of constancy of the covariance matrix based on this decomposition. Our work may therefore be also seen as a variant of or an extension to Yang (2014).

3 Log-likelihood and score of the auxiliary model

The log-likelihood of the auxiliary TV-STC-GARCH model for observation t equals

$$\begin{aligned}\ell_t(\boldsymbol{\theta}) &= k - \frac{1}{2}\sum_{i=1}^{N}\ln g_{it} - \frac{1}{2}\sum_{i=1}^{N}\ln h_{it} - \frac{1}{2}\sum_{i=1}^{N-1}\ln \psi_i(t/T) \\ &\quad -\frac{1}{2}\ln(N - \sum_{k=1}^{N-1}\psi_k(t/T)) - \frac{1}{2}\sum_{i=1}^{N-1}\psi_i^{-1}(t/T)w_{it}^2 \\ &\quad -\frac{1}{2}(N - \sum_{k=1}^{N-1}\psi_k(t/T))^{-1}w_{Nt}^2, \end{aligned} \tag{9}$$

where $\psi_i(t/T) = \psi_{0i} + \psi_{1i}t/T + \psi_{2i}(t/T)^2$, and $w_{it} = \boldsymbol{q}_i'\boldsymbol{z}_t$, $i = 1, \ldots, N$, with $\boldsymbol{z}_t = \boldsymbol{D}_t^{-1}\boldsymbol{S}_t^{-1}\boldsymbol{\varepsilon}_t$, so $\mathrm{cov}(\boldsymbol{z}_t) = \boldsymbol{P}_t$. The vector $\boldsymbol{\theta} = (\boldsymbol{\theta}_g', \boldsymbol{\theta}_h', \boldsymbol{\theta}_\psi')'$, where its components are defined as follows: $\boldsymbol{\theta}_g = (\boldsymbol{\theta}_{g1}', \ldots, \boldsymbol{\theta}_{gN}')'$ with $\boldsymbol{\theta}_{gi} = (\boldsymbol{\delta}_i', \boldsymbol{\gamma}_i', \boldsymbol{c}_i')'$, where $\boldsymbol{\delta}_i = (\delta_{i1}, \ldots, \delta_{ir_i})'$, $\boldsymbol{\gamma}_i = (\gamma_{i1}, \ldots, \gamma_{ir_i})'$, and $\boldsymbol{c}_i = (\boldsymbol{c}_{i1}', \ldots, \boldsymbol{c}_{ir_i}')'$; $\boldsymbol{\theta}_h = (\boldsymbol{\theta}_{h1}', \ldots, \boldsymbol{\theta}_{hN}')'$ with $\boldsymbol{\theta}_{hi} = (\alpha_{i0}, \alpha_{i1}, \kappa_{i1}, \beta_{i1})'$; and $\boldsymbol{\theta}_\psi = (\boldsymbol{\psi}_0', \boldsymbol{\psi}_1', \boldsymbol{\psi}_2')'$, contains the parameters of (8). Let $\boldsymbol{e}_i$ be the ith column of the $N \times N$ identity matrix and $\mathbf{1}_N = (1, \ldots, 1)'$ an $N \times 1$ vector of ones. For notational purposes define the following $N - 1 \times 3$ parameter matrix

$$\boldsymbol{\Psi} = \begin{bmatrix} \psi_{01} & \psi_{11} & \psi_{21} \\ \cdots & & \\ \psi_{0,N-1} & \psi_{1,N-1} & \psi_{2,N-1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\psi}_0 & \boldsymbol{\psi}_1 & \boldsymbol{\psi}_2 \end{bmatrix}$$

and let $\boldsymbol{\psi} = \mathrm{vec}(\boldsymbol{\Psi}) = (\boldsymbol{\psi}_0', \boldsymbol{\psi}_1', \boldsymbol{\psi}_2')'$ and $\overline{\boldsymbol{\psi}} = \mathrm{vec}(\boldsymbol{\Psi}') = (\overline{\boldsymbol{\psi}}_1', \ldots, \overline{\boldsymbol{\psi}}_{N-1}')'$, where $\overline{\boldsymbol{\psi}}_j = (\psi_{0j}, \psi_{1j}, \psi_{2j})'$, $j = 1, \ldots, N-1$. We now state the following result:

Theorem 1. *Consider the auxiliary TV-STC-GARCH model* (2) *with* (8) *whose log-likelihood for observation t is defined in* (9)*. The blocks of the average score of the auxiliary log-likelihood are*

$$\frac{1}{T}\sum_{t=1}^{T}\frac{\partial \ell_t(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}_{gi}} = \frac{1}{2T}\sum_{t=1}^{T}(\frac{1}{g_{it}}\frac{\partial g_{it}}{\partial \boldsymbol{\theta}_{gi}} + \frac{1}{h_{it}}\frac{\partial h_{it}}{\partial \boldsymbol{\theta}_{gi}})\{\boldsymbol{e}_i'\boldsymbol{z}_t\boldsymbol{z}_t'(\boldsymbol{P}_t^A)^{-1}\boldsymbol{e}_i - 1\} \tag{10}$$

and

$$\frac{1}{T}\sum_{t=1}^{T}\frac{\partial \ell_t(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}_{hi}} = \frac{1}{2T}\sum_{t=1}^{T}\frac{1}{h_{it}}\frac{\partial h_{it}}{\partial \boldsymbol{\theta}_{hi}}\{\boldsymbol{e}_i'\boldsymbol{z}_t\boldsymbol{z}_t'(\boldsymbol{P}_t^A)^{-1}\boldsymbol{e}_i - 1\} \tag{11}$$

for $i = 1, \dots, N$*, and*

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T}\frac{\partial \ell_t(\boldsymbol{\theta})}{\partial \overline{\boldsymbol{\psi}}_i} &= \frac{1}{2T}\sum_{t=1}^{T}\{\frac{1}{\psi_i(t/T)}(\frac{w_{it}^2}{\psi_i(t/T)} - 1) \\ &\quad -\frac{1}{N-\sum_{k=1}^{N-1}\psi_k(t/T)}\left(\frac{w_{Nt}^2}{N-\sum_{k=1}^{N-1}\psi_k(t/T)} - 1\right)\}\boldsymbol{\tau}_t \end{aligned} \tag{12}$$

for $i = 1, \dots, N-1$*, where* $\boldsymbol{\tau}_t = (1, t/T, (t/T)^2)'$*. Under* H_0*,* (10)*,* (11) *and* (12) *become*

$$\frac{1}{T}\sum_{t=1}^{T}\frac{\partial \ell_t(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}_{gi}}|_{H_0} = \frac{1}{2T}\sum_{t=1}^{T}(\frac{1}{g_{it}}\frac{\partial g_{it}}{\partial \boldsymbol{\theta}_{gi}} + \frac{1}{h_{it}}\frac{\partial h_{it}}{\partial \boldsymbol{\theta}_{gi}})\{\boldsymbol{e}_i'\boldsymbol{z}_t\boldsymbol{z}_t'\boldsymbol{P}^{-1}\boldsymbol{e}_i - 1\},$$

$$\frac{1}{T}\sum_{t=1}^{T}\frac{\partial \ell_t(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}_{hi}}|_{H_0} = \frac{1}{2T}\sum_{t=1}^{T}\frac{1}{h_{it}}\frac{\partial h_{it}}{\partial \boldsymbol{\theta}_{hi}}\{\boldsymbol{e}_i'\boldsymbol{z}_t\boldsymbol{z}_t'\boldsymbol{P}^{-1}\boldsymbol{e}_i - 1\}$$

and

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T}\frac{\partial \ell_t(\boldsymbol{\theta})}{\partial \overline{\boldsymbol{\psi}}_i}|_{H_0} &= \frac{1}{2T}\sum_{t=1}^{T}\{\frac{1}{\psi_{0i}}(\frac{w_{it}^2}{\psi_{0i}} - 1) \\ &\quad -\frac{1}{N-\sum_{k=1}^{N-1}\psi_{0k}}\left(\frac{w_{Nt}^2}{N-\sum_{k=1}^{N-1}\psi_{0k}} - 1\right)\}\boldsymbol{\tau}_t.\end{aligned}$$

Proof. See Appendix B.

4 Information matrix

In order to form the test statistic, we need the information matrix of $L_T = \frac{1}{T}\sum_{t=1}^{T}\ell_t(\boldsymbol{\theta})$. Define $\partial g_{it}^0/\partial \boldsymbol{\theta}_{gi} = \partial g_{it}/\partial \boldsymbol{\theta}_{gi}|_{\boldsymbol{\theta}_{gi}=\boldsymbol{\theta}_{gi}^0}$ and $\partial h_{it}^0/\partial \boldsymbol{\theta}_{hi} = \partial h_{it}/\partial \boldsymbol{\theta}_{hi}|_{\boldsymbol{\theta}_{hi}=\boldsymbol{\theta}_{hi}^0}$, where $\boldsymbol{\theta}_{gi}^0 = (\boldsymbol{\delta}_i^{0\prime}, \boldsymbol{\gamma}_i^{0\prime}, \boldsymbol{c}_i^{0\prime})'$ and $\boldsymbol{\theta}_{hi}^0 = (\alpha_{i0}^0, \alpha_{i1}^0, \kappa_{i1}^0, \beta_{i1}^0)'$ are the true parameter vectors. Let $\psi_{01}^0 = \lambda_1^0, \dots, \psi_{0,N-1}^0 = \lambda_{N-1}^0$, $\psi_{0N}^0 = N - \sum_{k=1}^{N-1}\psi_{0k}^0$, be the true eigenvalues, so the matrix of true eigenvalues equals $\boldsymbol{\Psi}^0 = \mathrm{diag}(\psi_{01}^0, \ldots, \psi_{0N}^0)$. The true correlation matrix is denoted by $\boldsymbol{P}^0$. The information matrix is divided into blocks as follows:

$$\boldsymbol{J} = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \mathsf{E} \frac{\partial \ell_t}{\partial \boldsymbol{\theta}} \frac{\partial \ell_t}{\partial \boldsymbol{\theta}'}|_{\mathrm{H}_0} = \begin{bmatrix} \boldsymbol{J}_{\theta_g\theta_g} & \boldsymbol{J}_{\theta_g\theta_h} & \boldsymbol{J}_{\theta_g\overline{\psi}} \\ & \boldsymbol{J}_{\theta_h\theta_h} & \boldsymbol{J}_{\theta_h\overline{\psi}} \\ & & \boldsymbol{J}_{\overline{\psi\psi}} \end{bmatrix}. \tag{13}$$

The following result defines the blocks of (13).

Theorem 2. *The blocks of the information matrix* (13) *are as follows: The* (i,j) *block,* $i \neq j$*, of* $\boldsymbol{J}_{\theta_g\theta_g}$ *equals*

$$\begin{aligned}
[\boldsymbol{J}_{\theta_g\theta_g}]_{ij} &= \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \mathsf{E} \frac{\partial \ell_t}{\partial \boldsymbol{\theta}_{gi}} \frac{\partial \ell_t}{\partial \boldsymbol{\theta}'_{gj}}|_{H_0} \\
&= \frac{1}{4}[\int_0^1 \frac{1}{g_{ir}^0 g_{jr}^0} \frac{\partial g_{ir}^0}{\partial \boldsymbol{\theta}_{gi}} \frac{\partial g_{jr}^0}{\partial \boldsymbol{\theta}'_{gj}} dr \\
&\quad + \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \{ \frac{1}{g_{it}^0 h_{it}^0} \frac{\partial g_{it}^0}{\partial \boldsymbol{\theta}_{gi}} \frac{\partial h_{it}^0}{\partial \boldsymbol{\theta}'_{gi}} + \frac{1}{h_{it}^0} \frac{\partial h_{it}^0}{\partial \boldsymbol{\theta}_{gi}} (\frac{1}{g_{jt}^0} \frac{\partial g_{jt}^0}{\partial \boldsymbol{\theta}'_{gj}} + \frac{1}{h_{jt}^0} \frac{\partial h_{jt}^0}{\partial \boldsymbol{\theta}'_{gj}}) \}] \\
&\quad \times \boldsymbol{e}_i'(\boldsymbol{P}^0)^{-1} \boldsymbol{e}_j \boldsymbol{e}_i' \boldsymbol{P}^0 \boldsymbol{e}_j
\end{aligned}$$

and

$$\begin{aligned}
[\boldsymbol{J}_{\theta_g\theta_g}]_{ii} &= \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \mathsf{E} \frac{\partial \ell_t}{\partial \boldsymbol{\theta}_{gi}} \frac{\partial \ell_t}{\partial \boldsymbol{\theta}'_{gi}}|_{H_0} \\
&= \frac{1}{4}[\int_0^1 (\frac{1}{(g_{ir}^0)^2} \frac{\partial g_{ir}^0}{\partial \boldsymbol{\theta}_{gi}} \frac{\partial g_{ir}^0}{\partial \boldsymbol{\theta}'_{gi}} dr + \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \frac{1}{h_{it}^0} \frac{\partial h_{it}^0}{\partial \boldsymbol{\theta}_{gi}} (\frac{1}{g_{it}^0} \frac{\partial g_{it}^0}{\partial \boldsymbol{\theta}'_{gi}} + \frac{1}{h_{it}^0} \frac{\partial h_{it}^0}{\partial \boldsymbol{\theta}'_{gi}})] \\
&\quad \times \{ \boldsymbol{e}_i'(\boldsymbol{P}^0)^{-1} \boldsymbol{e}_i + 1 \}.
\end{aligned}$$

The (i,j) *block,* $i \neq j$*, of* $\boldsymbol{J}_{\theta_h\theta_h}$ *equals*

$$\begin{aligned}
[\boldsymbol{J}_{\theta_h\theta_h}]_{ij} &= \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \mathsf{E} \frac{\partial \ell_t}{\partial \boldsymbol{\theta}_{hi}} \frac{\partial \ell_t}{\partial \boldsymbol{\theta}'_{hj}}|_{H_0} \\
&= \lim_{T\to\infty} \frac{1}{4T} \sum_{t=1}^{T} \frac{1}{h_{it}^0 h_{jt}^0} \frac{\partial h_{it}^0}{\partial \boldsymbol{\theta}_{hi}} \frac{\partial h_{jt}^0}{\partial \boldsymbol{\theta}'_{hj}} \boldsymbol{e}_i'(\boldsymbol{P}^0)^{-1} \boldsymbol{e}_j \boldsymbol{e}_i' \boldsymbol{P}^0 \boldsymbol{e}_j
\end{aligned}$$

and

$$\begin{aligned}
[\boldsymbol{J}_{\theta_h\theta_h}]_{ii} &= \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \mathsf{E} \frac{\partial \ell_t}{\partial \boldsymbol{\theta}_{hi}} \frac{\partial \ell_t}{\partial \boldsymbol{\theta}'_{hi}}|_{H_0} \\
&= \lim_{T\to\infty} \frac{1}{4T} \sum_{t=1}^{T} \frac{1}{(h_{it}^0)^2} \frac{\partial h_{it}^0}{\partial \boldsymbol{\theta}_{hi}} \frac{\partial h_{it}^0}{\partial \boldsymbol{\theta}'_{hi}} \{ \boldsymbol{e}_i'(\boldsymbol{P}^0)^{-1} \boldsymbol{e}_i + 1 \}.
\end{aligned}$$

The (i,j) sub-block, $i \neq j$, of $\boldsymbol{J}_{\theta_g\theta_h}$ equals

$$
\begin{aligned}
[\boldsymbol{J}_{\theta_g\theta_h}]_{ij} &= \lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\mathsf{E}\frac{\partial\ell_t}{\partial\boldsymbol{\theta}_{gi}}\frac{\partial\ell_t}{\partial\boldsymbol{\theta}'_{hj}}|_{H_0} \\
&= \lim_{T\to\infty}\frac{1}{4T}\sum_{t=1}^{T}(\frac{1}{g^0_{it}}\frac{\partial g^0_{it}}{\partial\boldsymbol{\theta}_{gi}}+\frac{1}{h^0_{it}}\frac{\partial h^0_{it}}{\partial\boldsymbol{\theta}_{gi}})\frac{1}{h^0_{jt}}\frac{\partial h^0_{jt}}{\partial\boldsymbol{\theta}'_{hj}}\{\boldsymbol{e}'_i(\boldsymbol{P}^0)^{-1}\boldsymbol{e}_j\boldsymbol{e}'_i\boldsymbol{P}^0\boldsymbol{e}_j\}
\end{aligned}
$$

and

$$
\begin{aligned}
[\boldsymbol{J}_{\theta_g\theta_h}]_{ii} &= \lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\mathsf{E}\frac{\partial\ell_t}{\partial\boldsymbol{\theta}_{gi}}\frac{\partial\ell_t}{\partial\boldsymbol{\theta}'_{hi}}|_{H_0} \\
&= \lim_{T\to\infty}\frac{1}{4T}\sum_{t=1}^{T}(\frac{1}{g^0_{it}}\frac{\partial g^0_{it}}{\partial\boldsymbol{\theta}_{gi}}+\frac{1}{h^0_{it}}\frac{\partial h^0_{it}}{\partial\boldsymbol{\theta}_{gi}})\frac{1}{h^0_{it}}\frac{\partial h^0_{it}}{\partial\boldsymbol{\theta}'_{hi}}\{\boldsymbol{e}'_i(\boldsymbol{P}^0)^{-1}\boldsymbol{e}_i+1\}.
\end{aligned}
$$

Furthermore, the (i,j) sub-block of $\boldsymbol{J}_{\theta_g\overline{\psi}}$, $i,j=1,\ldots,N$, has the form

$$
\begin{aligned}
[\boldsymbol{J}_{\theta_g\overline{\psi}}]_{ij} &= \lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\mathsf{E}\frac{\partial\ell_t}{\partial\boldsymbol{\theta}_{gi}}\frac{\partial\ell_t}{\partial\overline{\boldsymbol{\psi}}'_j}|_{H_0} \\
&= \frac{1}{4}(\int_0^1\frac{1}{g^0_{ir}}\frac{\partial g^0_{ir}}{\partial\boldsymbol{\theta}_{gi}}\boldsymbol{r}'\,dr+\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\frac{1}{h^0_{it}}\frac{\partial h^0_{it}}{\partial\boldsymbol{\theta}_{gi}}\boldsymbol{\tau}'_t) \\
&\quad\times\{\frac{1}{\psi_{0j}}\boldsymbol{e}'_i\boldsymbol{q}_j\boldsymbol{q}'_i\boldsymbol{e}_j-\frac{1}{N-\sum_{k=1}^{N-1}\psi_{0k}}\boldsymbol{e}'_i\boldsymbol{q}_N\boldsymbol{q}'_i\boldsymbol{e}_N\},
\end{aligned}
$$

where $\boldsymbol{r}=(1,r,r^2)'$, and the corresponding sub-block of $\boldsymbol{J}_{\theta_h\overline{\psi}}$ equals

$$
\begin{aligned}
[\boldsymbol{J}_{\theta_h\overline{\psi}}]_{ij} &= \lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\mathsf{E}\frac{\partial\ell_t}{\partial\boldsymbol{\theta}_{hi}}\frac{\partial\ell_t}{\partial\overline{\boldsymbol{\psi}}'_j}|_{H_0} \\
&= \lim_{T\to\infty}\frac{1}{4T}\sum_{t=1}^{T}\frac{1}{h^0_{it}}\frac{\partial h^0_{it}}{\partial\boldsymbol{\theta}_{hi}}\boldsymbol{\tau}'_t\{\frac{1}{\psi_{0j}}\boldsymbol{e}'_i\boldsymbol{q}_j\boldsymbol{q}'_i\boldsymbol{e}_j-\frac{1}{N-\sum_{k=1}^{N-1}\psi_{0k}}\boldsymbol{e}'_i\boldsymbol{q}_N\boldsymbol{q}'_i\boldsymbol{e}_N\}.
\end{aligned}
$$

Finally,

$$
\boldsymbol{J}_{\overline{\psi\psi}}=\mathsf{E}\frac{\partial\ell_t}{\partial\overline{\boldsymbol{\psi}}}\frac{\partial\ell_t}{\partial\overline{\boldsymbol{\psi}}'}|_{H_0}=\frac{1}{2}(\boldsymbol{\Psi}_0^{-2}+\frac{1}{(N-\sum_{k=1}^{N-1}\psi_{0k})^2}\mathbf{1}_{N-1}\mathbf{1}'_{N-1})\otimes\begin{bmatrix}1 & 1/2 & 1/3\\ & 1/3 & 1/4\\ & & 1/5\end{bmatrix},
$$

where $\boldsymbol{\Psi}_0=\mathrm{diag}(\psi_{01},\ldots,\psi_{0,N-1})$.

Proof. See Appendix B.

