Asymptotics of partial sums of linear processes with changing memory parameter*

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Abstract

We study the limit distribution of partial sums of nonstationary linear process $\{X_t, t = 1, \dots, n\}$ with long memory and changing memory parameter $d_{t,n} \in (0, \infty)$. Two classes of linear processes are investigated, namely, (I) the class of FARIMA-type truncated moving averages with time-varying fractional integration parameter and (II) the class of time-varying fractionally integrated processes introduced in Philippe et al. (2006, 2008). The cases of fast changing memory parameter ($d_{t,n} = d_t$ does not depend on n) and slowly changing memory parameter ($d_{t,n} = d(t/n)$ for some function $d(\tau), \tau \in [0, 1]$) are discussed. In the case of fast changing memory, the limit partial sums process is a type II fractional Brownian motion (fBm) with the Hurst parameter equal to the global maximum of (d_t) for class (I) processes, and the mean value of (d_t) for class (II) processes. In the case of slowly changing memory, the limit of partial sums for both classes (I) and (II) is degenerated and "localized" at the global maximum of the memory function $d(\cdot)$; however, a nondegenerate limit of the partial sums process is shown to exist when time is suitably rescaled in the vicinity of the maximum point.

Keywords: Linear process; Time-varying memory; Partial sums' limits; Type II fractional Brownian motion.

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1 Introduction

Let $\{X_1, \dots, X_n\}$ be a given sample from discrete time series $\{X_t, t \in \mathbb{Z}\}$. The partial sums process is usually defined as the process

$$S_n(\tau) = \sum_{t=1}^{\lfloor n\tau \rfloor} X_t$$

indexed by $\tau \in [0, 1]$, where $\lfloor a \rfloor$ is the largest integer less or equal to $a \in \mathbb{R}$. It is well-known that the range and fluctuations of partial sums as $n \to \infty$ can discriminate between stationary and nonstationary behavior of the time series, characterize the intensity of long memory, the presence of trends, change-points and other features of interest. Finding the asymptotic distribution of partial sums is important for time series analysis and inference. Examples and applications of partial sums limits can be found in the monograph Whitt (2002).

An important class of stationary time series models form causal linear processes

$$X_t = \sum_{s \le t} \psi_{t-s} \zeta_s, \tag{1.1}$$

where $\psi_j, j \ge 0$ are moving average coefficients, $\sum_{j=0}^{\infty} \psi_j^2 < \infty$, and $\{\zeta_s\}$ are standardized uncorrelated random variables (weak white noise). It is well-known that every zero-mean covariance stationary process $\{X_t\}$ with spectral density $f(x)$ satisfying $\int_{-\pi}^{\pi} \log f(x) \mathrm{d}x > -\infty$ admits (Wold's) representation (1.1). The covariance structure of (1.1) is determined by the moving average coefficients, $\psi_j, j \ge 0$. In particular, if the ψ_j's decay regularly as j^{d-1} for some $0 < d < 1/2$, the covariance function of $\{X_t\}$ in (1.1) is nonsummable (decays as j^{2d-1}), meaning that (1.1) has long memory and d is called the long memory parameter of $\{X_t\}$ (Giraitis et al. (2012)). In addition, if innovations $\{\zeta_s\}$ are i.i.d.$(0, 1)$, the normalized partial sums process of $\{X_t\}$ in (1.1) tends to a fractional Brownian motion with Hurst parameter $H = d + 1/2$, see Davydov (1970), also Giraitis et al. (2012, Proposition 4.4.4).

Various economic and physical time series indicate that the long memory parameter in real data may change with time. The natural question arises to model changing long memory and construct various inferential procedures for such models. Testing for a change or nonconstancy in long memory parameter was discussed in Beran and Terrin (1996), Horváth and Shao (1999), Sibbertsen and Kruse (2009), Yamaguchi (2011), Lavancier et al. (2012) and other papers.

The present paper discusses partial sums' limits of nonstationary linear processes $\{X_t\}$ with changing memory intensity. Two classes of nonstationary "time-varying" generalizations of (1.1) are discussed. The first class is obtained by taking a parametric class $\{\psi_j(d), j = 0, 1, \dots, d \in (0, \infty)\}$ of moving average coefficients such that for any $d \in (0, \infty)$ we have

$$\psi_j(d) \sim \kappa(d) j^{d-1} \qquad (j \to \infty) \tag{1.2}$$

for some $\kappa(d) > 0$, and replacing a constant value d by a function $d_t = d_{t,n}$ of $t \in \{1, \ldots, n\}$ (possibly depending also on n), viz.,

$$X_t := \sum_{s=1}^{t} \psi_{t-s}(d_t)\zeta_s, \qquad t = 1, \ldots, n \tag{1.3}$$

with $\{\zeta_s, s \in \mathbb{Z}\}$ being a sequence of martingale differences with zero mean and unit variance. A typical example of the above parametric class is FARIMA$(0, d, 0)$ corresponding to

$$\psi_j(d) = \frac{d}{1} \cdot \frac{d+1}{2} \cdots \frac{d-1+j}{j} = \frac{\Gamma(d+j)}{j!\Gamma(d)} \quad (j \geq 1), \qquad \psi_0(d) = 1. \tag{1.4}$$