5 Test statistic

Under regularity conditions, Silvennoinen and Teräsvirta (in press) showed that the maximum likelihood estimators of the parameters of the null model (time-varying GARCH equations and constant correlations) are consistent and asymptotically normal. Rewrite (13) as

$$\boldsymbol{J} = \lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} \mathsf{E}\frac{\partial \ell_t}{\partial \boldsymbol{\theta}}\frac{\partial \ell_t}{\partial \boldsymbol{\theta}'}|_{\mathrm{H}_0} = \begin{bmatrix} \boldsymbol{J}_{\theta_g\theta_g} & \boldsymbol{J}_{\theta_g\theta_h} & \boldsymbol{J}_{\theta_g\psi_0} & \boldsymbol{J}_{\theta_g\overline{\psi}_{12}} \\ & \boldsymbol{J}_{\theta_h\theta_h} & \boldsymbol{J}_{\theta_h\psi_0} & \boldsymbol{J}_{\theta_h\overline{\psi}_{12}} \\ & & \boldsymbol{J}_{\psi_0\psi_0} & \boldsymbol{J}_{\psi_0\overline{\psi}_{12}} \\ & & & \boldsymbol{J}_{\overline{\psi}_{12}\overline{\psi}_{12}} \end{bmatrix}, \tag{14}$$

where $\overline{\boldsymbol{\psi}}_{12} = (\overline{\boldsymbol{\psi}}'_{12,1}, \ldots, \overline{\boldsymbol{\psi}}'_{12,N-1})$ with $\overline{\boldsymbol{\psi}}_{12,j} = (\psi_{1j}, \psi_{2j})'$, $j = 1, \ldots, N-1$. Then

$$\boldsymbol{J}_{\psi_0\psi_0} = \frac{1}{2}(\boldsymbol{\Psi}_0^{-2} + \frac{1}{(N - \sum_{k=1}^{N-1}\psi_{0k})^2}\mathbf{1}_{N-1}\mathbf{1}'_{N-1}),$$

$$\boldsymbol{J}_{\psi_0\overline{\psi}_{12}} = \frac{1}{2}(\boldsymbol{\Psi}_0^{-2} + \frac{1}{(N - \sum_{k=1}^{N-1}\psi_{0k})^2}\mathbf{1}_{N-1}\mathbf{1}'_{N-1}) \otimes \begin{bmatrix} 1/2 & 1/3 \end{bmatrix}$$

and

$$\boldsymbol{J}_{\overline{\psi}_{12}\overline{\psi}_{12}} = \frac{1}{2}(\boldsymbol{\Psi}_0^{-2} + \frac{1}{(N - \sum_{k=1}^{N-1}\psi_{0k})^2}\mathbf{1}_{N-1}\mathbf{1}'_{N-1}) \otimes \begin{bmatrix} 1/3 & 1/4 \\ & 1/5 \end{bmatrix}.$$

Let

$$\boldsymbol{J}_{00} = \begin{bmatrix} \boldsymbol{J}_{\theta_g\theta_g} & \boldsymbol{J}_{\theta_g\theta_h} & \boldsymbol{J}_{\theta_g\psi_0} \\ & \boldsymbol{J}_{\theta_h\theta_h} & \boldsymbol{J}_{\theta_h\psi_0} \\ & & \boldsymbol{J}_{\psi_0\psi_0} \end{bmatrix}$$

and

$$\boldsymbol{J}_{0\overline{\psi}_{12}} = \begin{bmatrix} \boldsymbol{J}_{\theta_g\overline{\psi}_{12}} \\ \boldsymbol{J}_{\theta_h\overline{\psi}_{12}} \\ \boldsymbol{J}_{\psi_0\overline{\psi}_{12}} \end{bmatrix}.$$

Using the Lagrange multiplier principle and the assumption that $\boldsymbol{z}_t$ is multivariate normal, we obtain the following statistic for testing H'_0: $\overline{\boldsymbol{\psi}}_{12,j} = \mathbf{0}$, $j = 1, \ldots, N-1$:

$$\begin{aligned} LM \; = \; & \frac{T}{4}\{\frac{1}{T}\sum_{t=1}^{T}(\widehat{x}^0_{1t}\boldsymbol{\tau}'_{12t}, \ldots, \widehat{x}^0_{Nt}\boldsymbol{\tau}'_{12t})\}\{\boldsymbol{J}_{\overline{\psi}_{12}\overline{\psi}_{12}} - \boldsymbol{J}_{\overline{\psi}_{12}0}\boldsymbol{J}_{00}^{-1}\boldsymbol{J}_{0\overline{\psi}_{12}}\} \\ & \times\{\frac{1}{T}\sum_{t=1}^{T}(\widehat{x}^0_{1t}\boldsymbol{\tau}'_{12t}, \ldots, \widehat{x}^0_{Nt}\boldsymbol{\tau}'_{12t})'\}, \end{aligned} \tag{15}$$

where $\boldsymbol{\tau}_{12t} = (t/T, (t/T)^2)'$, and

$$\widehat{x}_{jt} = \frac{1}{\widehat{\psi}_{0j}}(\frac{\widehat{w}^2_{jt}}{\widehat{\psi}_{0j}} - 1) - \frac{1}{N - \sum_{k=1}^{N-1}\widehat{\psi}_{0j}}(\frac{\widehat{w}^2_{Nt}}{N - \sum_{k=1}^{N-1}\widehat{\psi}_{0j}} - 1), \tag{16}$$

where $\widehat{\psi}_{0j}$ is the estimate of ψ_{0j} under H'_0. In addition, in (16) $\widehat{w}_{it} = \widehat{\boldsymbol{q}}'_i\widehat{\boldsymbol{S}}_t^{-1}\widehat{\boldsymbol{D}}_t^{-1}\boldsymbol{\varepsilon}_t$, where $\widehat{\boldsymbol{S}}_t = \mathrm{diag}(\widehat{g}^{1/2}_{1t}, \ldots, \widehat{g}^{1/2}_{Nt})$ contains square roots of the estimated deterministic components,

$\widehat{\boldsymbol{D}}_t = \text{diag}(\widehat{h}_{1t}^{1/2}, \ldots, \widehat{h}_{Nt}^{1/2})$ contains the estimated conditional standard deviations of $\boldsymbol{\phi}_t$, and $\widehat{\boldsymbol{q}}_j$ is the jth eigenvector of the estimated correlation matrix $\widehat{\boldsymbol{P}}$ under H_0'. Based on the results in Silvennoinen and Teräsvirta (in press), this statistic has an asymptotic χ^2-distribution with $2(N-1)$ degrees of freedom when H_0' holds. To make it operational, the blocks of the information matrix in (15) have to be replaced by their consistent estimators.

If the transition function (6) is assumed monotonic in t/T, that is, $t/T - c_2 \equiv 1$, the second-order component can be omitted from the approximation (7), and the $N-1$-dimensional null hypothesis becomes $\boldsymbol{\psi}_1 = \mathbf{0}$. If this assumption holds, the power of the test increases compared to the situation in which the second-order component is included in the test.

As already discussed, the matrix $\boldsymbol{P}_t = \boldsymbol{Q}\boldsymbol{\Lambda}_t\boldsymbol{Q}'$ is a correlation matrix only when $\boldsymbol{P}_t = \boldsymbol{P}$, that is, when $\boldsymbol{\Lambda}_t \equiv \boldsymbol{\Lambda}$. There is one exception to this rule, however. When all correlations are equal, the time-varying matrix $\boldsymbol{P}_t = \boldsymbol{Q}\boldsymbol{\Lambda}_t\boldsymbol{Q}'$ remains a correlation matrix when $\boldsymbol{\Lambda}_t$ is defined as in (5). In the GARCH context this type of equicorrelation is discussed in Engle and Kelly (2012).

The test statistic (15) can be applied in the general case in which the GARCH component is multiplicative and contains a smooth deterministically varying component. The purpose of this component, $\boldsymbol{S}_t$ in (2), is to account for nonstationarity in variance that manifests itself in changing amplitudes of the volatility clusters that ARCH and GARCH models are designed to explain. A cruder way of describing this type of variability is to assume that there are breaks in the variance. This alternative does not fit into the present analysis, however, because breaks at unknown points of time make the log-likelihood ill-behaved. Nevertheless, the statistic (15) does have power against that alternative, although the standard asymptotic theory does not cover it.

If it is assumed that $\boldsymbol{S}_t = \boldsymbol{I}_N$ and that the GARCH process is weakly stationary, the test statistic continues to be valid. This simplifies the expressions, while the null hypothesis remains unchanged. Setting $\boldsymbol{D}_t = \boldsymbol{I}_N$ makes it possible to test constancy of $\boldsymbol{P}_t$ before specifying the conditional variances. This is discussed in Silvennoinen and Teräsvirta (in press). If both $\boldsymbol{D}_t = \boldsymbol{S}_t = \boldsymbol{I}_N$, the test is a parsimonious test of constancy of a correlation matrix against the alternative that the correlations change over time. In that case, $\boldsymbol{P}_t$ may be a covariance matrix and not necessarily a correlation matrix. The statistic (15) must, however, be modified because the restriction that the eigenvalues sum up to N does not hold for the covariance matrix. With this modification, the test can for instance be used for testing constancy of the error covariance matrix of a vector autoregressive model against deterministically changing covariances; see also Yang (2014).

6 Simulations

In this section we investigate the properties of our test via several simulations. The finer details of the various experiments as well as the tabulated results are found in Appendix C.

We first simulate the size of our test. For this purpose, we choose $N = 2, 5, 10, 20$ and $T = 500, 1000, 2000$ in (2). All GARCH(1,1) equations are standard symmetric GARCH ones, parameterised such that the persistence is 0.95 and kurtosis of $\varepsilon_{it} = 4$, $i = 1, \ldots, N$, or in the next set up, kurtosis of $\varepsilon_{it} = 6$, $i = 1, \ldots, N$. For these simulations,

$g_{it} \equiv 1$, and the unconditional variance is fixed to one by defining $\alpha_{i0} = 1 - \alpha_{i1} - \beta_{i1}$. The correlation matrix is an equicorrelation matrix (Engle and Kelly 2012) with either $\rho = 0.33$ or $\rho = 0.67$, and we call the model the Constant Equicorrelation (CEC-) GARCH model. Finally, $\boldsymbol{\zeta}_t \sim \text{iid}\,\mathcal{N}(\mathbf{0}, \boldsymbol{I}_N)$.

The test statistic has been derived such that the highest order in the Taylor expansion equals two. In simulations, we include the orders up to four. This is done to find out how the empirical size of the test behaves when flexibility of the statistic (and the dimension of the null hypothesis) to cover more variable and nonmonotonic shifts in correlations is increased. In practice this means that (7) becomes

$$\boldsymbol{\Psi}_t = \boldsymbol{\Psi}_{(0)} + \sum_{i=1}^{4} \boldsymbol{\Psi}_{(i)}(t/T)^i + \boldsymbol{\Psi}_{(R4)},$$

where $\boldsymbol{\Psi}_{(R4)}$ is the residual. The null hypothesis is H_0': $\boldsymbol{\Psi}_{(1)}(t/T) = \ldots = \boldsymbol{\Psi}_{(4)}(t/T)^4 = \mathbf{0}$.

The p-value size discrepancy, see Davidson and MacKinnon (1998), results for $g_{it} = 1$ and for kurtosis of ε_{it} equal to 4 and 6, when $\rho = 0.33$ appear in Figure 1 and Table C.1. Although estimating GARCH equations when $T = 500$ cannot be recommended in practice, this sample size is included in simulations to find out how the test behaves in that situation. The empirical size of test is very close to its nominal size. In particular, the change in kurtosis does not have any effect on the empirical size. The only exception where the test is slightly oversized is the design in which $T = 500$ and the order of the polynomial is four.

We move on to the strongly correlated situation, that is, $\rho = 0.67$. The size discrepancies are in Figure 2, see also Table C.2. The story remains, for most parts, similar to that of the weakly correlated system. Now the test is somewhat oversized when $T = 500$ and the Taylor polynomial is at least equal to two. The equicorrelation matrix becomes gradually more ill-conditioned as its dimension grows but is still reasonably accurately inverted when $N = 20$.

Furthermore, we consider a situation where we replace the equicorrelation matrix with a positive definite matrix comprised of equicorrelation blocks. The block-equicorrelation structure (Engle and Kelly 2012) imposes different equicorrelations between and within blocks of series. We choose $N = 12$ and $N = 16$ and blocks of size four. The chosen correlation strengths mimic those of the equicorrelated (weak and strong) levels while maintaining similar condition numbers to ensure fair comparison.[1] The only difference in Table C.3 compared to the equicorrelation case (Tables C.1 and C.2) is that the test is slightly oversized when the order of the polynomial exceeds one.

The remaining results address misspecified GARCH equations. Such misspecification may show up in the covariance as time-variation, even if the correlations happen to be constant. The purpose of these simulations is to find out how well our test is able to detect the resulting time-variation in the eigenvalues.

When g_{it} is time-varying but this variance is ignored, the model is indeed misspecified. In these simulations, the GARCH equations are TV-GARCH equations with $\delta_0 = 1$, $\delta_1 = 3$, $\gamma = 20$ and $c = 0.5$, with equicorrelation coefficient equal to 0.33 and 0.67. The slope parameter γ_i has been calibrated such that the monotonically increasing $G_i(t/T, \gamma_i, c_i)$

[1]See Appendix C for details.

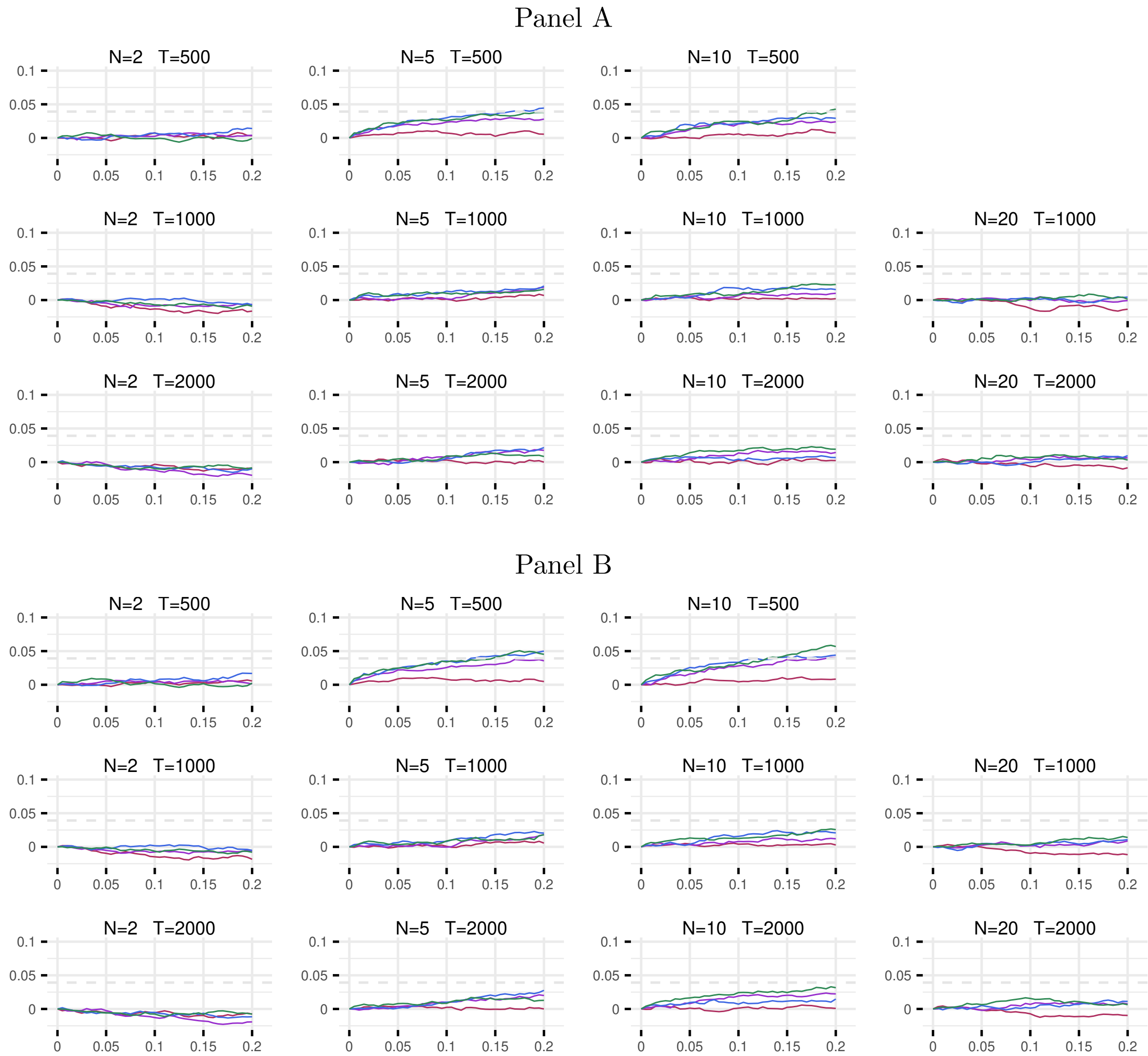

Figure 1: p-value size discrepancy of the test statistic (15) of orders 1 (red), 2 (purple), 3 (blue), and 4 (green). The test is based on the correctly specified DGP, which is CEC-GARCH with persistence of 0.95, kurtosis of 4 (Panel A) and 6 (Panel B), and equicorrelation of 0.33. The dashed line indicates the upper 95% confidence level of $1.96/\sqrt{2500}$.

remains practically equal to zero until $t/T = 0.25$ and (almost) reaches one when $t/T = 0.75$. This means that there is a rather mild shift in the (local) unconditional variance in these equations over time, resulting in the amplitude of clusters doubling in size over time. The error covariance matrix is thereby time-varying, whereas the error correlation matrix is constant over time. The reported rejection frequencies in Table C.4 indicate that the test detects time-variation even for weakly correlated system (see also Figure 3), and even more so with the correlation of 0.67 (Table C.5), which stresses the importance of specifying the GARCH equations properly before testing constancy of correlations. The rejection frequency increases with the sample size and the dimension of the system,

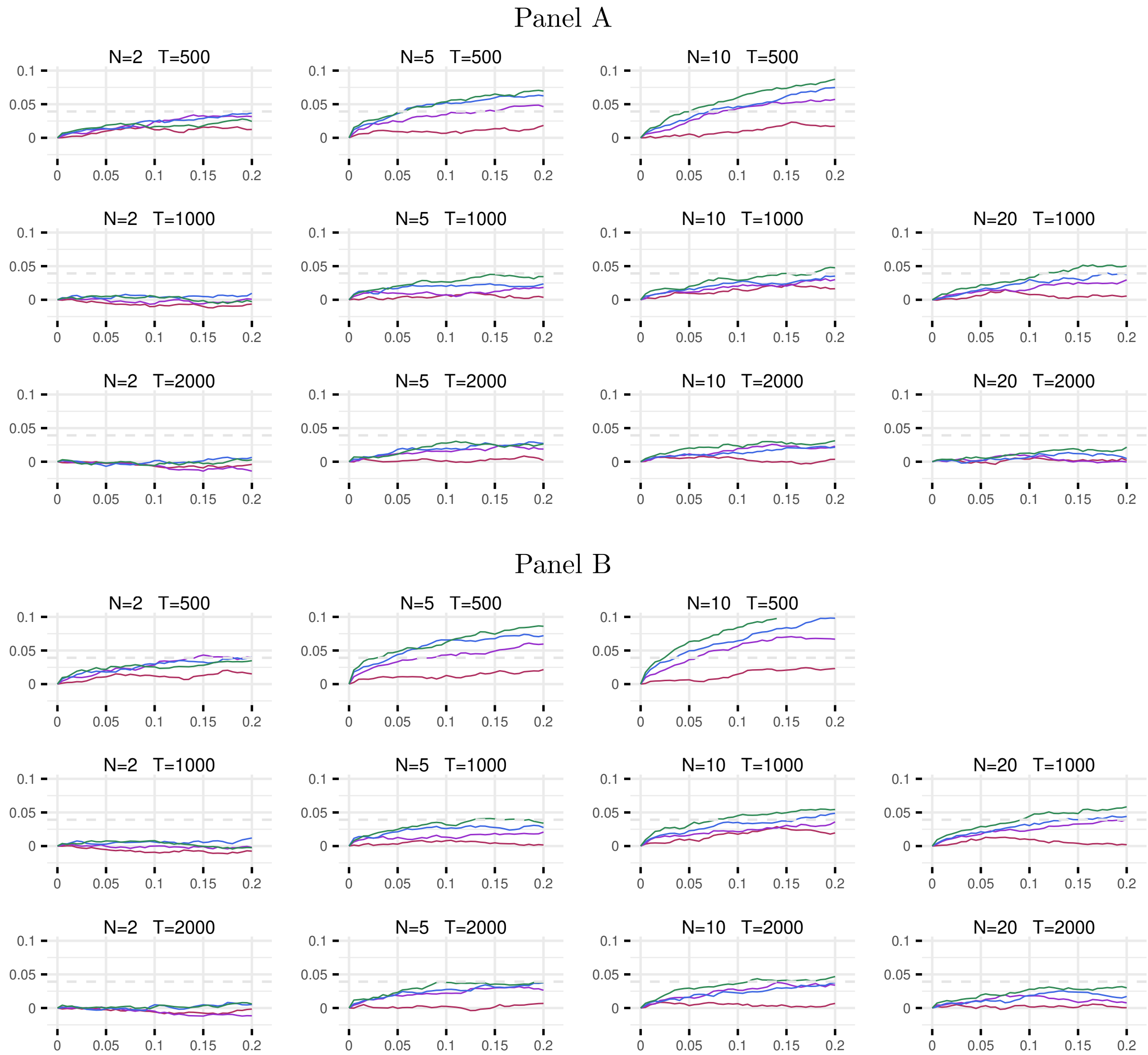

Figure 2: p-value size discrepancy of the test statistic (15) of orders 1 (red), 2 (purple), 3 (blue), and 4 (green). The test is based on the correctly specified DGP, which is CEC-GARCH with persistence of 0.95, kurtosis of 4 (Panel A) and 6 (Panel B), and equicorrelation of 0.67. The dashed line indicates the upper 95% confidence level of $1.96/\sqrt{2500}$.

and becomes overwhelming when more information against the null hypothesis becomes available. We also experimented with higher values of δ_1, but because the feature is already well illustrated for $\delta_1 = 3$, we do not report any additional results here.

GARCH can also be misspecified such that asymmetry in the form of GJR-GARCH is ignored. The simulation design concerning this sets the $\alpha_1 = 0$ leaving the asymmetric component κ_1 solely responsible for the effect of the past shocks. The parameterisation follows the targets of the previous simulations, that is, the implied kurtosis of four and six, unconditional variance of one and persistence is kept at 0.95. When the equicorrelation is 0.33, there is positive size distortion for $N \geq 5$, and for each N, an increase in sample size

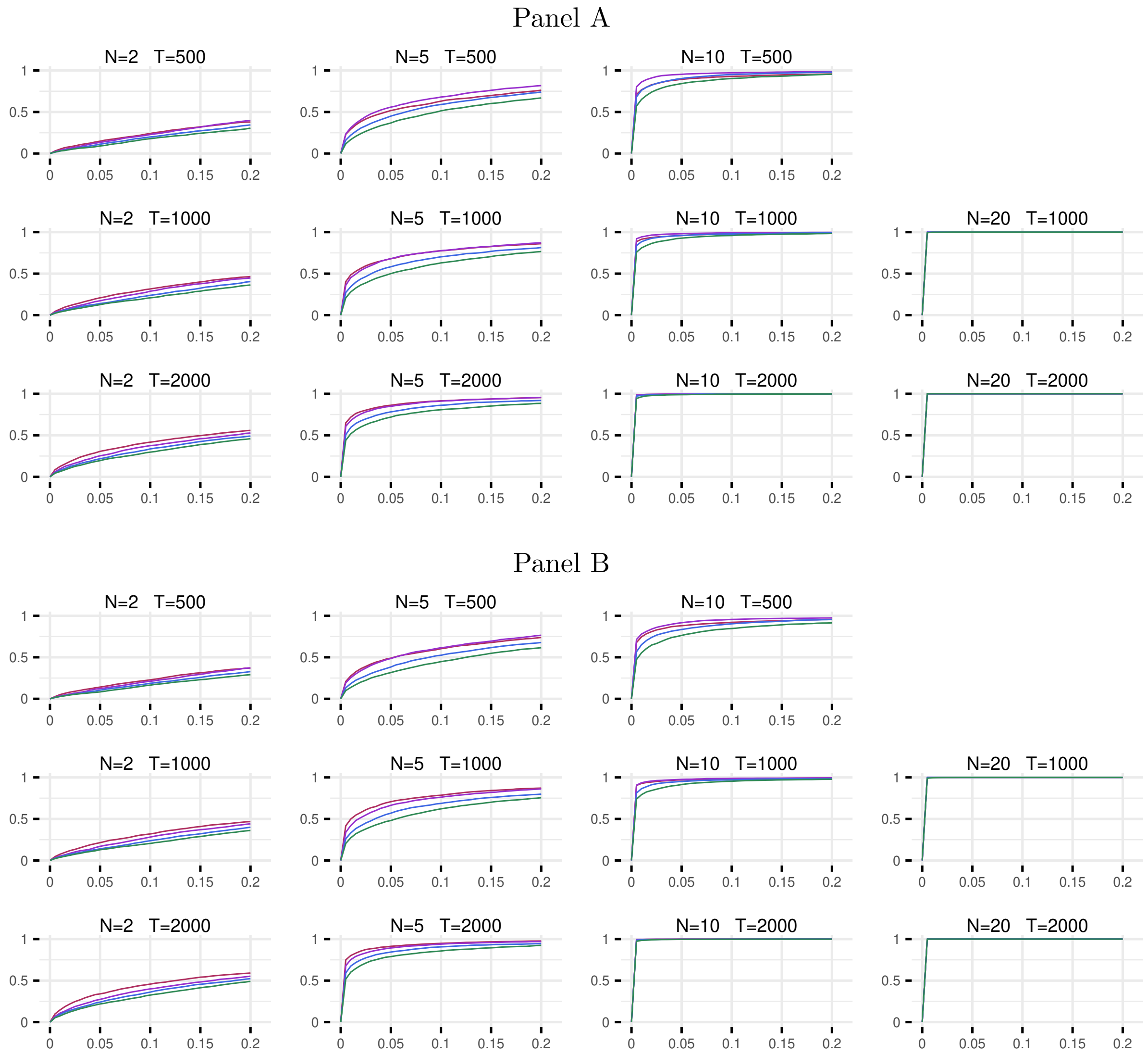

Figure 3: Rejection frequencies of the test statistic (15) of orders 1 to 4. The test is based on CEC-GARCH, while the DGP is TV-CEC-GARCH with persistence of 0.95, kurtosis of 4 (Panel A) and 6 (Panel B), equicorrelation of 0.33, and TV-parameters $\delta_0 = 1$, $\delta_1 = 3$, $c = 0.5$, and $\gamma = 20$ with $s_t = t/T$.

makes very little difference in terms of improving the size. This is seen from the rejection frequencies reported in Table C.6, see also Figure 4. The size distortion is already present when $N = 2$ for the 0.67 equicorrelated case, see Table C.7. In situations where the past shocks feed into the volatility via both symmetric and asymmetric channels, the size distortion is milder than in the extreme case discussed here, and will lie somewhere between the results here and those in Tables C.1 and C.2. Regardless, it may be concluded that a misspecification in the GARCH equation has a minor impact on constant correlation detection in comparison to the case when the deterministic shift in GARCH is erroneously ignored.

Finally, Tables C.8-C.9 and Figure 5 show what happens when, instead of normal,

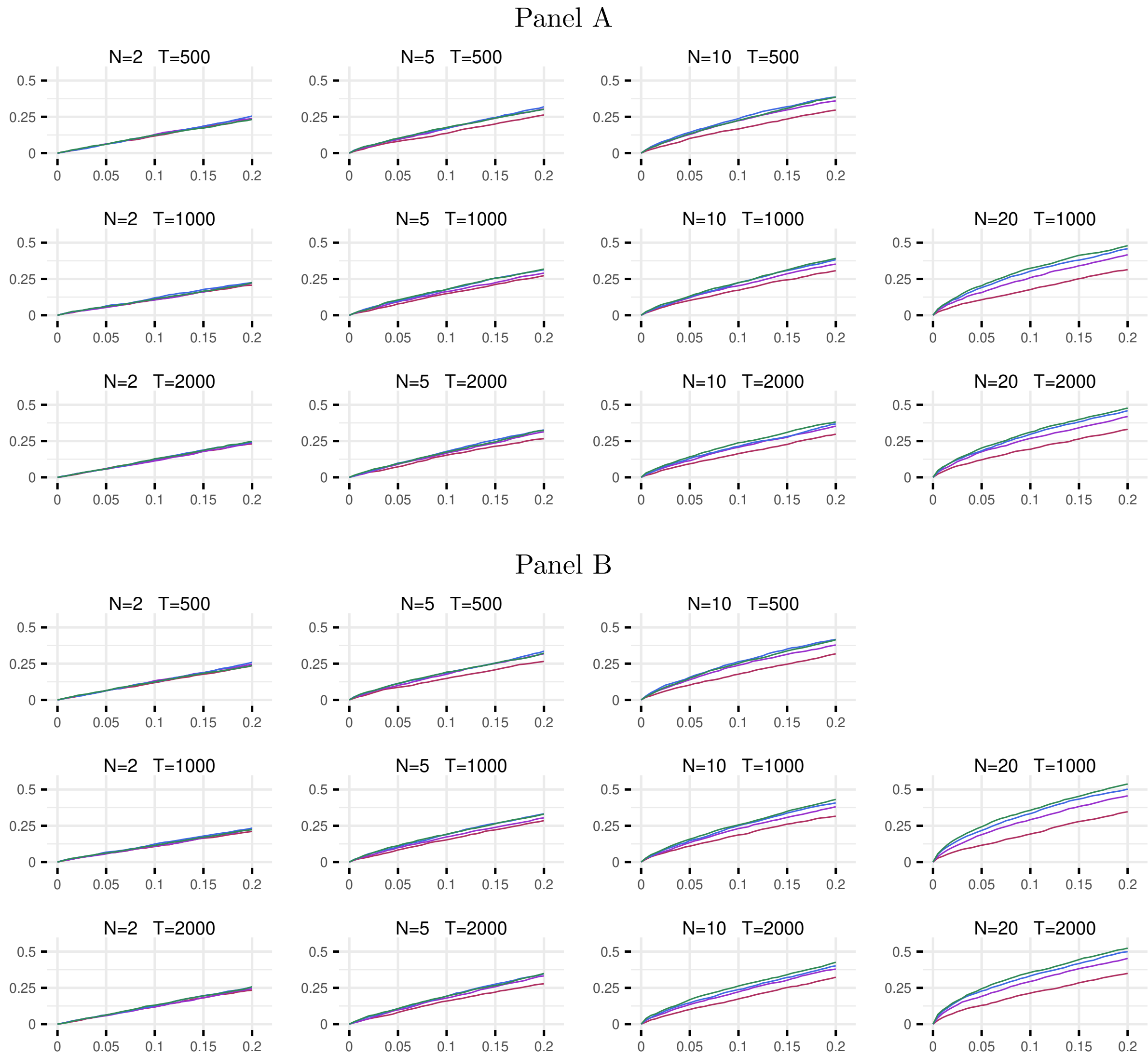

Figure 4: Rejection frequencies of the test statistic (15) of orders 1 (red), 2 (purple), 3 (blue), and 4 (green). The test is based on CEC-GARCH, while the DGP is CEC-GJR-GARCH with persistence of 0.95, kurtosis of 4 (Panel A) and 6 (Panel B), and equicorrelation of 0.33.

the error vectors are t-distributed with $df = 5$ and $df = 8$. Not accounting for this and assuming that the errors are multinormal, causes positive size distortion. Again, the distortion is not very large compared to what is observed in connection with ignoring the time-variation. It increases when the tails grow fatter (degrees of freedom decrease from eight to five) and when the order of the polynomial in the test grows. It may be noted, however, that this design may not be completely realistic. In practice it is quite possible that the GARCH residuals of equation i may seem to follow a t-distribution just because the GARCH component is misspecified, for example by ignoring the deterministic component g_{it}. Here we simulate the case in which the standard GARCH equation with normal errors for some unknown reason does not adequately describe the conditional

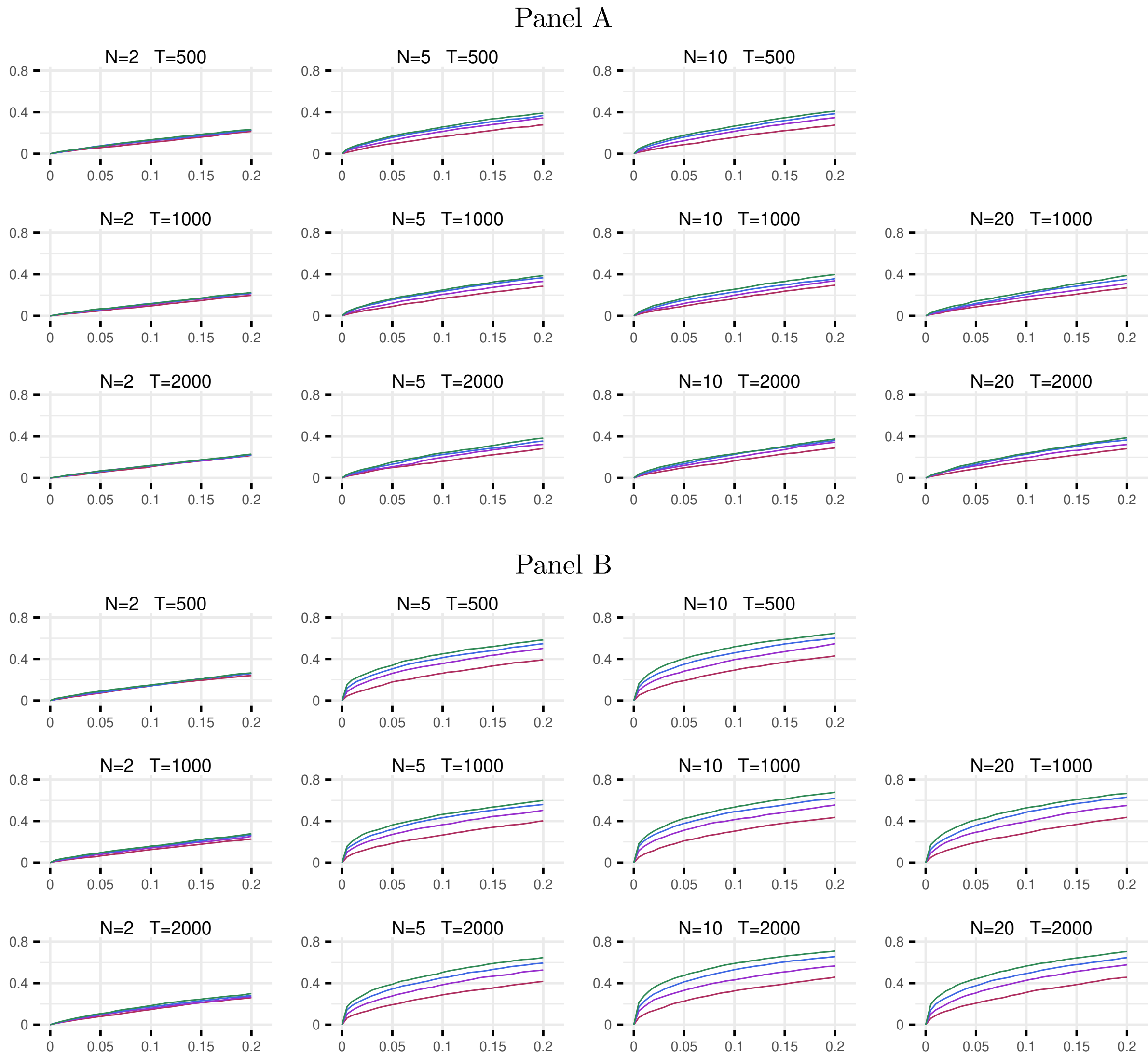

Figure 5: Rejection frequencies of the test statistic (15) of orders 1 (red), 2 (purple), 3 (blue), and 4 (green). The test is based on CEC-GARCH with normal errors, while the DGP has t-distributed errors, with persistence of 0.95, $df = 8$ (Panel A) and $df = 5$ (Panel B), and equicorrelation of 0.33.

variances. Once again, these results suggest that the GARCH equations have to be correctly specified before testing constancy of correlations can be attempted.

It is worth mentioning that when (3) is valid, the error covariance matrix is nonconstant even when the correlations are constant. In that case, the test by Yang (2014) would no doubt reject the null hypothesis of a constant error *covariance* matrix, whereas our test, after modelling the time-varying error variances, would not reject constancy of the error *correlation* matrix.

The observations from these simulations underline the need for testing adequacy (constancy, asymmetry) of the GARCH equations before embarking on testing constancy of correlations. Tests against multiplicative time-varying (TV-)GARCH are discussed in

Amado and Teräsvirta (2017) and Hall et al. (2021). Estimation of TV-GARCH equations is considered in Amado and Teräsvirta (2013). For other univariate tests, see e.g. Bollerslev (1986), Hagerud (1997) and Lundbergh and Teräsvirta (2002). For multivariate ones, see Eklund and Teräsvirta (2007).

7 Application

In order to demonstrate the use of the test we select 26 stocks that have been included in the Dow Jones index during the whole observation period from 2 January 2001 to 31 December 2020 and consider their daily returns. The names, symbols and respective categories of the stocks are listed in Table A.1 in Appendix A. We split the observation period into two halves such that the returns from 2001 to the end of 2010 form the first period and the rest belong to the second one. Both samples contain approximately 2500 observations. The first part of the sample includes the periods of turbulence due to the dot-com bubble and GFC, the second is tranquil with a lead-up into the recent Covid-19 events. To perform the tests we first determine the number of transitions in the multiplicative time-varying (MTV) GJR-GARCH equations (it can be zero) using the sequential procedure described in Hall et al. (2021). The 26 estimated GARCH equations (or their g_{it} specification test results) are not reported here, but the plots of the multiplicative component (3) together with the daily returns appear in Figures 6 – 10.

The first-order test clearly rejects the null of constant correlations. Increasing the order of the polynomial in the test statistic up does not affect the conclusions. Although the dimension of the null hypothesis increases from 25 to 100, all tests strongly reject the hypothesis of stable correlations for both observation periods. The p-values of the test are practically zero. If this had been attempted for the HST-test, the corresponding degrees of freedom would have increased from 325 to 1300.

Even if the main purpose of this example is to demonstrate the use of our tests for a relatively large set of stocks, we also consider stability of the pairwise correlations. The magnitudes of the resulting p-values from the pairwise tests applied to the first part of the sample can be found in Tables A.2 and A.3 for the polynomial orders of one and two, respectively. Tables A.4 and A.5 contain the corresponding ones for the second part of the sample. In the former, the evidence of time-variation in the correlations is very clear. In the latter, there are more cases where the first-order test fails to reject constancy of correlations. The second-order test, however, does find evidence of time-varying correlations between most pairs of stocks. It appears that the change during the second period can often be nonmonotonic rather than monotonic.

It is clear that our test is the only alternative when the number of assets is large. When it is small so that both tests are available, we can make comparisons and see how much power may be lost when our parsimonious test is applied instead of the HST-test. To this end, groups of three to four stocks are subsequently examined. The results are consistent in most all cases. Two exceptions are discussed next.

The four stocks representing consumer staples (WMT, WBA), services (VZ), and energy (XOM) form the first example. Our test for 2001-2010 results in p-values of 0.0134 and 0.0000 for test orders one (three degrees of freedom, df) and two (six df), respectively.

In 2011–2020, the corresponding p-values are 0.1059 and 0.0001. For the HST-test, the p-values for 2001-2010 are 0.0000 for both polynomial orders (six and 12 df), and for 2011-2020 they are 0.0001 and 0.0000, respectively. The obvious conclusion is that the parsimonious test should mainly be used when the other test is no longer applicable.

The three information technology companies AAPL, IBM and INTC, form an example of the smallest collection of stocks such that our test differs from the HST-test. For 2001–2010, the p-values for the first and second-order versions of our tests with two and four df are 0.0103 and 0.0000. The p-values of the corresponding HST-test (three and six df) are 0.0413 and 0.0000. For the second part of the sample, the p-values of our parsimonious test are 0.3459 and 0.0001, compared to 0.0000 for both orders for the HST-test. Here we note the rather rare occasion (2001–2010, first-order test) in which our test is more powerful than its competitor. An explanation may be found in Table A.2. It is seen that only one of the p-values of the three pairwise tests lies below 0.1. Thus, in the HST-test only one pair weighs towards a rejection, whereas the evidence against the null is more spread out in the test based on the eigenvalues, and the test has one df less than the HST-test.

In this example, the alternative to constancy of correlations is that the correlations vary as a function of time. However, both the parsimonious and the HST test are conditioned on the choice of the transition variable which need not be deterministic. This means that they are equally useful for practitioners who may wish to examine correlation stability over some other indicator than time. The underlying theoretical foundations of the tests are unaffected by such considerations, and hence the integrity of the tests is not compromised.

8 Conclusions

In this paper we derive a test for testing constancy of the correlation matrix in the multivariate time-varying GARCH model. It bears some similarity to the test of constancy of the error covariance matrix in a multivariate model by Yang (2014). However, there are substantial differences between the two tests. In Yang's test, the model for covariances need not be a GARCH model, whereas our test is designed for a class of multivariate GARCH models. It is based on the decomposition of the error covariance matrix into variances and the correlation matrix as in Bollerslev (1990). The advantage of this decomposition is that one can test constancy of the conditional variances one by one as described in Amado and Teräsvirta (2017) or Hall et al. (2021), and estimate the time-varying variances before considering the constancy of correlations. This makes it possible to examine potential nonconstancy in the error correlation matrix such that time-variation in variances has already been taken care of. The simulation results emphasise the importance of correct specification of GARCH equations before constancy of the correlation matrix is under test.

The test is intended for use in situations in which the number of variables, typically asset returns, is large, and where for this reason the test by Hall et al. (2021) is either not available or suffers from numerical problems. Our simulations evidence the test is reasonably well-behaved as long as the conditional variances are correctly specified. The

Dow-Jones example illustrates the use of the test in the entire 26-dimensional system, as well as conducting 325 pairwise tests and tests on some selected subgroups. Pairwise tests, while not the main topic of this paper, would help locate those pairs whose correlations are constant. This in turn would help specifying and estimating the final model.