The fact that the moving average representation in (1.3) is truncated at negative $s \leq 0$ is not very important for our discussion, provided $d_t \in (0, 1/2)$. However, truncated FARIMA$(0, d, 0)$ series exist for all $d \in \mathbb{R}$ and the limit of their partial sums (a type II fractional Brownian motion) is well-defined for any $d > -.5$ (Marinucci and Robinson, 1999). Moreover, the truncation in (1.3) seems rather natural in the context of changing d since otherwise the limit behavior of partial sums depends on the behavior of the d_t's as $t \to -\infty$ and not only on its values in the interval $1 \leq t \leq n$. See also Marinucci and Robinson (1999) for econometric considerations in favor of the truncated series. Davidson and Hashimzade (2009) discuss the differences between the distributions of type I and type II fractionally integrated processes and the importance to distinguish whether the pre-sample shocks are included in the lag structure of the model, or suppressed, as this can lead to a significant distortion of the limiting distribution.

The second class of time-varying linear processes with changing memory parameter was defined in Philippe et al. (2006, 2008):

$$X_t^a = \sum_{s=1}^{t} a_{t-s}(t)\zeta_s, \qquad X_t^b = \sum_{s=1}^{t} b_{t-s}(t)\zeta_s, \tag{1.5}$$

where

$$\begin{aligned} a_j(t) &:= \frac{d_{t-1}}{1} \cdot \frac{d_{t-2}+1}{2} \cdot \frac{d_{t-3}+2}{3} \cdots \frac{d_{t-j}-1+j}{j}, \\ b_j(t) &:= \frac{d_{t-1}}{1} \cdot \frac{d_{t-j}+1}{2} \cdot \frac{d_{t-j+1}+2}{3} \cdots \frac{d_{t-2}-1+j}{j}, \qquad j \geq 1, \end{aligned} \tag{1.6}$$

$a_0(t) = b_0(t) := 1$, are defined for $t = 1, 2, \ldots, j = 0, 1, \ldots, t-1$ and a given sequence $\{d_t\} = \{d_t,\, t = 1, 2, \ldots\}$ of real numbers. In the case when $d_t \equiv d$ are constants, the coefficients $a_j(t)$ and $b_j(t)$ in (1.6) coincide with FARIMA$(0, d, 0)$ coefficients in (1.4). Similarly to (1.4), they satisfy the following orthogonality relations

$$\sum_{j=0}^{n} b_j^-(t) a_{n-j}(t-j) = 0, \qquad t, n = 1, 2, \ldots,\ n < t,$$

where $b^-_j(t) := (-1)^j(\frac{d_{t-1}}{j!})\prod_{k=1}^{j-1}(d_{t-k-1} - j + k),\ j \ge 1$ are defined as in (1.6) with $\{d_t,\, t = 1, 2, \dots\}$ replaced by $\{-d_t,\, t = 1, 2, \dots\}$.

Partial sums' limits of time-varying fractionally integrated processes in (1.5) were discussed in Philippe et al. (2006, 2008), Bružaitė et al. (2007), Doukhan et al. (2007), Lavancier et al. (2012), albeit most these papers focused on "nontruncated" versions of (1.5) and "fast changing" memory parameter $\{d_t\}$ (see below). Before proceeding further, let us note an important distinction between the filter coefficients $\psi_j(d_t)$ and $a_j(t), b_j(t)$ in (1.6) (corresponding to the same sequence $\{d_t\}$): when time changes $t \to t+1$, this evokes an "instantaneous" change $\psi_j(d_t) \to \psi_j(d_{t+1})$ of all filter coefficients including the decay rate as $j \to \infty$, leading to an "instantaneous change" of the memory intensity of the series (1.3) at the next time moment $t + 1$. On the other hand, a similar change $t \to t + 1$ affects the filter coefficients in (1.6) in a much lesser way and does not essentially alter their asymptotic behavior as $j \to \infty$ (see Sec. 2). These differences between filter coefficients are reflected in a different asymptotic behavior of partial sums for models in (1.3) and in (1.5).