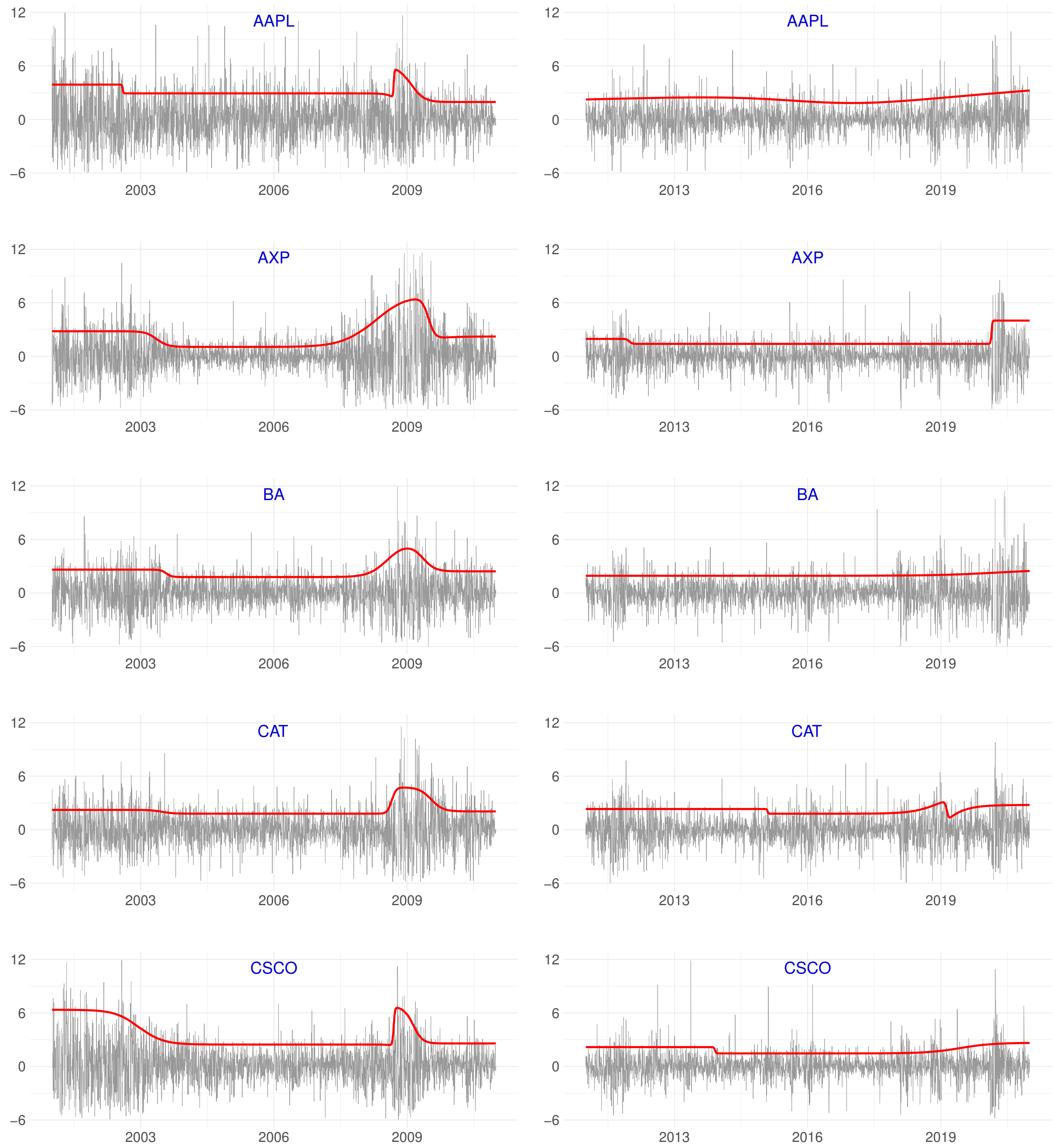

Figure 6: Daily returns of the Dow Jones stocks from Apple (AAPL) to Cisco (CSCO) (grey) and the corresponding deterministic component (red) from the MTV-GJR-GARCH equation for the period 2 January 2001 – 31 December 2010 (left column) and for 3 January 2011 – 31 December 2020 (right column)

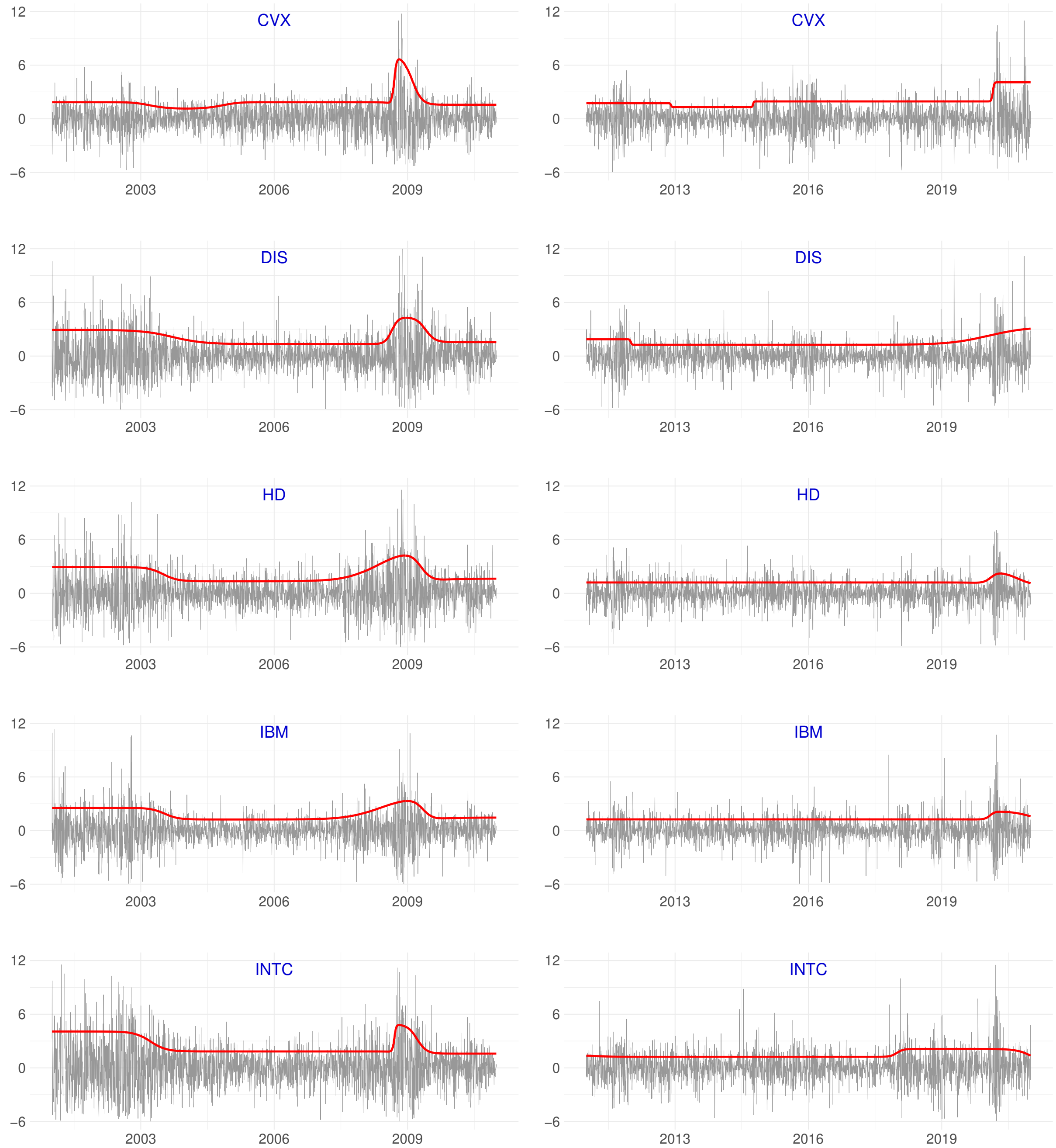

Figure 7: Daily returns of the Dow Jones stocks from Chevron (CVX) to Intel (INTC) (grey) and the corresponding deterministic component (red) from the MTV-GJR-GARCH equation for the period 2 January 2001 – 31 December 2010 (left column) and for 3 January 2011 – 31 December 2020 (right column)

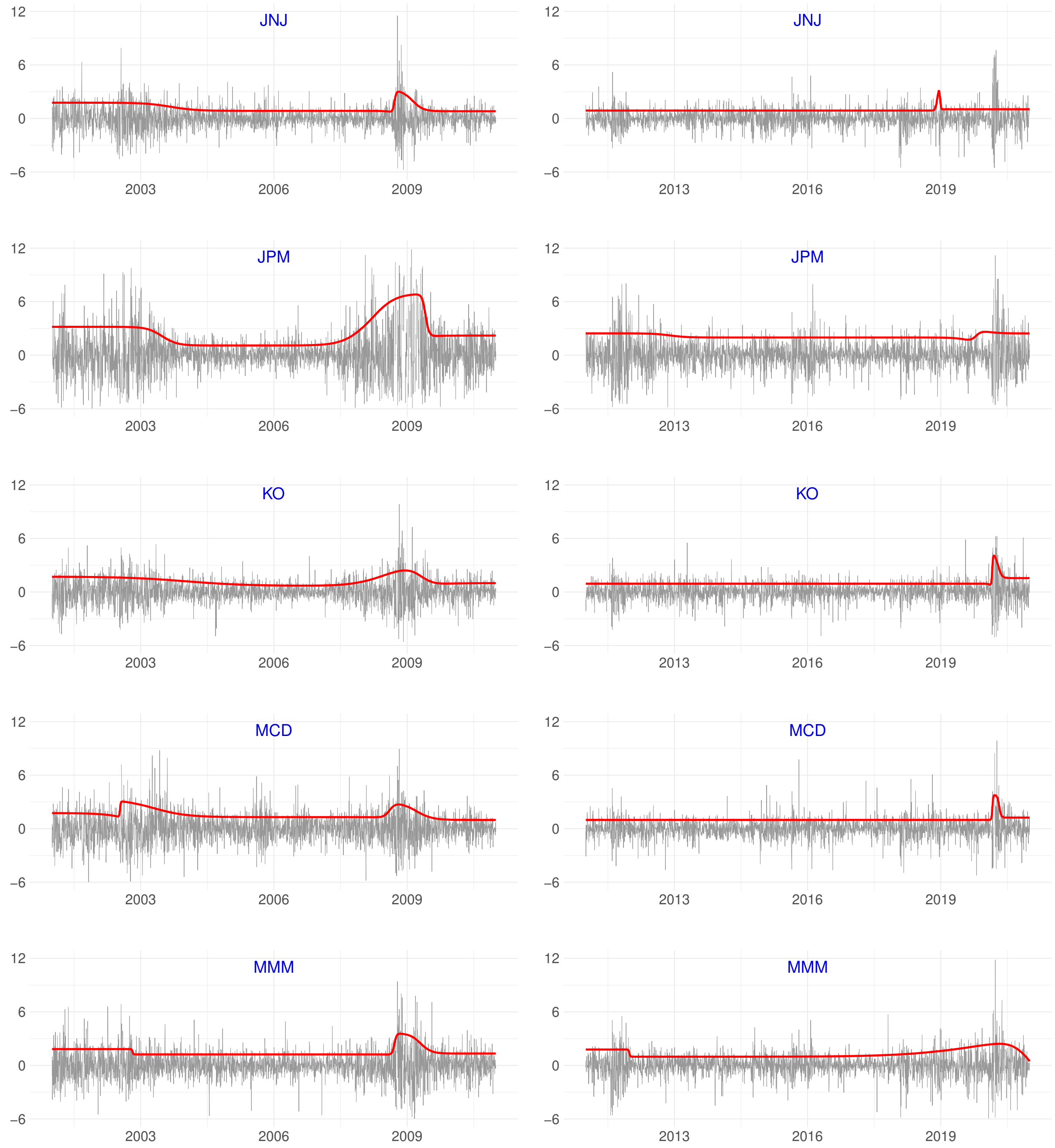

Figure 8: Daily returns of the Dow Jones stocks from Johnson and Johnson (JNJ) to 3M (MMM) (grey) and the corresponding deterministic component (red) from the MTV-GJR-GARCH equation for the period 2 January 2001 – 31 December 2010 (left column) and for 3 January 2011 – 31 December 2020 (right column)

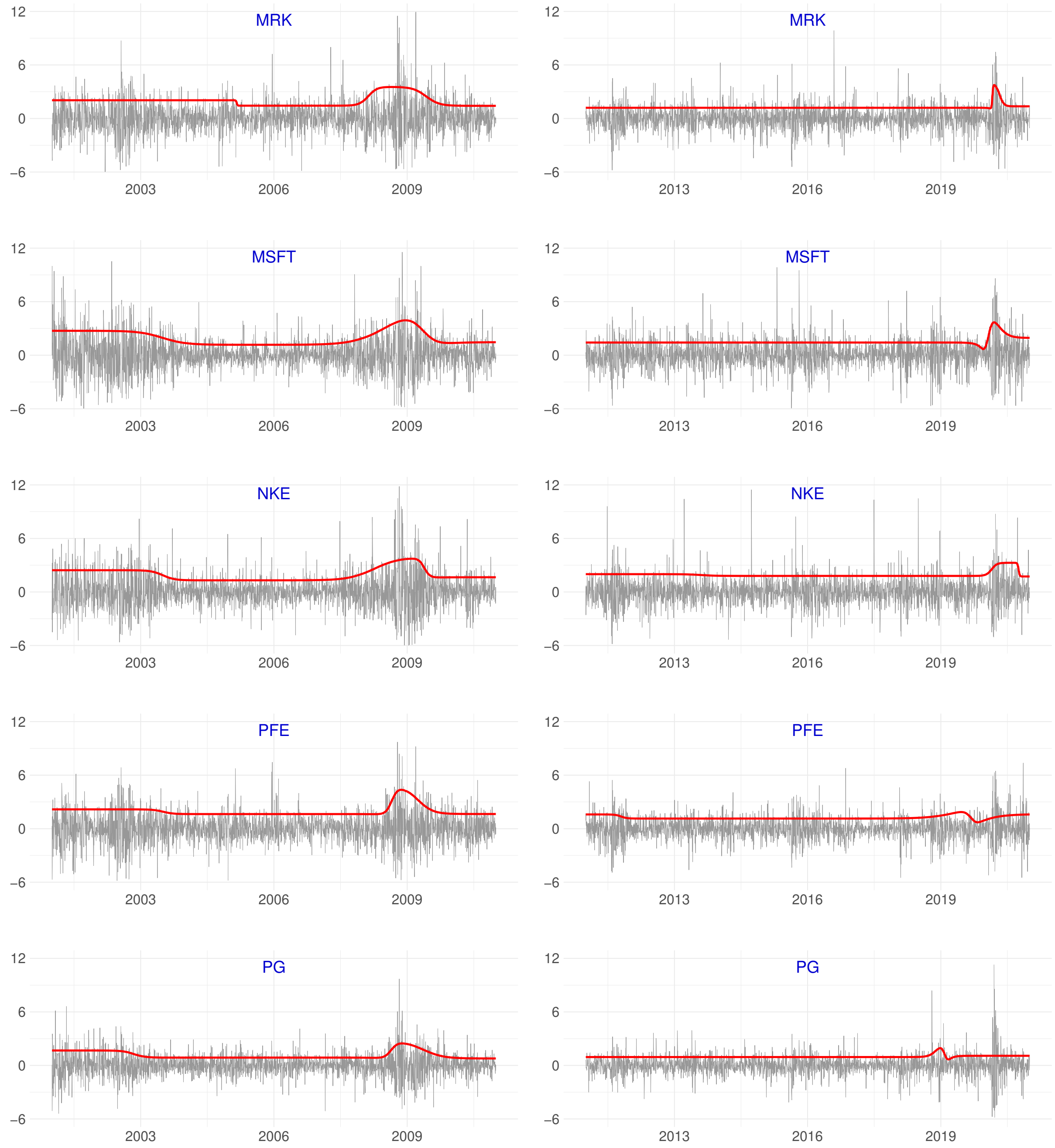

Figure 9: Daily returns of the Dow Jones stocks from Merck (MRK) to Procter & Gamble (PG) (grey) and the corresponding deterministic component (red) from the MTV-GJR-GARCH equation for the period 2 January 2001 – 31 December 2010 (left column) and for 3 January 2011 – 31 December 2020 (right column)

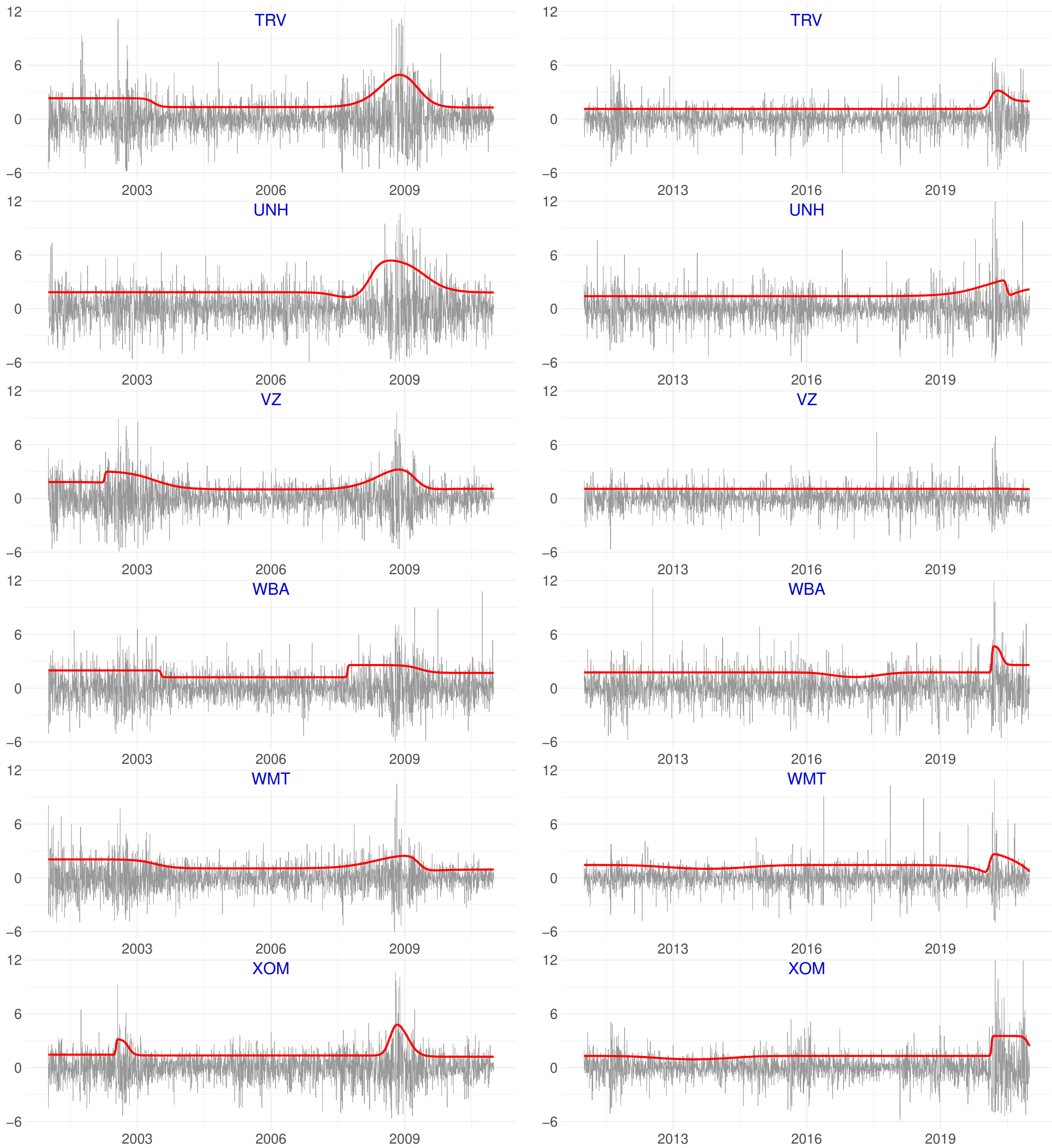

Figure 10: Daily returns of the Dow Jones stocks from Travelers Companies (TRV) to Exxon (XOM) (grey) and the corresponding deterministic component (red) from the MTV-GJR-GARCH equation for the period 2 January 2001 – 31 December 2010 (left column) and for 3 January 2011 – 31 December 2020 (right column)

Author Contributions

All authors contributed equally. All authors have read and agreed to the published version of the manuscript.

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Conflict of Interest

The authors declare no conflict of interest.

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Appendices

A Application

AAPL	Apple Inc.	Information technology
AXP	American Express Company	Financial services
BA	The Boeing Company	Aerospace and defense
CAT	Caterpillar Inc.	Construction and Mining
CSCO	Cisco Systems, Inc.	Information technology
CVX	Chevron Corporation	Petroleum industry
DIS	The Walt Disney Company	Broadcasting and entertainment
HD	The Home Depot, Inc.	Home Improvement
IBM	International Business Machines Corporation	Information technology
INTC	Intel Corporation	Semiconductor industry
JNJ	Johnson & Johnson	Pharmaceutical industry
JPM	JPMorgan Chase & Co.	Financial services
KO	The Coca-Cola Company	Soft Drink
MCD	McDonald's Corporation	Food industry
MMM	3M Company	Conglomerate
MRK	Merck & Co., Inc.	Pharmaceutical industry
MSFT	Microsoft Corporation	Information technology
NKE	Nike, Inc.	Apparel
PFE	Pfizer Inc.	Pharmaceutical industry
PG	The Procter & Gamble Company	Fast-moving consumer goods
TRV	The Travelers Companies, Inc.	Insurance
UNH	UnitedHealth Group Incorporated	Managed health care
VZ	Verizon Communications Inc.	Telecommunication
WBA	Walgreens Boots Alliance, Inc.	Retailing
WMT	Walmart Inc.	Retailing
XOM	Exxon Mobil Corporation	Energy

Table A.1: The 26 stocks that have been continuously part of Dow Jones Industrial Average from 2 January 2001 to 31 December 2020.

	AAPL	AXP	BA	CAT	CSCO	CVX	DIS	HD	IBM	INTC	JNJ	JPM	KO	MCD	MMM	MRK	MSFT	NKE	PFE	PG	TRV	UNH	VZ	WBA	WMT
AXP	**																								
BA	***	***																							
CAT	***	***	***																						
CSCO			***	***																					
CVX	***	***	***	***	***																				
DIS	*		***	***	***	***																			
HD	***	***	***	**	**	***	***																		
IBM	***	*	***	***		***	*	***																	
INTC			**	***	***	***		*																	
JNJ	***	**	***	***	***	***	***	***	***	***															
JPM	*	**	***			***	*	***			***														
KO	***	**	***	***	***	***	***	*	***	***	***	***													
MCD	***	***	***	***	**	***	***	***	***	***	***	***	***												
MMM	***	**	***	*	***	***	***	**	***	***	***	***	**	***											
MRK	***	***	***	***	***	***	***	***	***	***	***	***	***	***	***										
MSFT			***		***	***			**	***	***		***	***	***	***									
NKE	***	***	***	***	***	***	***	***	***	***	***	***	***	***	***	***	***								
PFE	***	***	***	***	***	***	***	***	***	***	**	***	***	***	***	***	***	***							
PG	***	***	***	***	***	***	***	***	***	***	***	***	**	***	**	***	***	***	**						
TRV	***	***	***	***	**	***	***	***	***	***	***	***	***	***	***	***	***	***	***	***					
UNH	***	**	***	**	***	***	***	***	***	***	***	***	***	***	**	***	***	***	***	***	**				
VZ	**		***	***	*	***		***	***	**	***		***	***	***	***		***	***	***	***	***			
WBA	***		***	**	***	***	***	*	***	**	***	**	***	***		**	**	***	*	***	***	*	**		
WMT		***				**		***			***	**	***	***	**	***	**		**	***	**			*	
XOM	***	***	***	***	***	***	***	**	***	***	***	***	***	***	***	***	***	***	***	***	***	***	***	**	

Table A.2: Pairwise tests of constant correlation. Time period 2 January 2001 – 31 December 2010. Order of the test = 1. The entries are the magnitudes of the p-values of the test: ∗∗∗($p < 0.01$), ∗∗($0.01 < p < 0.05$), and ∗ ($0.05 < p < 0.10$)

	AAPL	AXP	BA	CAT	CSCO	CVX	DIS	HD	IBM	INTC	JNJ	JPM	KO	MCD	MMM	MRK	MSFT	NKE	PFE	PG	TRV	UNH	VZ	WBA	WMT
AXP	***																								
BA	***	***																							
CAT	***	***	***																						
CSCO	***	***	***	***																					
CVX	***	***	***	***	***																				
DIS	***	***	***	***	***	***																			
HD	***	***	***	***	**	***	***																		
IBM	***	***	***	***	**	***	***	***																	
INTC	***	***	***	***	***	***	***	***	***																
JNJ	***	***	***	***	***	***	***	***	***	***															
JPM	***	***	***	***		***	***	***	***	***	***														
KO	**	**	***	***	***	***	***		***	**	***	***													
MCD	***	***	***	***	**	***	***	***	***	***	***	***	***												
MMM	***	***	***	***	***	***	***	***	***	***	***	***	*	***											
MRK	***	***	***	***	***	***	***	***	***	***	***	***	***	***	***										
MSFT	***	***	***	***	***	***	***	***	***	***	***	***	***	***	***	***									
NKE	***	***	***	***	***	***	***	***	***	***	***	***	***	***	***	***	***								
PFE	***	***	***	***	***	***	***	***	***	***	***	***	***	***	***	***	***	***							
PG	***	**	***	***	***	***	***	***	***	***	***	***	***	***	***	***	***	***	**						
TRV	***	***	***	***		***	***	***	***	***	***	***	***	***	***	***	***	***	***	***					
UNH	***	**	***	**	***	***	***	***	***	***	***	***	***	***	**	***	***	***	***	***	**				
VZ	***	**	***	***		***	**	***	***	**	***		***	***	***	***		***	***	***	***	***			
WBA	***	**	***	***	***	***	***		***	**	***	**	***	***	*	***	**	***	***	***	***		**		
WMT		***	**	*		***		***			***	***	**	***	**	***	***			***	*				
XOM	***	***	***	***	***	***	***	***	***	***	***	***	***	***	***	***	***	***	***	***	***	***	***	***	***

Table A.3: Pairwise tests of constant correlation. Time period 2 January 2001 – 31 December 2010. Order of the test = 2. The entries are the magnitudes of the p-values of the test: $***(p < 0.01)$, $**(0.01 < p < 0.05)$, and $*$ $(0.05 < p < 0.10)$

	AAPL	AXP	BA	CAT	CSCO	CVX	DIS	HD	IBM	INTC	JNJ	JPM	KO	MCD	MMM	MRK	MSFT	NKE	PFE	PG	TRV	UNH	VZ	WBA	WMT
AXP																									
BA		***																							
CAT			**																						
CSCO	***	***		*																					
CVX					**																				
DIS		**	***	***		***																			
HD	**		***		***		***																		
IBM		**	***		***		*																		
INTC	***		***	**	***		**																		
JNJ		***	***	***	***	***	***	***	***	**															
JPM		**					**	*			***														
KO		**	***	**		***	***	***	***	***	***	***													
MCD		**	***	***		***	***		***	***	***	***	***												
MMM		***	***			***	***	*		**	***	**	***	***											
MRK		**	***	***	*	***	***			***		***		***	***										
MSFT	***	*	**	**	***		*	***		**		**	**		***										
NKE	***		***		***						**		**	***			**								
PFE		***	***	***	*	***	***	***	*	***	***	***	***	***	***	*		**							
PG	*	***	***	**		***	***		*	*	***	***		**	***	**			***						
TRV			***	***		***	***	**		***	***		*		***				***	***					
UNH	**	**	*		***		*						*			*	***				*				
VZ		***	***	***		**	***	**		***	**	***		*	***		***	**	**		**				
WBA		*			***	*						**	*	***					**	***		***			
WMT		***	***	**		**	***		*		***	***	***	***		***			***	**	***		***		
XOM			**		***		***	**			***		***	***	***	***	**		***	***	***		***	**	**

Table A.4: Pairwise tests of constant correlation. Time period 3 January 2011 – 31 December 2020. Order of the test = 1. The entries are the magnitudes of the p-values of the test: $***(p < 0.01)$, $**(0.01 < p < 0.05)$, and $*$ $(0.05 < p < 0.10)$

	AAPL	AXP	BA	CAT	CSCO	CVX	DIS	HD	IBM	INTC	JNJ	JPM	KO	MCD	MMM	MRK	MSFT	NKE	PFE	PG	TRV	UNH	VZ	WBA	WMT
AXP	***																								
BA																									
CAT	***	***	***																						
CSCO	***	***		***																					
CVX	***	***	***	***	***																				
DIS	***	***	***	***	**	***																			
HD	***	***	***	***	***	***	***																		
IBM	***	***	***	***	***	***	***	*																	
INTC	***	***	***	***	***	**	**		***																
JNJ	***	***	***	***	***	***	***	***	***	**															
JPM		***	***	***		***	***	*	**		***														
KO	**	***	***	***		***	***	***	***	***	***	***													
MCD	*	***	***	***		***	***	**	***	***	***	**	***												
MMM		***	***	***	**	***	***	**	***	**	***	***	***	***											
MRK	**	***	***	***	*	***	***	**	***	**	***	***	***	***	***										
MSFT	***	***	***	***	***	***	***	***	***	**	***	*	***	*	***	***									
NKE	***	***	***	***	***	***			***	***	***	*	***	***	***	**	***								
PFE		***	***	***		***	***	***	***	**	***	***	***	***	***		*	*							
PG	**	***	***	***		***	***	***	**		***	***	**	*	***	***			***						
TRV		***	***	***		***	***	*	**	**	***		*		***	***		**	***	***					
UNH	***	***		***	***	***	**	**	***	*	**		***			*	***	*							
VZ	**	***	***	***	*	***	***	**	***	***	***	***	*	*	***	**	**	**	***	*	**	***			
WBA		***	***	***	***	***			***			***	**	***	***				*	**		***			
WMT	**	***	***	***		***	***	***	*	**	***	***	***	***	***	***	***		***	***	***		***		
XOM	***	***	***	***	***	***	***	***	***	*	***	***	***	***	***	***	***	***	***	***	***	***	***	***	***

Table A.5: Pairwise tests of constant correlation. Time period 3 January 2011 – 31 December 2020. Order of the test = 2. The entries are the magnitudes of the p-values of the test: $***(p < 0.01)$, $**(0.01 < p < 0.05)$, and $*$ $(0.05 < p < 0.10)$

B Proofs

Proof of Theorem 1. First, recall from (2) that $z_{it} = \varepsilon_{it}/(h_{it}^{1/2} g_{it}^{1/2})$. Then

$$\begin{aligned}\frac{\partial z_{it}}{\partial \theta_{gi}} &= \frac{\partial}{\partial \theta_{gi}}(\frac{\varepsilon_{it}}{h_{it}^{1/2} g_{it}^{1/2}}) \\ &= -\frac{1}{2}\frac{\varepsilon_{it}}{h_{it}^{1/2} g_{it}^{3/2}}\frac{\partial g_{it}}{\partial \theta_{gi}} - \frac{1}{2}\frac{\varepsilon_{it}}{h_{it}^{3/2} g_{it}^{1/2}}\frac{\partial h_{it}}{\partial \theta_{gi}} \\ &= -\frac{1}{2}\frac{\varepsilon_{it}}{h_{it}^{1/2} g_{it}^{1/2}}(\frac{1}{g_{it}}\frac{\partial g_{it}}{\partial \theta_{gi}} + \frac{1}{h_{it}}\frac{\partial h_{it}}{\partial \theta_{gi}}) \\ &= -\frac{z_{it}}{2}(\frac{1}{g_{it}}\frac{\partial g_{it}}{\partial \theta_{gi}} + \frac{1}{h_{it}}\frac{\partial h_{it}}{\partial \theta_{gi}}). \end{aligned} \tag{17}$$

Blocks of the score for the observation t are as follows. First,

$$\begin{aligned}\frac{\partial \ell_t}{\partial \boldsymbol{\theta}_{gi}} &= -(\frac{1}{2g_{it}}\frac{\partial g_{it}}{\partial \boldsymbol{\theta}_{gi}} + \frac{1}{2h_{it}}\frac{\partial h_{it}}{\partial \boldsymbol{\theta}_{gi}}) - \frac{1}{2}\sum_{j=1}^{N} 2w_{jt}\frac{\partial w_{jt}}{\partial \boldsymbol{\theta}_{gi}}\psi_j^{-1}(t/T) \\ &= -\frac{1}{2}(\frac{1}{g_{it}}\frac{\partial g_{it}}{\partial \boldsymbol{\theta}_{gi}} + \frac{1}{h_{it}}\frac{\partial h_{it}}{\partial \boldsymbol{\theta}_{gi}}) - \sum_{j=1}^{N} w_{jt}\boldsymbol{q}_j'\frac{\partial \boldsymbol{z}_t}{\partial \boldsymbol{\theta}_{gi}}\psi_j^{-1}(t/T)\end{aligned}$$

and, using (17),

$$\begin{aligned}\frac{\partial \ell_t}{\partial \boldsymbol{\theta}_{gi}} =& -\frac{1}{2}(\frac{1}{g_{it}}\frac{\partial g_{it}}{\partial \boldsymbol{\theta}_{gi}} + \frac{1}{h_{it}}\frac{\partial h_{it}}{\partial \boldsymbol{\theta}_{gi}}) + \frac{1}{2}\sum_{j=1}^{N} w_{jt}\boldsymbol{q}_j'(0,\ldots,0,z_{it},0,\ldots,0)' \\ &\times (\frac{1}{h_{it}}\frac{\partial h_{it}}{\partial \theta_{gi}} + \frac{1}{g_{it}}\frac{\partial g_{it}}{\partial \boldsymbol{\theta}_{gi}})\psi_j^{-1}(t/T) \\ =& \frac{1}{2}(\frac{1}{g_{it}}\frac{\partial g_{it}}{\partial \boldsymbol{\theta}_{gi}} + \frac{1}{h_{it}}\frac{\partial h_{it}}{\partial \boldsymbol{\theta}_{gi}})(\sum_{j=1}^{N} z_{it}w_{jt}\boldsymbol{q}_j'\boldsymbol{e}_i\psi_j^{-1}(t/T) - 1) \\ =& \frac{1}{2}(\frac{1}{g_{it}}\frac{\partial g_{it}}{\partial \boldsymbol{\theta}_{gi}} + \frac{1}{h_{it}}\frac{\partial h_{it}}{\partial \boldsymbol{\theta}_{gi}})(\boldsymbol{e}_i'\boldsymbol{z}_t\boldsymbol{z}_t'\sum_{j=1}^{N} \boldsymbol{q}_j\psi_j^{-1}(t/T)\boldsymbol{q}_j'\boldsymbol{e}_i - 1) \\ =& \frac{1}{2}(\frac{1}{g_{it}}\frac{\partial g_{it}}{\partial \boldsymbol{\theta}_{gi}} + \frac{1}{h_{it}}\frac{\partial h_{it}}{\partial \boldsymbol{\theta}_{gi}})(\boldsymbol{e}_i'\boldsymbol{z}_t\boldsymbol{z}_t'(\boldsymbol{P}_t^A)^{-1}\boldsymbol{e}_i - 1),\end{aligned} \tag{18}$$

where $\boldsymbol{P}_t^A = \boldsymbol{Q}\boldsymbol{\Psi}_t\boldsymbol{Q}'$. In a similar fashion,

$$\begin{aligned}\frac{\partial \ell_t}{\partial \boldsymbol{\theta}_{hi}} &= -\frac{1}{2h_{it}}\frac{\partial h_{it}}{\partial \boldsymbol{\theta}_{hi}} - \frac{1}{2}\sum_{j=1}^{N} 2w_{jt}\frac{\partial w_{jt}}{\partial \boldsymbol{\theta}_{gi}}\psi_j^{-1}(t/T) \\ &= \frac{1}{2}\frac{1}{h_{it}}\frac{\partial h_{it}}{\partial \boldsymbol{\theta}_{hi}}(\boldsymbol{e}_i'\boldsymbol{z}_t\boldsymbol{z}_t'(\boldsymbol{P}_t^A)^{-1}\boldsymbol{e}_i - 1).\end{aligned}$$

Next,

$$\frac{\partial \ell_t}{\partial \overline{\boldsymbol{\psi}}_j} = -\frac{1}{2}\frac{\partial}{\partial \overline{\boldsymbol{\psi}}_j}[\sum_{k=1}^{N-1}\ln \psi_k(t/T) + \ln(N - \sum_{k=1}^{N-1}\psi_k(t/T)) + \sum_{k=1}^{N-1} w_{kt}^2 \psi_k^{-1}(t/T)$$
$$+ w_{Nt}^2(N - \sum_{k=1}^{N-1}\psi_k(t/T))^{-1}]$$
$$= \frac{1}{2}\{\frac{1}{\psi_j(t/T)}(\frac{w_{jt}^2}{\psi_j(t/T)} - 1) - \frac{1}{N - \sum_{k=1}^{N-1}\psi_k(t/T)}(\frac{w_{Nt}^2}{N - \sum_{k=1}^{N-1}\psi_k(t/T)} - 1)\boldsymbol{\tau}_t$$

for $j = 1, \ldots, N-1$. Under H_0': $\boldsymbol{\psi}_1 = \boldsymbol{\psi}_2 = \mathbf{0}$, and so the corresponding blocks are

$$\frac{\partial \ell_t}{\partial \boldsymbol{\theta}_{gi}}|_{\mathrm{H}_0} = \frac{1}{2}(\frac{1}{g_{it}}\frac{\partial g_{it}}{\partial \boldsymbol{\theta}_{gi}} + \frac{1}{h_{it}}\frac{\partial h_{it}}{\partial \boldsymbol{\theta}_{gi}})(\boldsymbol{e}_i'\boldsymbol{z}_t\boldsymbol{z}_t'(\boldsymbol{P}^0)^{-1}\boldsymbol{e}_i - 1),$$

where $\boldsymbol{P}^0 = \boldsymbol{Q}\boldsymbol{\Psi}_0\boldsymbol{Q}'$,

$$\frac{\partial \ell_t}{\partial \boldsymbol{\theta}_{hi}}|_{\mathrm{H}_0} = \frac{1}{2}\frac{1}{h_{it}}\frac{\partial h_{it}}{\partial \boldsymbol{\theta}_{hi}}(\boldsymbol{e}_i'\boldsymbol{z}_t\boldsymbol{z}_t'(\boldsymbol{P}^0)^{-1}\boldsymbol{e}_i - 1)$$

and

$$\frac{\partial \ell_t(\boldsymbol{\theta})}{\partial \overline{\boldsymbol{\psi}}_j}|_{\mathrm{H}_0} = \frac{1}{2}\{\frac{1}{\psi_{0j}}(\frac{w_{jt}^2}{\psi_{0j}} - 1) - \frac{1}{N - \sum_{k=1}^{N-1}\psi_{0k}}(\frac{w_{Nt}^2}{N - \sum_{k=1}^{N-1}\psi_{0k}} - 1)\}\boldsymbol{\tau}_t.$$

This completes the proof. ■

In order to prove Theorem 2 we formulate and prove five lemmas.

Lemma 3. *For $i \neq j$,*

$$\mathsf{E}\frac{\partial \ell_t}{\partial \boldsymbol{\theta}_{gi}}\frac{\partial \ell_t}{\partial \boldsymbol{\theta}_{gj}'}|_{H_0} = \frac{1}{4}(\frac{1}{g_{it}^0}\frac{\partial g_{it}^0}{\partial \boldsymbol{\theta}_{gi}} + \frac{1}{h_{it}^0}\frac{\partial h_{it}^0}{\partial \boldsymbol{\theta}_{gi}})(\frac{1}{g_{jt}^0}\frac{\partial g_{jt}^0}{\partial \boldsymbol{\theta}_{gj}'} + \frac{1}{h_{jt}^0}\frac{\partial h_{jt}^0}{\partial \boldsymbol{\theta}_{gj}'})\boldsymbol{e}_i'(\boldsymbol{P}^0)^{-1}\boldsymbol{e}_j\boldsymbol{e}_i'\boldsymbol{P}^0\boldsymbol{e}_j.$$

When $i = j$, $i = 1, \ldots, N$,

$$\mathsf{E}\frac{\partial \ell_t}{\partial \boldsymbol{\theta}_{gi}}\frac{\partial \ell_t}{\partial \boldsymbol{\theta}_{gi}'}|_{H_0} = \frac{1}{4}(\frac{1}{g_{it}^0}\frac{\partial g_{it}^0}{\partial \boldsymbol{\theta}_{gi}} + \frac{1}{h_{it}^0}\frac{\partial h_{it}^0}{\partial \boldsymbol{\theta}_{gi}})(\frac{1}{g_{it}^0}\frac{\partial g_{it}^0}{\partial \boldsymbol{\theta}_{gi}'} + \frac{1}{h_{it}^0}\frac{\partial h_{it}^0}{\partial \boldsymbol{\theta}_{gi}'})\{\boldsymbol{e}_i'(\boldsymbol{P}^0)^{-1}\boldsymbol{e}_i\boldsymbol{e}_i'\boldsymbol{P}^0\boldsymbol{e}_i + 1\}.$$

Proof. From (18) it follows that we have to consider

$$\mu_1 = \mathsf{E}(\boldsymbol{e}_i'\boldsymbol{z}_t\boldsymbol{z}_t'(\boldsymbol{P}^0)^{-1}\boldsymbol{e}_i - 1)(\boldsymbol{e}_j'\boldsymbol{z}_t\boldsymbol{z}_t'(\boldsymbol{P}^0)^{-1}\boldsymbol{e}_j - 1). \tag{19}$$

Write

$$\boldsymbol{e}_i'\boldsymbol{z}_t\boldsymbol{z}_t'(\boldsymbol{P}^0)^{-1}\boldsymbol{e}_i = (\boldsymbol{e}_i'(\boldsymbol{P}^0)^{-1} \otimes \boldsymbol{e}_i')\mathrm{vec}(\boldsymbol{z}_t\boldsymbol{z}_t'),$$

so (19) becomes

$$\begin{aligned}\mu_1 =& ((\boldsymbol{e}_i'\boldsymbol{P}^0)^{-1} \otimes \boldsymbol{e}_i')\mathsf{E}\mathrm{vec}(\boldsymbol{z}_t\boldsymbol{z}_t')\mathrm{vec}(\boldsymbol{z}_t\boldsymbol{z}_t')'((\boldsymbol{P}^0)^{-1}\boldsymbol{e}_j \otimes \boldsymbol{e}_j)\\ &- \boldsymbol{e}_i'\mathsf{E}\boldsymbol{z}_t\boldsymbol{z}_t'(\boldsymbol{P}^0)^{-1}\boldsymbol{e}_i - \boldsymbol{e}_j'\mathsf{E}\boldsymbol{z}_t\boldsymbol{z}_t'(\boldsymbol{P}^0)^{-1}\boldsymbol{e}_j + 1.\end{aligned} \tag{20}$$

Consider the first term on the right-hand size of (20). From Anderson (2003, p. 64) one obtains

$$\mathsf{E}\text{vec}(\boldsymbol{z}_t\boldsymbol{z}_t')\text{vec}(\boldsymbol{z}_t\boldsymbol{z}_t')' = (\boldsymbol{P}^0 \otimes \boldsymbol{P}^0) + (\boldsymbol{I}_N \otimes \boldsymbol{P}^0)\boldsymbol{K}(\boldsymbol{I}_N \otimes \boldsymbol{P}^0) + \text{vec}(\boldsymbol{P}^0)\text{vec}(\boldsymbol{P}^0)', \quad (21)$$

where $\boldsymbol{K}$ is an $N^2 \times N^2$ commutation matrix, see Magnus and Neudecker (1979). Applying (21) to the right-hand size of (20) yields $\mu_1 = \mu_{11} + \mu_{12} + \mu_{13}$, where

$$\begin{aligned}\mu_{11} &= (\boldsymbol{e}_i'(\boldsymbol{P}^0)^{-1} \otimes \boldsymbol{e}_i')(\boldsymbol{P}^0 \otimes \boldsymbol{P}^0)((\boldsymbol{P}^0)^{-1}\boldsymbol{e}_j \otimes \boldsymbol{e}_j) \\ &= (\boldsymbol{e}_i'(\boldsymbol{P}^0)^{-1} \otimes \boldsymbol{e}_i')(\boldsymbol{e}_j \otimes \boldsymbol{P}^0\boldsymbol{e}_j) = \boldsymbol{e}_i'(\boldsymbol{P}^0)^{-1}\boldsymbol{e}_j\boldsymbol{e}_i'\boldsymbol{P}^0\boldsymbol{e}_j, \end{aligned} \quad (22)$$

$$\begin{aligned}\mu_{12} &= (\boldsymbol{e}_i'(\boldsymbol{P}^0)^{-1} \otimes \boldsymbol{e}_i')(\boldsymbol{I}_N \otimes \boldsymbol{P}^0)\boldsymbol{K}(\boldsymbol{I}_N \otimes \boldsymbol{P}^0)((\boldsymbol{P}^0)^{-1}\boldsymbol{e}_j \otimes \boldsymbol{e}_j) \\ &= (\boldsymbol{e}_i'(\boldsymbol{P}^0)^{-1} \otimes \boldsymbol{e}_i'\boldsymbol{P}^0)\boldsymbol{K}((\boldsymbol{P}^0)^{-1}\boldsymbol{e}_j \otimes \boldsymbol{P}^0\boldsymbol{e}_j) \\ &= (\boldsymbol{e}_i'(\boldsymbol{P}^0)^{-1} \otimes \boldsymbol{e}_i'\boldsymbol{P}^0)(\boldsymbol{P}^0\boldsymbol{e}_j \otimes (\boldsymbol{P}^0)^{-1}\boldsymbol{e}_j) = 0 \end{aligned} \quad (23)$$

for $i \neq j$, and 1 for $i = j$, $i = 1, \ldots, N$, and

$$\mu_{13} = (\boldsymbol{e}_i'\boldsymbol{P}^0)^{-1} \otimes \boldsymbol{e}_i')\text{vec}(\boldsymbol{P}^0)\text{vec}(\boldsymbol{P}^0)'((\boldsymbol{P}^0)^{-1}\boldsymbol{e}_i \otimes \boldsymbol{e}_i) = 1. \quad (24)$$

Furthermore,

$$\boldsymbol{e}_m'\mathsf{E}\boldsymbol{z}_t\boldsymbol{z}_t'(\boldsymbol{P}^0)^{-1}\boldsymbol{e}_m = \boldsymbol{e}_m'\boldsymbol{e}_m = 1$$

for $m = i, j$, so, in total

$$\mathsf{E}(\boldsymbol{e}_i'\boldsymbol{z}_t\boldsymbol{z}_t'(\boldsymbol{P}^0)^{-1}\boldsymbol{e}_i - 1)(\boldsymbol{e}_j'\boldsymbol{z}_t\boldsymbol{z}_t'(\boldsymbol{P}^0)^{-1}\boldsymbol{e}_j - 1) = \boldsymbol{e}_i'(\boldsymbol{P}^0)^{-1}\boldsymbol{e}_j\boldsymbol{e}_i'\boldsymbol{P}^0\boldsymbol{e}_j$$

for $i \neq j$. When $i = j$, consider

$$\mu_1 = \mathsf{E}(\boldsymbol{e}_i'\boldsymbol{z}_t\boldsymbol{z}_t'(\boldsymbol{P}^0)^{-1}\boldsymbol{e}_i - 1)^2 = 1 - 2\mathsf{E}\boldsymbol{e}_i'\boldsymbol{z}_t\boldsymbol{z}_t'\boldsymbol{P}^0\boldsymbol{e}_i + \mathsf{E}(\boldsymbol{e}_i'\boldsymbol{z}_t\boldsymbol{z}_t'(\boldsymbol{P}^0)^{-1}\boldsymbol{e}_i)^2.$$

Then the three terms in $\mathsf{E}(\boldsymbol{e}_i'\boldsymbol{z}_t\boldsymbol{z}_t'(\boldsymbol{P}^0)^{-1}\boldsymbol{e}_i)^2$ corresponding to (22), (23) and (24) become $\boldsymbol{e}_i'(\boldsymbol{P}^0)^{-1}\boldsymbol{e}_i\boldsymbol{e}_i'\boldsymbol{P}^0\boldsymbol{e}_i$, 1 and 1, respectively, and the result follows. ■

Lemma 4. *For $i \neq j$,*

$$\mathsf{E}\frac{\partial \ell_t}{\partial \boldsymbol{\theta}_{gi}}\frac{\partial \ell_t}{\partial \boldsymbol{\theta}_{hj}'}|_{H_0} = \frac{1}{4}(\frac{1}{g_{it}^0}\frac{\partial g_{it}^0}{\partial \boldsymbol{\theta}_{gi}} + \frac{1}{h_{it}^0}\frac{\partial h_{it}^0}{\partial \boldsymbol{\theta}_{gi}})\frac{1}{h_{jt}^0}\frac{\partial h_{jt}^0}{\partial \boldsymbol{\theta}_{hj}'}\boldsymbol{e}_i'(\boldsymbol{P}^0)^{-1}\boldsymbol{e}_j\boldsymbol{e}_i'\boldsymbol{P}^0\boldsymbol{e}_j.$$

When $i = j$,

$$\mathsf{E}\frac{\partial \ell_t}{\partial \boldsymbol{\theta}_{gi}}\frac{\partial \ell_t}{\partial \boldsymbol{\theta}_{hi}'}|_{H_0} = \frac{1}{4}(\frac{1}{g_{it}^0}\frac{\partial g_{it}^0}{\partial \boldsymbol{\theta}_{gi}} + \frac{1}{h_{it}^0}\frac{\partial h_{it}^0}{\partial \boldsymbol{\theta}_{gi}})\frac{1}{h_{it}^0}\frac{\partial h_{it}^0}{\partial \boldsymbol{\theta}_{hi}'}\{\boldsymbol{e}_i'(\boldsymbol{P}^0)^{-1}\boldsymbol{e}_i + 1\}.$$

Proof. Similar to the proof of Lemma 3 and therefore omitted. ■

Lemma 5. *For $i \neq j$,*

$$\mathsf{E}\frac{\partial \ell_t}{\partial \boldsymbol{\theta}_{hi}}\frac{\partial \ell_t}{\partial \boldsymbol{\theta}_{hj}'}|_{H_0} = \frac{1}{4}\frac{1}{h_{it}^0 h_{jt}^0}\frac{\partial h_{it}^0}{\partial \boldsymbol{\theta}_{hi}}\frac{\partial h_{jt}^0}{\partial \boldsymbol{\theta}_{hj}'}\boldsymbol{e}_i'(\boldsymbol{P}^0)^{-1}\boldsymbol{e}_j\boldsymbol{e}_i'(\boldsymbol{P}^0)^{-1}\boldsymbol{e}_j.$$

When $i = j$,

$$\mathsf{E}\frac{\partial \ell_t}{\partial \boldsymbol{\theta}_{hi}}\frac{\partial \ell_t}{\partial \boldsymbol{\theta}_{hi}'}|_{H_0} = \frac{1}{4}\frac{1}{(h_{it}^0)^2}\frac{\partial h_{it}^0}{\partial \boldsymbol{\theta}_{hi}}\frac{\partial h_{it}^0}{\partial \boldsymbol{\theta}_{hi}'}\{\boldsymbol{e}_i'(\boldsymbol{P}^0)^{-1}\boldsymbol{e}_i + 1\}.$$

Proof. Similar to the proof of Lemma 3 and therefore omitted. ■

Lemma 6. *The expectation* $\mathsf{E}\frac{\partial \ell_t}{\partial \boldsymbol{\theta}_{gi}}\frac{\partial \ell_t}{\partial \overline{\boldsymbol{\psi}}_j'}|_{H_0}$ *equals*

$$\mathsf{E}\frac{\partial \ell_t}{\partial \boldsymbol{\theta}_{gi}}\frac{\partial \ell_t}{\partial \overline{\boldsymbol{\psi}}_j'}|_{H_0} = \frac{1}{2}(\frac{1}{g_{it}^0}\frac{\partial g_{it}^0}{\partial \boldsymbol{\theta}_{gi}} + \frac{1}{h_{it}^0}\frac{\partial h_{it}^0}{\partial \boldsymbol{\theta}_{gi}})\boldsymbol{\tau}_t' \times \{\psi_{0j}^{-1}\boldsymbol{e}_i'\boldsymbol{q}_j\boldsymbol{e}_j'\boldsymbol{q}_i + (N - \sum_{k=1}^{N-1}\psi_{0k})^{-1}\boldsymbol{e}_i'\boldsymbol{q}_N\boldsymbol{e}_N'\boldsymbol{q}_i\} \quad (25)$$

and, similarly,

$$\mathsf{E}\frac{\partial \ell_t}{\partial \boldsymbol{\theta}_{hi}}\frac{\partial \ell_t}{\partial \overline{\boldsymbol{\psi}}_j'}|_{H_0} = \frac{1}{2}\frac{1}{h_{it}}\frac{\partial h_{it}^0}{\partial \boldsymbol{\theta}_{hi}}\boldsymbol{\tau}_t'\{\psi_{0j}^{-1}\boldsymbol{e}_i'\boldsymbol{q}_j\boldsymbol{e}_j'\boldsymbol{q}_i + (N - \sum_{k=1}^{N-1}\psi_{0k})^{-1}\boldsymbol{e}_i'\boldsymbol{q}_N\boldsymbol{e}_N'\boldsymbol{q}_i\}, \quad (26)$$

$j = 1, \ldots, N-1.$

Proof. In order to prove (25), consider the expectation

$$\begin{aligned}
\mu_2 =& \mathsf{E}(\boldsymbol{e}_i'\boldsymbol{z}_t\boldsymbol{z}_t'(\boldsymbol{P}^0)^{-1}\boldsymbol{e}_i - 1)\{(\boldsymbol{q}_j \otimes \boldsymbol{q}_j)'\mathrm{vec}(\boldsymbol{z}_t\boldsymbol{z}_t')\frac{1}{\psi_{0j}} - 1\} \\
=& (\boldsymbol{e}_i'(\boldsymbol{P}^0)^{-1} \otimes \boldsymbol{e}_i')\mathsf{E}\mathrm{vec}(\boldsymbol{z}_t\boldsymbol{z}_t')\mathrm{vec}(\boldsymbol{z}_t\boldsymbol{z}_t')'(\boldsymbol{q}_j \otimes \boldsymbol{q}_j)\frac{1}{\psi_{0j}} \\
&- \boldsymbol{e}_i'\mathsf{E}\boldsymbol{z}_t\boldsymbol{z}_t'(\boldsymbol{P}^0)^{-1}\boldsymbol{e}_i - (\boldsymbol{q}_j \otimes \boldsymbol{q}_j)'\mathrm{vec}(\mathsf{E}\boldsymbol{z}_t\boldsymbol{z}_t')\frac{1}{\psi_{0j}} + 1 \\
=& (\boldsymbol{e}_i'(\boldsymbol{P}^0)^{-1} \otimes \boldsymbol{e}_i')\mathsf{E}\mathrm{vec}(\boldsymbol{z}_t\boldsymbol{z}_t')\mathrm{vec}(\boldsymbol{z}_t\boldsymbol{z}_t')'(\boldsymbol{q}_j \otimes \boldsymbol{q}_j)\frac{1}{\psi_{0j}} - 1. \quad (27)
\end{aligned}$$

Inserting (21) to (27) yields $\mu_2 = \mu_{21} + \mu_{22} + \mu_{23} - 1$, where

$$\begin{aligned}
\mu_{21} &= (\boldsymbol{e}_i'(\boldsymbol{P}^0)^{-1} \otimes \boldsymbol{e}_i')(\boldsymbol{P}^0 \otimes \boldsymbol{P}^0)(\boldsymbol{q}_j \otimes \boldsymbol{q}_j)\frac{1}{\psi_{0j}} \\
&= (\boldsymbol{e}_i' \otimes \boldsymbol{e}_i'\boldsymbol{P}^0)(\boldsymbol{q}_j \otimes \boldsymbol{q}_j)\frac{1}{\psi_{0j}} \\
&= \boldsymbol{e}_i'\boldsymbol{q}_j\boldsymbol{e}_i'\boldsymbol{P}^0\boldsymbol{q}_j\frac{1}{\psi_{0j}} = \boldsymbol{e}_i'\boldsymbol{q}_j\boldsymbol{q}_i'\boldsymbol{e}_j,
\end{aligned}$$

$$\begin{aligned}
\mu_{22} &= (\boldsymbol{e}_i'(\boldsymbol{P}^0)^{-1} \otimes \boldsymbol{e}_i')(\boldsymbol{I}_N \otimes \boldsymbol{P}^0)\boldsymbol{K}(\boldsymbol{I}_N \otimes \boldsymbol{P}^0)(\boldsymbol{q}_j \otimes \boldsymbol{q}_j)\frac{1}{\psi_{0j}} \\
&= (\boldsymbol{e}_i'(\boldsymbol{P}^0)^{-1} \otimes \boldsymbol{e}_i'\boldsymbol{P}^0)\boldsymbol{K}(\boldsymbol{q}_j \otimes \boldsymbol{P}^0\boldsymbol{q}_j)\frac{1}{\psi_{0j}} \\
&= (\boldsymbol{e}_i'(\boldsymbol{P}^0)^{-1} \otimes \boldsymbol{e}_i'\boldsymbol{P}^0)(\boldsymbol{P}^0\boldsymbol{q}_j \otimes \boldsymbol{q}_j)\frac{1}{\psi_{0j}} = \boldsymbol{e}_i'\boldsymbol{q}_j\boldsymbol{q}_i'\boldsymbol{e}_j
\end{aligned}$$

and

$$\mu_{23} = (\boldsymbol{e}_i'(\boldsymbol{P}^0)^{-1} \otimes \boldsymbol{e}_i')\text{vec}(\boldsymbol{P}^0)\text{vec}(\boldsymbol{P}^0)'(\boldsymbol{q}_j \otimes \boldsymbol{q}_j)\frac{1}{\psi_{0j}}$$
$$= (\boldsymbol{e}_i'\boldsymbol{e}_i)(\boldsymbol{q}_j'\boldsymbol{P}^0\boldsymbol{q}_j)\frac{1}{\psi_{0j}} = 1.$$

In total,

$$(\boldsymbol{e}_i'(\boldsymbol{P}^0)^{-1} \otimes \boldsymbol{e}_i')\mathsf{E}\text{vec}(\boldsymbol{z}_t\boldsymbol{z}_t')\text{vec}(\boldsymbol{z}_t\boldsymbol{z}_t')'(\boldsymbol{q}_j \otimes \boldsymbol{q}_j)\frac{1}{\psi_{0j}} = 2\boldsymbol{e}_i'\boldsymbol{q}_j\boldsymbol{q}_i'\boldsymbol{e}_j + 1. \tag{28}$$

Thus, from (28),

$$\mathsf{E}(\boldsymbol{e}_i'\boldsymbol{z}_t\boldsymbol{z}_t'(\boldsymbol{P}^0)^{-1}\boldsymbol{e}_i - 1)(\boldsymbol{q}_j \otimes \boldsymbol{q}_j)'\text{vec}(\boldsymbol{z}_t\boldsymbol{z}_t')\frac{1}{\psi_{0j}} - 1 = 2\boldsymbol{e}_i'\boldsymbol{q}_j\boldsymbol{q}_i'\boldsymbol{e}_j.$$

Equation (26) is proved in a similar fashion. ∎

Lemma 7.

$$\mathsf{E}\frac{\partial \ell_t}{\partial \overline{\boldsymbol{\psi}}}\frac{\partial \ell_t}{\partial \overline{\boldsymbol{\psi}}'} = \frac{1}{2}(\boldsymbol{\Psi}_0^{-2} + \frac{1}{(N - \sum_{k=1}^{N-1}\psi_{0k})^2}\mathbf{1}_{N-1}\mathbf{1}_{N-1}') \otimes (\boldsymbol{\tau}_t\boldsymbol{\tau}_t'),$$

where $\overline{\boldsymbol{\psi}} = (\overline{\boldsymbol{\psi}}_1', \ldots, \overline{\boldsymbol{\psi}}_{N-1}')'$ *with* $\overline{\boldsymbol{\psi}}_j = (\psi_{0j}, \psi_{1j}, \psi_{2,j})'$, $j = 1, \ldots, N-1$, *and* $\boldsymbol{\Psi}_0 = \text{diag}(\psi_{01}, \ldots, \psi_{0,N-1})$.

Proof. Under H_0', $w_{jt} = \boldsymbol{q}_j'\boldsymbol{z}_t \sim \text{iid}\mathcal{N}(0, \psi_{0j})$ and $w_{jt}^2/\psi_{0j} \sim \chi^2(1)$. Thus,

$$\frac{1}{\psi_{0j}^2}\mathsf{E}(\frac{w_{jt}^2}{\psi_{0j}} - 1)^2 = \frac{2}{\psi_{0j}^2},$$

$j = 1, \ldots, N$, where $\psi_{0N} = N - \sum_{k=1}^{N-1}\psi_{0k}$, and

$$\frac{1}{\psi_{0l}\psi_{0j}}\mathsf{E}(\frac{w_{it}^2}{\psi_{0l}} - 1)(\frac{w_{jt}^2}{\psi_{0j}} - 1) = 0$$

for $l \neq j$. Then,

$$\mathsf{E}\frac{\partial \ell_t}{\partial \psi_{kj}}\frac{\partial \ell_t}{\partial \psi_{ij}}|_{\text{H}_0} = \frac{1}{4}(\frac{t}{T})^{k+i}[\frac{1}{\psi_{0j}^2}\mathsf{E}(\frac{w_{jt}^2}{\psi_{0j}} - 1)^2 + \frac{1}{(N - \sum_{k=1}^{N-1}\psi_{0k})^2}\mathsf{E}(\frac{w_{Nt}^2}{N - \sum_{k=1}^{N-1}\psi_{0k}} - 1)^2]$$
$$= \frac{1}{2}(\frac{t}{T})^{k+i}(\frac{1}{\psi_{0j}^2} + \frac{1}{(N - \sum_{k=1}^{N-1}\psi_{0k})^2})$$

for $j = 1, \ldots, N-1$, and

$$\mathsf{E}\frac{\partial \ell_t}{\partial \psi_{kl}}\frac{\partial \ell_t}{\partial \psi_{ij}}|_{\text{H}_0} = \frac{1}{2}(\frac{t}{T})^{k+i}\frac{1}{(N - \sum_{k=1}^{N-1}\psi_{0k})^2}$$

for $l \neq j$. In matrix form,

$$\mathsf{E}\frac{\partial \ell_t}{\partial \overline{\boldsymbol{\psi}}}\frac{\partial \ell_t}{\partial \overline{\boldsymbol{\psi}}'} = \frac{1}{2}(\boldsymbol{\Psi}_0^{-2} + \frac{1}{(N - \sum_{k=1}^{N-1}\psi_{0k})^2}\mathbf{1}_{N-1}\mathbf{1}_{N-1}') \otimes (\boldsymbol{\tau}_t\boldsymbol{\tau}_t'),$$

which is the desired result. ∎

Proof of Theorem 2. In order to prove the theorem, begin by considering the limit of $\frac{1}{T}\sum_{t=1}^{T}\mathsf{E}\frac{\partial \ell_t}{\partial \boldsymbol{\theta}_{gi}}\frac{\partial \ell_t}{\partial \boldsymbol{\theta}'_{gj}}$ as $T \to \infty$. From Lemma 3, rescaling time to the unit interval and denoting $t/T = [Tr]/T$, $0 < r \leq 1$, one obtains

$$\frac{1}{T}\sum_{t=1}^{T}\mathsf{E}\frac{\partial \ell_t}{\partial \boldsymbol{\theta}_{gi}}\frac{\partial \ell_t}{\partial \boldsymbol{\theta}'_{gj}}|_{\mathrm{H}_0} = \frac{1}{4T}\sum_{t=1}^{T}(\frac{1}{g_{it}^0}\frac{\partial g_{it}^0}{\partial \boldsymbol{\theta}_{gi}} + \frac{1}{h_{it}^0}\frac{\partial h_{it}^0}{\partial \boldsymbol{\theta}_{gi}})(\frac{1}{g_{jt}^0}\frac{\partial g_{jt}^0}{\partial \boldsymbol{\theta}'_{gj}} + \frac{1}{h_{jt}^0}\frac{\partial h_{jt}^0}{\partial \boldsymbol{\theta}'_{gj}})\boldsymbol{e}'_i(\boldsymbol{P}^0)^{-1}\boldsymbol{e}_j\boldsymbol{e}'_i\boldsymbol{P}^0\boldsymbol{e}_j$$
$$= [\frac{1}{4T}\sum_{t=1}^{T}\frac{1}{(g_{it}^0)^2}\frac{\partial g_{it}^0}{\partial \boldsymbol{\theta}_{gi}}\frac{\partial g_{jt}^0}{\partial \boldsymbol{\theta}'_{gj}} + \frac{1}{4T}\sum_{t=1}^{T}\{\frac{1}{g_{it}^0 h_{jt}^0}\frac{\partial g_{it}^0}{\partial \boldsymbol{\theta}_{gi}}\frac{\partial h_{jt}^0}{\partial \boldsymbol{\theta}'_{gj}}$$
$$+ \frac{1}{h_{it}^0}\frac{\partial h_{it}^0}{\partial \boldsymbol{\theta}_{gi}}(\frac{1}{g_{jt}^0}\frac{\partial g_{jt}^0}{\partial \boldsymbol{\theta}'_{gj}} + \frac{1}{h_{jt}^0}\frac{\partial h_{jt}^0}{\partial \boldsymbol{\theta}'_{gj}})\}]\boldsymbol{e}'_i(\boldsymbol{P}^0)^{-1}\boldsymbol{e}_j\boldsymbol{e}'_i\boldsymbol{P}^0\boldsymbol{e}_j. \quad (29)$$

Consider the first term on the r.h.s. of (29) and denote $t/T = [Tr]/T$, $0 < r \leq 1$. One obtains

$$\frac{1}{4}\sum_{t=1}^{T}\int_{t/T}^{(t+1)/T}\frac{1}{(g_{i[Tr]/T}^0)^2}\frac{\partial g_{i[Tr]/T}^0}{\partial \boldsymbol{\theta}_{gi}}\frac{\partial g_{i[Tr]/T}^0}{\partial \boldsymbol{\theta}'_{gi}}\mathrm{d}r = \frac{1}{4}\int_{1/T}^{(T+1)/T}\frac{1}{(g_{i[Tr]/T}^0)^2}\frac{\partial g_{i[Tr]/T}^0}{\partial \boldsymbol{\theta}_{gi}}\frac{\partial g_{i[Tr]/T}^0}{\partial \boldsymbol{\theta}'_{gi}}\mathrm{d}r$$
$$\to \frac{1}{4}\int_{0}^{1}\frac{1}{(g_{ir}^0)^2}\frac{\partial g_{ir}^0}{\partial \boldsymbol{\theta}_{gi}}\frac{\partial g_{ir}^0}{\partial \boldsymbol{\theta}'_{gi}}\mathrm{d}r$$

as $T \to \infty$. Consequently,

$$\frac{1}{T}\sum_{t=1}^{T}\mathsf{E}\frac{\partial \ell_t}{\partial \boldsymbol{\theta}_{gi}}\frac{\partial \ell_t}{\partial \boldsymbol{\theta}'_{gj}}|_{\mathrm{H}_0} \to \frac{1}{4}[\int_{0}^{1}\frac{1}{(g_{ir}^0)^2}\frac{\partial g_{ir}^0}{\partial \boldsymbol{\theta}_{gi}}\frac{\partial g_{ir}^0}{\partial \boldsymbol{\theta}'_{gi}}\mathrm{d}r + \lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\{\frac{1}{g_{it}^0 h_{jt}^0}\frac{\partial g_{it}^0}{\partial \boldsymbol{\theta}_{gi}}\frac{\partial h_{jt}^0}{\partial \boldsymbol{\theta}'_{gj}}$$
$$+ \frac{1}{h_{it}^0}\frac{\partial h_{it}^0}{\partial \boldsymbol{\theta}_{gi}}(\frac{1}{g_{jt}^0}\frac{\partial g_{jt}^0}{\partial \boldsymbol{\theta}'_{gj}} + \frac{1}{h_{jt}^0}\frac{\partial h_{jt}^0}{\partial \boldsymbol{\theta}'_{gj}})\}]\{\boldsymbol{e}'_i(\boldsymbol{P}^0)^{-1}\boldsymbol{e}_j\boldsymbol{e}'_i\boldsymbol{P}^0\boldsymbol{e}_j - 1\}$$
$$= [\boldsymbol{J}_{\theta_g\theta_g}]_{ij}$$

and when $i = j$,

$$\frac{1}{T}\sum_{t=1}^{T}\mathsf{E}\frac{\partial \ell_t}{\partial \boldsymbol{\theta}_{gi}}\frac{\partial \ell_t}{\partial \boldsymbol{\theta}'_{gi}}|_{\mathrm{H}_0} \to \frac{1}{4}[\int_{0}^{1}\frac{1}{(g_{ir}^0)^2}\frac{\partial g_{ir}^0}{\partial \boldsymbol{\theta}_{gi}}\frac{\partial g_{ir}^0}{\partial \boldsymbol{\theta}'_{gi}}\mathrm{d}r + \lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}(\frac{1}{h_{it}^0 g_{ir}^0}\frac{\partial g_{it}^0}{\partial \boldsymbol{\theta}_{gi}}\frac{\partial h_{ir}^0}{\partial \boldsymbol{\theta}'_{gi}}$$
$$+ \frac{1}{h_{it}^0}\frac{\partial h_{it}^0}{\partial \boldsymbol{\theta}_{gi}}(\frac{1}{g_{jt}^0}\frac{\partial g_{it}^0}{\partial \boldsymbol{\theta}'_{gi}} + \frac{1}{h_{jt}^0}\frac{\partial h_{it}^0}{\partial \boldsymbol{\theta}'_{gi}})]\{\boldsymbol{e}'_i(\boldsymbol{P}^0)^{-1}\boldsymbol{e}_i + 1\} = [\boldsymbol{J}_{\theta_g\theta_g}]_{ii}$$

as $T \to \infty$. In a similar fashion, from Lemma 5, for $i \neq j$ one obtains

$$\frac{1}{T}\sum_{t=1}^{T}\mathsf{E}\frac{\partial \ell_t}{\partial \boldsymbol{\theta}_{hi}}\frac{\partial \ell_t}{\partial \boldsymbol{\theta}'_{hj}}|_{\mathrm{H}_0} \to \lim_{T\to\infty}\frac{1}{4T}\sum_{t=1}^{T}\frac{1}{h_{it}^0 h_{jt}^0}\frac{\partial h_{it}^0}{\partial \boldsymbol{\theta}_{hi}}\frac{\partial h_{jt}^0}{\partial \boldsymbol{\theta}'_{hj}}\boldsymbol{e}'_i(\boldsymbol{P}^0)^{-1}\boldsymbol{e}_j\boldsymbol{e}'_i\boldsymbol{P}^0\boldsymbol{e}_j = [\boldsymbol{J}_{\theta_h\theta_h}]_{ij}$$

and when $i = j$,

$$\frac{1}{T}\sum_{t=1}^{T}\mathsf{E}\frac{\partial \ell_t}{\partial \boldsymbol{\theta}_{hi}}\frac{\partial \ell_t}{\partial \boldsymbol{\theta}'_{hi}}|_{\mathrm{H}_0} \to \lim_{T\to\infty}\frac{1}{4T}\sum_{t=1}^{T}\frac{1}{(h_{it}^0)^2}\frac{\partial h_{it}^0}{\partial \boldsymbol{\theta}_{hi}}\frac{\partial h_{it}^0}{\partial \boldsymbol{\theta}'_{hi}}\{\boldsymbol{e}'_i(\boldsymbol{P}^0)^{-1}\boldsymbol{e}_i + 1\} = [\boldsymbol{J}_{\theta_h\theta_h}]_{ii}$$

as $T \to \infty$. Applying Lemma 4, for $i \neq j$, one obtains

$$\frac{1}{T}\sum_{t=1}^{T} \mathsf{E}\frac{\partial \ell_t}{\partial \boldsymbol{\theta}_{gi}}\frac{\partial \ell_t}{\partial \boldsymbol{\theta}'_{hj}}|_{\mathrm{H}_0} \to \lim_{T\to\infty}\frac{1}{4T}\sum_{t=1}^{T}(\frac{1}{g_{it}^0}\frac{\partial g_{it}^0}{\partial \boldsymbol{\theta}_{gi}} + \frac{1}{h_{it}^0}\frac{\partial h_{it}^0}{\partial \boldsymbol{\theta}_{gi}})\frac{1}{h_{jt}^0}\frac{\partial h_{jt}^0}{\partial \boldsymbol{\theta}'_{gj}}\boldsymbol{e}'_i(\boldsymbol{P}^0)^{-1}\boldsymbol{e}_j\boldsymbol{e}'_i\boldsymbol{P}^0\boldsymbol{e}_j$$
$$= [\boldsymbol{J}_{\theta_g\theta_h}]_{ij}$$

and when $i = j$,

$$\frac{1}{T}\sum_{t=1}^{T} \mathsf{E}\frac{\partial \ell_t}{\partial \boldsymbol{\theta}_{gi}}\frac{\partial \ell_t}{\partial \boldsymbol{\theta}'_{hi}}|_{\mathrm{H}_0} \to \lim_{T\to\infty}\frac{1}{4T}\sum_{t=1}^{T}(\frac{1}{g_{it}^0}\frac{\partial g_{it}^0}{\partial \boldsymbol{\theta}_{gi}} + \frac{1}{h_{it}^0}\frac{\partial h_{it}^0}{\partial \boldsymbol{\theta}_{gi}})\frac{1}{h_{it}^0}\frac{\partial h_{it}^0}{\partial \boldsymbol{\theta}'_{gi}}\{\boldsymbol{e}'_i(\boldsymbol{P}^0)^{-1}\boldsymbol{e}_i + 1\}$$
$$= [\boldsymbol{J}_{\theta_g\theta_h}]_{ii}$$

as $T \to \infty$. Applying Lemma 6 and using the same arguments as above, one obtains

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T} \mathsf{E}\frac{\partial \ell_t}{\partial \boldsymbol{\theta}_{gi}}\frac{\partial \ell_t}{\partial \overline{\boldsymbol{\psi}}'_j}|_{\mathrm{H}_0} &= \frac{1}{2T}\sum_{t=1}^{T}(\frac{1}{g_{it}^0}\frac{\partial g_{it}^0}{\partial \boldsymbol{\theta}_{gi}} + \frac{1}{h_{it}^0}\frac{\partial h_{it}^0}{\partial \boldsymbol{\theta}_{gi}})\boldsymbol{\tau}'_t \\ &\quad \times (\frac{1}{\psi_{0j}}\boldsymbol{e}'_i\boldsymbol{q}_j\boldsymbol{q}'_i\boldsymbol{e}_j - \frac{1}{N - \sum_{k=1}^{N-1}\psi_{0k}}\boldsymbol{e}'_i\boldsymbol{q}_N\boldsymbol{q}'_i\boldsymbol{e}_N) \\ &\to \frac{1}{2}(\int_0^1 \frac{1}{g_{ir}^0}\frac{\partial g_{ir}^0}{\partial \boldsymbol{\theta}_{gi}}\boldsymbol{r}'\mathrm{d}r + \lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\frac{1}{h_{it}^0}\frac{\partial h_{it}^0}{\partial \boldsymbol{\theta}_{gi}}\boldsymbol{\tau}'_t) \\ &\quad \times (\frac{1}{\psi_{0j}}\boldsymbol{e}'_i\boldsymbol{q}_j\boldsymbol{q}'_i\boldsymbol{e}_j - \frac{1}{N - \sum_{k=1}^{N-1}\psi_{0k}}\boldsymbol{e}'_i\boldsymbol{q}_N\boldsymbol{q}'_i\boldsymbol{e}_N) = [\boldsymbol{J}_{\theta_g\overline{\psi}}]_{ij},\end{aligned}$$

where $\boldsymbol{r} = (1, r, r^2)'$, and

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T} \mathsf{E}\frac{\partial \ell_t}{\partial \boldsymbol{\theta}_{hi}}\frac{\partial \ell_t}{\partial \overline{\boldsymbol{\psi}}'_j}|_{\mathrm{H}_0} &\to \frac{1}{2T}\sum_{t=1}^{T}\frac{1}{h_{it}^0}\frac{\partial h_{it}^0}{\partial \boldsymbol{\theta}_{hi}}\boldsymbol{\tau}'_t \\ &\quad \times (\frac{1}{\psi_{0j}}\boldsymbol{e}'_i\boldsymbol{q}_j\boldsymbol{q}'_i\boldsymbol{e}_j - \frac{1}{N - \sum_{k=1}^{N-1}\psi_{0k}}\boldsymbol{e}'_i\boldsymbol{q}_N\boldsymbol{q}'_i\boldsymbol{e}_N) = [\boldsymbol{J}_{\theta_h\overline{\psi}}]_{ij}\end{aligned}$$

as $T \to \infty$. Finally, from Lemma 7 it follows that

$$\begin{aligned}\sum_{t=1}^{T} \mathsf{E}\frac{\partial \ell_t}{\partial \overline{\boldsymbol{\psi}}}\frac{\partial \ell_t}{\partial \overline{\boldsymbol{\psi}}'} &\to (\boldsymbol{\Psi}_0^{-2} + \frac{1}{(N - \sum_{k=1}^{N-1}\psi_{0k})^2}\mathbf{1}_{N-1}\mathbf{1}'_{N-1}) \otimes \int_0^1 \begin{bmatrix} 1 & r & r^2 \\ & r^2 & r^3 \\ & & r^4 \end{bmatrix} \mathrm{d}r \\ &= (\boldsymbol{\Psi}_0^{-2} + \frac{1}{(N - \sum_{k=1}^{N-1}\psi_{0k})^2}\mathbf{1}_{N-1}\mathbf{1}'_{N-1}) \otimes \begin{bmatrix} 1 & 1/2 & 1/3 \\ & 1/3 & 1/4 \\ & & 1/5 \end{bmatrix}\end{aligned}$$

as $t \to T$ and $T \to \infty$. This completes the proof of Theorem 2. ■

C Simulation details and results

In this Section we present further details of the various simulations discussed in the paper, as well as further investigations of the proposed test. All simulations presented here are based on 2500 replications. The version of R used is 4.1.0. We have developed our own R package, 'mtvgarch' to support this research into multivariate, time-varying GARCH models. The package is not currently available on CRAN, but is available upon request. The version used on this paper is 0.8.54. The code is maintained in a private GitHub repository.

The processing of the simulations is very compute-resource intensive. The mtvgarch package uses the doParallel package (available on CRAN) to parallelise the processing. This should be done using a MPP (massively-parallel-processing) array, but will also work on a multi-core desktop PC. A minimum of 8 logical processors and 32GB RAM is sufficient to run most simulations, but will be slow and will not handle the higher dimensional cases. We recommended reducing (or removing) the parallelisation when the execution of the code results in CPU or Memory usage approaching 100 percent. MS Azure Virtual Machines (VM) were used to do a lot of the processing. The operating system was Windows Server 2019 and the size was Standard-F8s-v2 with 8 vCPU's and 64GB RAM.[2] Our VM's took approximately 10 hours to process simulations where $N = 20$ and $T = 2000$.

Size

Tables C.1 and C.2 contain the size simulations, where the DGP is a CEC-GARCH (constant equicorrelation). For each series, $g_t = 1$ and the parameterisation of the GARCH equation is such that the persistence is 0.95 and kurtosis is set to 4 in the first experiment, and to 6 in the second. That is, $\alpha_1 = 0.1104$, $\beta_1 = 0.8396$ in the former, and $\alpha_1 = 0.1561$, $\beta_1 = 0.7939$ for the latter. The GARCH intercept is set to $1 - \alpha_1 - \beta_1$ to standardise the unconditional variance to unity. The equicorrelation coefficient is equal to 0.33 in the former and 0.67 in the latter. The transition variable in the test is a linear time trend. The dimension N ranges from 2 to 20, and sample size T from 500 to 2000. Note that the smallest size $T = 500$ is no longer feasible for $N = 20$.

We extend the previous set-up by defining the constant correlation matrix as having a block structure. The system of N series is divided into subgroups consisting of four series. Group i is described as having an equicorrelated state with ρ_i as the correlation parameter, $i = 1, \ldots, N/4$. The correlation between groups i and j is defined as $\sqrt{\rho_i \rho_j}$. This structure ensures positive definiteness of the resulting correlation matrix. We consider $N = 12$ (three groups of four series), first with $\boldsymbol{\rho} = (0.2, 0.3, 0.4)$ and then $\boldsymbol{\rho} = (0.25, 0.5, 0.75)$. The condition numbers of the resulting matrices are similar (7.13 and 25.69) to the ones for equicorrelated matrices of the same dimension with correlation 0.33 (condition number $C = 7$) and 0.67 (condition number $C = 25$), respectively. This is important, because

[2]The Fsv2-series currently run on the 3rd Generation Intel Xeon Platinum 8370C (Ice Lake), the Intel Xeon Platinum 8272CL (Cascade Lake) processors or the Intel Xeon Platinum 8168 (Skylake) processors. It features a sustained all core Turbo clock speed of 3.4 GHz and a maximum single-core turbo frequency of 3.7 GHz.

there is an introduced error associated with the matrix inversions that take place during the computation of the test statistic, and the error gets larger the higher the dimension N and the closer the matrix is to singularity. Setting the condition numbers equal will ensure the error is at par across the models and the results are comparable. We further extend the set up to $N = 16$ (four groups of four series), with $\boldsymbol{\rho} = (0.10, 0.20, 0.35, 0.45)$ and $\boldsymbol{\rho} = (0.25, 0.35, 0.55, 0.75)$. The condition numbers for these matrices are 9.20 and 32.21, again aligning with those of equicorrelated systems of the same size ($C = 9$ when $\rho = 0.33$, and $C = 33$ when $\rho = 0.67$). The GARCH parameterisation is the same as in the first experiment, targeting persistence 0.95 and kurtosis 4 and 6. The simulation results are presented in Table C.3.

Misspecified variance

The next experiment investigates the effect of neglecting the TV-component. That is, the DGP is a TV-CEC-GARCH, but the baseline volatility shift is ignored at the estimation stage, and only a CEC-GARCH model is estimated. As before, the GARCH equations are set up with persistence of 0.95, kurtosis of 4 and 6, $\alpha_0 = 1 - \alpha_1 - \beta_1$ for each series, and two strengths of equicorrelation are examined, 0.33 and 0.67. The TV-component has a transition located at the center of the sample, and the transition variable is a linear time trend. The speed of the transition is set to $\gamma = 20$, which translates to a transition that gradually begins at the first quartile and finishes at the third. The magnitude of the increase in the volatility from the initial level of $\delta_0 = 1$ is set to $\delta_1 = 3$ in the first simulation, and to $\delta_1 = 8$ in the second one, which effectively doubles and triples the standard deviations, respectively. The results for $\delta_1 = 3$ are presented in Tables C.4 and C.5. Because the result indicates a very strong tendency to reject the null even at the rather modest increase in variance, the results for $\delta_1 = 8$ are omitted.

Another experiment investigates the sensitivity of the test to the correctness of the GARCH model specification. In this case the GARCH equation is misspecified such that it includes an asymmetry component (GJR-GARCH), but this is ignored when the model is estimated and the test statistic computed. To keep the results comparable, the parameters are chosen such that the implied kurtosis levels are 4 and 6, in addition to keeping the unconditional variance equal to one ($\alpha_0 = 1 - \alpha_1 - 0.5\kappa_1 - \beta_1$ for each series) and persistence at 0.95. We choose to look at the extreme case where the effect of the past shock is inherited only from the negative shocks (i.e. $\alpha_1 = 0$). This yields $\kappa_1 = 0.14, \beta_1 = 0.88$ for the case of kurtosis$= 4$ and $\kappa_1 = 0.198, \beta_1 = 0.851$ when kurtosis$= 6$. The results are tabulated in Tables C.6 and C.7.

Misspecified error distribution

As the last investigation we look into the effect of nonnormality of the error distribution. We choose t-distribution with $df = 8$ and $df = 5$. To create multivariate t-distributed data, the individual noise series are first standardised to have unit variance. The correlation matrix is again an equicorrelated one, with $\rho = 0.33$ and $\rho = 0.67$, as before. The resulting data is thus correlated multivariate t, with t-distributed (standardised) marginals. The GARCH parameters are chosen such that the fourth moment still exists, and the resulting kurtosis is reasonable. We also wish to keep the persistence at 0.95 to allow for comparison with the normal cases discussed earlier, and the GARCH intercept is set to $1 - \alpha_1 - \beta_1$. To this end, we choose $\alpha_1 = 0.06$ and $\beta_1 = 0.89$ when $df = 8$ (the implied kurtosis is 5.17), and $\alpha_1 = 0.03$ and $\beta_1 = 0.92$ for $df = 5$ (the implied kurtosis is 9.72). The results of the size simulations are presented in Tables C.8 and C.9.

		Order 1			Order 2			Order 3			Order 4		
N	T	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%
					Panel A								
2	500	0.010	0.048	0.108	0.009	0.053	0.102	0.008	0.049	0.107	0.013	0.054	0.099
2	1000	0.012	0.041	0.087	0.010	0.046	0.091	0.012	0.048	0.102	0.009	0.049	0.093
2	2000	0.008	0.045	0.097	0.008	0.047	0.088	0.009	0.045	0.092	0.010	0.046	0.090
5	500	0.013	0.058	0.107	0.015	0.068	0.124	0.019	0.070	0.130	0.017	0.071	0.126
5	1000	0.010	0.051	0.101	0.014	0.052	0.102	0.014	0.060	0.113	0.017	0.057	0.110
5	2000	0.010	0.052	0.103	0.008	0.050	0.107	0.011	0.048	0.106	0.013	0.055	0.109
10	500	0.009	0.050	0.106	0.010	0.062	0.121	0.013	0.070	0.122	0.019	0.064	0.125
10	1000	0.010	0.054	0.101	0.010	0.056	0.107	0.012	0.054	0.118	0.012	0.058	0.108
10	2000	0.010	0.054	0.102	0.012	0.055	0.114	0.014	0.058	0.104	0.014	0.064	0.118
20	1000	0.012	0.050	0.088	0.009	0.052	0.099	0.009	0.052	0.103	0.009	0.051	0.101
20	2000	0.015	0.048	0.094	0.012	0.050	0.103	0.010	0.050	0.100	0.011	0.056	0.107
					Panel B								
2	500	0.010	0.049	0.104	0.012	0.056	0.104	0.010	0.052	0.106	0.014	0.058	0.102
2	1000	0.012	0.040	0.085	0.010	0.045	0.092	0.012	0.048	0.102	0.009	0.050	0.094
2	2000	0.008	0.045	0.094	0.007	0.048	0.087	0.010	0.044	0.095	0.009	0.044	0.092
5	500	0.013	0.059	0.108	0.018	0.072	0.126	0.020	0.074	0.135	0.021	0.075	0.134
5	1000	0.011	0.050	0.101	0.012	0.051	0.100	0.014	0.056	0.108	0.014	0.053	0.109
5	2000	0.010	0.054	0.102	0.008	0.054	0.110	0.011	0.051	0.110	0.015	0.058	0.108
10	500	0.010	0.053	0.104	0.010	0.066	0.128	0.016	0.076	0.133	0.021	0.072	0.133
10	1000	0.012	0.054	0.104	0.012	0.053	0.108	0.012	0.054	0.116	0.016	0.063	0.112
10	2000	0.010	0.051	0.101	0.012	0.058	0.119	0.016	0.060	0.107	0.016	0.066	0.124
20	1000	0.012	0.050	0.090	0.010	0.052	0.102	0.010	0.055	0.103	0.010	0.054	0.103
20	2000	0.014	0.048	0.093	0.012	0.048	0.107	0.009	0.055	0.101	0.013	0.056	0.116

Table C.1: Size of the test. The data is generated from a CEC-GARCH with persistence of 0.95, kurtosis of 4 (Panel A) and 6 (Panel B), and equicorrelation coefficient of 0.33. 2500 replications.

		Order 1			Order 2			Order 3			Order 4		
N	T	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%
Panel A													
2	500	0.011	0.060	0.112	0.015	0.063	0.124	0.016	0.064	0.124	0.018	0.069	0.117
2	1000	0.010	0.045	0.090	0.013	0.050	0.094	0.013	0.052	0.104	0.013	0.055	0.105
2	2000	0.008	0.048	0.093	0.010	0.048	0.094	0.012	0.043	0.102	0.012	0.048	0.094
5	500	0.016	0.059	0.107	0.023	0.074	0.134	0.026	0.086	0.152	0.028	0.088	0.154
5	1000	0.010	0.053	0.108	0.017	0.060	0.107	0.022	0.067	0.120	0.020	0.070	0.126
5	2000	0.013	0.050	0.102	0.012	0.062	0.116	0.014	0.068	0.120	0.017	0.061	0.128
10	500	0.012	0.056	0.108	0.017	0.075	0.144	0.020	0.079	0.147	0.024	0.089	0.160
10	1000	0.012	0.060	0.116	0.016	0.061	0.120	0.016	0.066	0.126	0.023	0.070	0.129
10	2000	0.014	0.056	0.104	0.012	0.062	0.120	0.014	0.060	0.113	0.017	0.070	0.124
20	1000	0.011	0.056	0.108	0.013	0.063	0.115	0.012	0.065	0.130	0.018	0.072	0.133
20	2000	0.012	0.048	0.104	0.011	0.052	0.108	0.012	0.051	0.107	0.014	0.056	0.113
Panel B													
2	500	0.012	0.061	0.112	0.017	0.071	0.128	0.021	0.068	0.130	0.021	0.072	0.126
2	1000	0.010	0.045	0.091	0.013	0.050	0.100	0.013	0.053	0.106	0.012	0.058	0.108
2	2000	0.009	0.048	0.094	0.010	0.050	0.095	0.012	0.049	0.105	0.012	0.050	0.102
5	500	0.017	0.061	0.113	0.025	0.083	0.143	0.031	0.094	0.166	0.036	0.098	0.162
5	1000	0.010	0.053	0.108	0.019	0.063	0.113	0.024	0.071	0.126	0.021	0.075	0.132
5	2000	0.013	0.050	0.102	0.016	0.068	0.122	0.018	0.071	0.128	0.022	0.073	0.138
10	500	0.014	0.056	0.115	0.024	0.086	0.156	0.032	0.100	0.164	0.036	0.113	0.184
10	1000	0.014	0.060	0.118	0.020	0.066	0.121	0.021	0.073	0.135	0.025	0.081	0.145
10	2000	0.015	0.054	0.106	0.016	0.066	0.126	0.018	0.066	0.122	0.020	0.080	0.136
20	1000	0.011	0.060	0.110	0.017	0.068	0.124	0.018	0.071	0.130	0.023	0.079	0.144
20	2000	0.011	0.052	0.102	0.014	0.060	0.118	0.015	0.059	0.119	0.019	0.069	0.128

Table C.2: Size of the test. The data is generated from a CEC-GARCH with persistence of 0.95, kurtosis of 4 (Panel A) and 6 (Panel B), and equicorrelation coefficient of 0.67. 2500 replications.

			Order 1			Order 2			Order 3			Order 4		
N	T	C	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%
Panel A														
12	1000	7.13	0.013	0.058	0.116	0.014	0.058	0.117	0.012	0.062	0.117	0.015	0.064	0.120
12	2000	7.13	0.009	0.054	0.100	0.011	0.051	0.108	0.015	0.060	0.111	0.018	0.064	0.119
16	1000	9.20	0.013	0.053	0.114	0.019	0.064	0.125	0.014	0.062	0.122	0.014	0.063	0.126
16	2000	9.20	0.010	0.052	0.109	0.014	0.064	0.119	0.010	0.068	0.118	0.018	0.067	0.121
12	1000	25.69	0.015	0.062	0.111	0.019	0.072	0.139	0.022	0.074	0.132	0.022	0.075	0.138
12	2000	25.69	0.010	0.052	0.103	0.017	0.063	0.124	0.012	0.075	0.125	0.020	0.075	0.135
16	1000	32.21	0.009	0.060	0.120	0.015	0.070	0.128	0.017	0.073	0.130	0.017	0.076	0.138
16	2000	32.21	0.010	0.055	0.112	0.014	0.071	0.116	0.011	0.069	0.124	0.016	0.070	0.133
Panel B														
12	1000	7.13	0.013	0.059	0.118	0.016	0.064	0.122	0.014	0.066	0.126	0.016	0.068	0.122
12	2000	7.13	0.008	0.053	0.098	0.014	0.064	0.116	0.018	0.064	0.122	0.027	0.076	0.132
16	1000	9.20	0.014	0.058	0.114	0.021	0.070	0.130	0.017	0.074	0.129	0.016	0.068	0.134
16	2000	9.20	0.010	0.053	0.107	0.017	0.070	0.129	0.013	0.075	0.124	0.024	0.078	0.136
12	1000	25.69	0.014	0.064	0.114	0.020	0.079	0.141	0.024	0.080	0.142	0.029	0.084	0.150
12	2000	25.69	0.011	0.053	0.107	0.020	0.069	0.140	0.016	0.081	0.136	0.029	0.089	0.159
16	1000	32.21	0.008	0.060	0.120	0.021	0.076	0.138	0.019	0.081	0.140	0.023	0.087	0.149
16	2000	32.21	0.011	0.058	0.114	0.021	0.079	0.134	0.017	0.076	0.138	0.028	0.093	0.157

Table C.3: Size of the test. The data is generated from a block-correlation-GARCH with persistence of 0.95, kurtosis of 4 (Panel A) and 6 (Panel B). The condition numbers (C) in the top and bottom section of each panel correspond to a weak and strong correlation, respectively. 2500 replications.

		Order 1			Order 2			Order 3			Order 4		
N	T	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%
Panel A													
2	500	0.052	0.148	0.241	0.037	0.134	0.229	0.030	0.112	0.199	0.031	0.091	0.178
2	1000	0.072	0.209	0.316	0.053	0.173	0.289	0.043	0.140	0.239	0.039	0.127	0.209
2	2000	0.125	0.307	0.417	0.095	0.253	0.376	0.079	0.218	0.336	0.064	0.195	0.297
5	500	0.293	0.517	0.630	0.310	0.559	0.680	0.221	0.450	0.590	0.168	0.368	0.514
5	1000	0.486	0.680	0.775	0.452	0.680	0.776	0.343	0.583	0.703	0.280	0.500	0.630
5	2000	0.718	0.862	0.915	0.678	0.849	0.911	0.595	0.782	0.862	0.517	0.719	0.808
10	500	0.768	0.894	0.928	0.861	0.954	0.972	0.762	0.902	0.951	0.651	0.842	0.902
10	1000	0.914	0.959	0.976	0.942	0.981	0.990	0.884	0.960	0.978	0.810	0.927	0.959
10	2000	0.990	0.995	0.998	0.990	0.997	0.999	0.980	0.993	0.997	0.963	0.989	0.995
20	1000	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.998	1.000	1.000
20	2000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Panel B													
2	500	0.050	0.141	0.228	0.036	0.123	0.211	0.028	0.104	0.186	0.028	0.083	0.164
2	1000	0.075	0.214	0.320	0.053	0.170	0.282	0.044	0.140	0.238	0.038	0.129	0.204
2	2000	0.146	0.340	0.458	0.109	0.274	0.400	0.093	0.241	0.361	0.075	0.219	0.326
5	500	0.276	0.490	0.604	0.254	0.486	0.615	0.183	0.383	0.524	0.137	0.316	0.448
5	1000	0.498	0.708	0.785	0.421	0.664	0.761	0.331	0.569	0.687	0.272	0.481	0.622
5	2000	0.802	0.913	0.948	0.752	0.893	0.939	0.675	0.844	0.906	0.602	0.789	0.856
10	500	0.746	0.880	0.921	0.779	0.916	0.956	0.652	0.835	0.903	0.554	0.763	0.846
10	1000	0.928	0.970	0.984	0.936	0.976	0.988	0.864	0.952	0.972	0.792	0.914	0.954
10	2000	0.996	1.000	1.000	0.996	0.999	0.999	0.991	0.998	0.999	0.984	0.997	0.998
20	1000	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.995	1.000	1.000
20	2000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table C.4: Rejection frequencies: misspecified deterministic variance component. The data is generated from a TV-CEC-GARCH with the GARCH equation parameterised such that persistence is 0.95 and kurtosis is 4 (Panel A) and 6 (Panel B), and the TV-component has $\delta_0 = 1$, $\delta_1 = 3$, $c = 0.5$ and $\gamma = 20$, and equicorrelation coefficient of 0.33. The test ignores the TV-component. 2500 replications.

		Order 1			Order 2			Order 3			Order 4		
N	T	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%
						Panel A							
2	500	0.227	0.430	0.537	0.226	0.465	0.600	0.182	0.405	0.547	0.158	0.353	0.488
2	1000	0.342	0.547	0.664	0.328	0.568	0.677	0.284	0.521	0.642	0.242	0.464	0.601
2	2000	0.502	0.699	0.777	0.486	0.698	0.790	0.449	0.645	0.750	0.413	0.612	0.712
5	500	0.688	0.828	0.876	0.818	0.924	0.955	0.764	0.901	0.942	0.695	0.860	0.919
5	1000	0.819	0.900	0.930	0.883	0.948	0.969	0.843	0.936	0.962	0.800	0.911	0.952
5	2000	0.941	0.973	0.983	0.962	0.985	0.991	0.944	0.978	0.987	0.919	0.972	0.983
10	500	0.890	0.934	0.955	0.974	0.993	0.995	0.964	0.988	0.996	0.948	0.984	0.990
10	1000	0.948	0.974	0.980	0.988	0.994	0.997	0.982	0.994	0.996	0.973	0.992	0.995
10	2000	0.991	0.994	0.996	0.996	0.999	0.999	0.995	0.999	0.999	0.991	0.997	0.998
20	1000	0.986	0.994	0.998	0.998	1.000	1.000	0.998	0.999	1.000	0.999	0.999	1.000
20	2000	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
						Panel B							
2	500	0.200	0.411	0.521	0.185	0.415	0.542	0.154	0.356	0.491	0.128	0.302	0.433
2	1000	0.345	0.556	0.672	0.308	0.548	0.657	0.270	0.506	0.620	0.235	0.445	0.582
2	2000	0.571	0.754	0.826	0.524	0.735	0.823	0.494	0.693	0.790	0.461	0.656	0.752
5	500	0.668	0.816	0.871	0.743	0.883	0.929	0.668	0.842	0.902	0.598	0.794	0.871
5	1000	0.836	0.913	0.942	0.868	0.939	0.962	0.816	0.916	0.950	0.780	0.894	0.934
5	2000	0.975	0.986	0.990	0.977	0.991	0.995	0.966	0.986	0.992	0.951	0.983	0.991
10	500	0.884	0.934	0.954	0.956	0.981	0.991	0.936	0.976	0.986	0.903	0.961	0.981
10	1000	0.957	0.979	0.984	0.982	0.992	0.996	0.974	0.990	0.994	0.960	0.987	0.991
10	2000	0.996	0.998	0.999	0.999	1.000	1.000	0.998	0.999	0.999	0.996	0.998	0.999
20	1000	0.992	0.998	0.999	0.999	1.000	1.000	0.997	1.000	1.000	0.997	0.999	1.000
20	2000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table C.5: Rejection frequencies: misspecified deterministic variance component. The data is generated from a TV-CEC-GARCH with the GARCH equation parameterised such that persistence is 0.95 and kurtosis is 4 (Panel A) and 6 (Panel B), and the TV-component has $\delta_0 = 1$, $\delta_1 = 3$, $c = 0.5$ and $\gamma = 20$, and equicorrelation coefficient of 0.67. The test ignores the TV-component. 2500 replications.

		Order 1			Order 2			Order 3			Order 4		
N	T	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%
Panel A													
2	500	0.010	0.054	0.104	0.011	0.056	0.113	0.011	0.054	0.109	0.012	0.054	0.109
2	1000	0.012	0.052	0.094	0.011	0.047	0.092	0.015	0.056	0.106	0.014	0.052	0.098
2	2000	0.009	0.053	0.103	0.011	0.049	0.100	0.010	0.052	0.109	0.009	0.051	0.112
5	500	0.018	0.071	0.120	0.025	0.082	0.149	0.028	0.089	0.149	0.028	0.090	0.156
5	1000	0.016	0.069	0.131	0.020	0.080	0.143	0.024	0.086	0.155	0.024	0.093	0.157
5	2000	0.017	0.064	0.134	0.015	0.079	0.147	0.019	0.086	0.157	0.022	0.083	0.153
10	500	0.022	0.089	0.146	0.032	0.113	0.197	0.038	0.126	0.209	0.034	0.117	0.199
10	1000	0.024	0.090	0.151	0.029	0.106	0.179	0.031	0.110	0.197	0.035	0.116	0.198
10	2000	0.019	0.082	0.144	0.029	0.101	0.182	0.030	0.114	0.188	0.039	0.124	0.210
20	1000	0.030	0.094	0.155	0.046	0.137	0.228	0.058	0.168	0.267	0.061	0.179	0.285
20	2000	0.034	0.106	0.170	0.049	0.155	0.236	0.058	0.160	0.261	0.063	0.178	0.274
Panel B													
2	500	0.011	0.056	0.105	0.012	0.056	0.116	0.013	0.056	0.109	0.014	0.057	0.109
2	1000	0.011	0.054	0.094	0.012	0.050	0.095	0.016	0.060	0.109	0.016	0.054	0.101
2	2000	0.012	0.055	0.105	0.012	0.051	0.104	0.012	0.056	0.112	0.009	0.055	0.115
5	500	0.020	0.076	0.130	0.024	0.088	0.156	0.030	0.098	0.163	0.032	0.100	0.169
5	1000	0.020	0.073	0.134	0.023	0.087	0.152	0.026	0.093	0.166	0.026	0.099	0.169
5	2000	0.017	0.071	0.140	0.016	0.086	0.158	0.025	0.094	0.170	0.026	0.094	0.169
10	500	0.025	0.090	0.156	0.038	0.123	0.210	0.044	0.138	0.232	0.034	0.133	0.222
10	1000	0.032	0.096	0.164	0.035	0.118	0.203	0.036	0.130	0.220	0.044	0.139	0.225
10	2000	0.026	0.089	0.153	0.039	0.119	0.196	0.040	0.129	0.210	0.053	0.148	0.231
20	1000	0.035	0.102	0.170	0.054	0.166	0.257	0.073	0.192	0.293	0.080	0.214	0.314
20	2000	0.040	0.114	0.188	0.065	0.169	0.259	0.078	0.201	0.293	0.086	0.215	0.313

Table C.6: Rejection frequencies: misspecified GARCH equation. The data is generated from a CEC-GJR-GARCH with the GJR-GARCH equation parameterised such that persistence is 0.95 and kurtosis is 4 (Panel A) and 6 (Panel B), and equicorrelation coefficient of 0.33. The test ignores the asymmetric GJR-component. 2500 replications.

		Order 1			Order 2			Order 3			Order 4		
N	T	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%
						Panel A							
2	500	0.020	0.078	0.138	0.022	0.096	0.168	0.028	0.102	0.183	0.033	0.111	0.183
2	1000	0.019	0.064	0.120	0.021	0.077	0.135	0.026	0.092	0.155	0.024	0.086	0.156
2	2000	0.020	0.078	0.127	0.020	0.080	0.152	0.024	0.088	0.158	0.022	0.097	0.172
5	500	0.044	0.118	0.190	0.066	0.166	0.264	0.080	0.190	0.279	0.084	0.198	0.298
5	1000	0.036	0.107	0.182	0.048	0.150	0.232	0.074	0.183	0.270	0.078	0.192	0.291
5	2000	0.027	0.109	0.187	0.047	0.142	0.241	0.059	0.168	0.274	0.072	0.185	0.293
10	500	0.066	0.164	0.233	0.108	0.246	0.345	0.137	0.287	0.391	0.145	0.306	0.416
10	1000	0.063	0.151	0.236	0.094	0.234	0.338	0.127	0.274	0.380	0.134	0.304	0.414
10	2000	0.061	0.152	0.227	0.084	0.213	0.303	0.110	0.250	0.361	0.137	0.290	0.395
20	1000	0.091	0.192	0.272	0.152	0.304	0.406	0.200	0.366	0.468	0.236	0.412	0.511
20	2000	0.097	0.194	0.267	0.158	0.301	0.398	0.196	0.358	0.472	0.229	0.406	0.526
						Panel B							
2	500	0.021	0.082	0.140	0.026	0.102	0.184	0.032	0.112	0.194	0.036	0.120	0.197
2	1000	0.020	0.069	0.124	0.023	0.082	0.146	0.029	0.102	0.164	0.029	0.100	0.170
2	2000	0.022	0.083	0.138	0.023	0.090	0.158	0.029	0.106	0.173	0.027	0.111	0.189
5	500	0.048	0.124	0.202	0.074	0.196	0.286	0.093	0.216	0.317	0.103	0.231	0.334
5	1000	0.040	0.119	0.199	0.062	0.170	0.252	0.089	0.208	0.305	0.100	0.225	0.321
5	2000	0.033	0.123	0.201	0.058	0.165	0.266	0.076	0.198	0.304	0.088	0.219	0.326
10	500	0.078	0.179	0.250	0.131	0.277	0.381	0.172	0.329	0.439	0.176	0.358	0.480
10	1000	0.075	0.175	0.257	0.122	0.271	0.374	0.163	0.314	0.422	0.180	0.350	0.468
10	2000	0.074	0.171	0.252	0.105	0.246	0.343	0.138	0.298	0.411	0.174	0.336	0.457
20	1000	0.108	0.220	0.308	0.193	0.344	0.450	0.250	0.424	0.532	0.301	0.485	0.596
20	2000	0.116	0.223	0.304	0.194	0.341	0.437	0.245	0.426	0.531	0.300	0.485	0.591

Table C.7: Rejection frequencies: misspecified GARCH equation. The data is generated from a CEC-GJR-GARCH with the GJR-GARCH equation parameterised such that persistence is 0.95 and kurtosis is 4 (Panel A) and 6 (Panel B), and equicorrelation coefficient of 0.67. The test ignores the asymmetric GJR-component. 2500 replications.

		Order 1			Order 2			Order 3			Order 4		
N	T	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%
						Panel A							
2	500	0.016	0.058	0.107	0.016	0.070	0.122	0.014	0.075	0.130	0.020	0.073	0.134
2	1000	0.011	0.049	0.095	0.015	0.059	0.109	0.016	0.061	0.118	0.016	0.068	0.116
2	2000	0.011	0.054	0.108	0.012	0.065	0.117	0.013	0.061	0.117	0.013	0.068	0.121
5	500	0.024	0.098	0.165	0.044	0.127	0.213	0.056	0.158	0.240	0.065	0.170	0.259
5	1000	0.027	0.090	0.167	0.039	0.121	0.206	0.053	0.155	0.235	0.060	0.164	0.249
5	2000	0.027	0.100	0.160	0.036	0.108	0.197	0.042	0.134	0.226	0.052	0.154	0.243
10	500	0.024	0.088	0.158	0.040	0.125	0.215	0.058	0.161	0.242	0.070	0.178	0.267
10	1000	0.029	0.094	0.167	0.044	0.119	0.200	0.049	0.151	0.229	0.059	0.172	0.256
10	2000	0.030	0.098	0.166	0.038	0.121	0.198	0.044	0.137	0.226	0.056	0.154	0.234
20	1000	0.022	0.086	0.152	0.026	0.107	0.183	0.038	0.120	0.206	0.047	0.144	0.229
20	2000	0.023	0.088	0.162	0.034	0.118	0.195	0.038	0.130	0.228	0.044	0.145	0.238
						Panel B							
2	500	0.016	0.072	0.141	0.022	0.076	0.144	0.022	0.081	0.142	0.027	0.092	0.150
2	1000	0.017	0.065	0.125	0.024	0.079	0.142	0.028	0.092	0.154	0.035	0.095	0.159
2	2000	0.022	0.080	0.146	0.020	0.095	0.159	0.026	0.100	0.172	0.029	0.106	0.184
5	500	0.064	0.180	0.262	0.117	0.262	0.356	0.156	0.305	0.413	0.198	0.340	0.450
5	1000	0.081	0.186	0.266	0.134	0.271	0.365	0.168	0.320	0.433	0.206	0.362	0.465
5	2000	0.090	0.194	0.288	0.138	0.277	0.385	0.185	0.345	0.454	0.224	0.390	0.505
10	500	0.074	0.191	0.292	0.137	0.283	0.394	0.180	0.350	0.460	0.214	0.404	0.519
10	1000	0.080	0.211	0.303	0.156	0.313	0.414	0.208	0.378	0.491	0.236	0.426	0.534
10	2000	0.098	0.227	0.327	0.172	0.332	0.432	0.223	0.414	0.530	0.276	0.474	0.591
20	1000	0.078	0.196	0.285	0.132	0.293	0.395	0.175	0.358	0.486	0.220	0.412	0.527
20	2000	0.089	0.208	0.312	0.143	0.306	0.428	0.199	0.375	0.493	0.254	0.443	0.566

Table C.8: Rejection frequencies: misspecified error distribution. The data is generated from a CEC-GARCH with the equicorrelation coefficient of 0.33. The errors are t-distributed with $df = 8$ (Panel A) and $df = 5$ (Panel B). 2500 replications.

		Order 1			Order 2			Order 3			Order 4		
N	T	1%	5%	10%	1%	5%	10%	1%	5%	10%	1%	5%	10%
Panel A													
2	500	0.026	0.078	0.144	0.033	0.100	0.161	0.036	0.107	0.180	0.042	0.113	0.185
2	1000	0.019	0.070	0.136	0.025	0.089	0.150	0.025	0.095	0.164	0.031	0.107	0.179
2	2000	0.020	0.072	0.140	0.025	0.086	0.152	0.028	0.091	0.163	0.031	0.105	0.175
5	500	0.040	0.115	0.190	0.064	0.174	0.258	0.087	0.201	0.288	0.100	0.220	0.325
5	1000	0.033	0.116	0.186	0.051	0.155	0.241	0.077	0.196	0.289	0.094	0.211	0.307
5	2000	0.035	0.117	0.190	0.051	0.152	0.240	0.063	0.176	0.269	0.076	0.203	0.304
10	500	0.031	0.100	0.184	0.054	0.154	0.240	0.076	0.192	0.286	0.096	0.218	0.324
10	1000	0.033	0.108	0.177	0.046	0.146	0.230	0.066	0.178	0.265	0.086	0.212	0.307
10	2000	0.039	0.118	0.206	0.052	0.160	0.246	0.058	0.174	0.277	0.076	0.193	0.290
20	1000	0.029	0.100	0.168	0.039	0.124	0.210	0.047	0.151	0.236	0.058	0.174	0.275
20	2000	0.021	0.092	0.170	0.035	0.123	0.214	0.050	0.156	0.251	0.056	0.173	0.269
Panel B													
2	500	0.038	0.118	0.190	0.056	0.153	0.230	0.065	0.168	0.251	0.080	0.191	0.276
2	1000	0.036	0.110	0.181	0.063	0.152	0.230	0.082	0.181	0.272	0.092	0.203	0.287
2	2000	0.053	0.134	0.210	0.065	0.171	0.257	0.084	0.202	0.286	0.100	0.214	0.320
5	500	0.104	0.228	0.324	0.176	0.344	0.446	0.229	0.400	0.510	0.273	0.448	0.565
5	1000	0.112	0.244	0.337	0.199	0.354	0.464	0.255	0.426	0.530	0.303	0.474	0.592
5	2000	0.128	0.261	0.356	0.197	0.359	0.480	0.262	0.451	0.561	0.332	0.531	0.633
10	500	0.103	0.238	0.338	0.190	0.349	0.462	0.240	0.432	0.544	0.304	0.494	0.605
10	1000	0.118	0.256	0.373	0.220	0.388	0.498	0.283	0.460	0.573	0.334	0.521	0.631
10	2000	0.145	0.278	0.380	0.235	0.398	0.512	0.305	0.492	0.599	0.365	0.579	0.677
20	1000	0.111	0.246	0.338	0.193	0.354	0.464	0.251	0.424	0.544	0.310	0.484	0.598
20	2000	0.110	0.250	0.356	0.193	0.376	0.492	0.260	0.464	0.585	0.328	0.545	0.657

Table C.9: Rejection frequencies: misspecified error distribution. The data is generated from a CEC-GARCH with the equicorrelation coefficient of 0.67. The errors are t-distributed with $df = 8$ (Panel A) and $df = 5$ (Panel B). 2500 replications.

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