Let us describe the main results of the paper. For models in (1.3) and (1.5) with changing memory parameter we study two characteristic situations of how fast a change occurs. If the d_t's for $t = 1, \dots, n$ do not depend on n, we say that the corresponding series $\{X_t, t = 1, \dots, n\}$ has *fast changing memory*. On the other hand, if the d_t's have the form $d_t = d(t/n)$ for $t = 1, \dots, n$ where $d(\tau), \tau \in [0, 1]$ is a given function, we say that $\{X_t, t = 1, \dots, n\}$ has *slowly changing memory.*

Sec. 2 discusses the case of fast changing memory. Theorem 4 states that if the sequence $\{d_t\}$ admits a Cesaro mean

$$\bar d := \lim_{n\to\infty} n^{-1}\sum_{t=1}^{n} d_t \ \in\ (0,\infty), \tag{1.7}$$

and satisfies some additional technical conditions, then partial sums of X^a_t and X^b_t in (1.5), normalized by $n^{\bar d+1/2}$, tend to a multiple of the Gaussian process

$$J_{\bar d}(\tau) := \int_0^\tau (\tau - x)^{\bar d}\, W(\mathrm{d}x), \quad \tau \in [0, 1], \tag{1.8}$$

where $\{W(\mathrm{d}x), x \in [0, 1]\}$ is a standard Gaussian white noise with zero mean and variance $\mathrm{d}x$. The process in (1.8) represents the "rough part" of a fractional Brownian motion (fBm) and is also called a *type II fBm* (Marinucci and Robinson, 1999). Theorem 4 below extends some results in Philippe et al. (2006, 2008), Bružaitė et al. (2007), Doukhan et al. (2007) and Lavancier et al. (2012).

Theorem 2 refers to model (1.3). In this case, the role of the asymptotic parameter $\bar d$ in the previous theorem is played by

$$d_+ := \limsup_{t\to\infty} d_t \ \in\ (0,\infty). \tag{1.9}$$

We also assume that $d_t < d_+$, for all $t \in \mathbb{N}^* = \{1, 2, \ldots\}$, and the existence of a properly normalized limit of the tail empirical process of the sequence $\{d_t\}$. In particular, these assumptions are satisfied a.s. for a realization of an i.i.d. sequence $\{d_t\}$ of r.v.'s taking values in an interval $[d_-, d_+) \subset (0,1)$ and having a continuous and positive probability density at d_+. We show that in a such case, the partial sums process of (1.3) converges to J_{d_+} at a slower normalization $n^{d_+ + 1/2}/\log n$. See Sec. 2 and Theorem 2 for precise formulations and details.

Sec. 3 discusses the case of slowly changing memory parameter $d_t = d(t/n)$, according to a given "memory function" $d(\tau), \tau \in [0,1]$ taking values in the interval $(0, 1/2)$. It is clear that in this case, the behavior and normalization of the partial sums process is determined by the behavior of the function $d(\cdot)$ at the maximum point $\tau_{\max} := \mathrm{argmax}(d(\tau) : \tau \in [0,1])$. In this paper we restrict ourselves to a model situation where the maximum is unique and the function $d(\cdot)$ takes a power form in its neighborhood: $d(\tau_{\max} \pm u) = d_{\max} - (\Delta_\pm + o(1))u^\gamma$ $(u \downarrow 0)$ for some $\Delta_\pm \ge 0, \gamma > 0, \tau_{\max} \in (0,1)$. Corollary 8 says that in such case, the partial sums process of $\{X_t\}$ in (1.3), normalized by $\big(\frac{n}{\log^{1/\gamma} n}\big)^{d_{\max}+1/2}$, tends to degenerated Gaussian processes Z, taking constant values on intervals $[0, \tau_{\max})$ and $(\tau_{\max}, 1]$ with possible discontinuities at $\tau_{\max} \pm 0$ having an explicit stochastic integral representation. However, if time is rescaled by factor $\log^{1/\gamma} n$ near the maximum point, the above partial sums process exhibits a nondegenerate behavior, see Theorems 5 and 7, below. Similar inference holds for the partial sums process of $\{X^a_t\}$ in (1.5) (see Corollary 8 (ii) and Theorem 7). The above mentioned "localization" of the partial sums process near the maximum point $d_{\max}$ implies that the truncation of the original series at negative $s \le 0$ does not affect the limit and therefore "stationarity" condition $d_{\max} < 1/2$ is needed in this case (see also Remark 3 below).

The proofs of our results use the scheme of discrete stochastic integrals in Proposition 1 below (see Surgailis (2003)). It is a convenient application of the martingale central limit theorem in Billingsley (1968, Theorem 23.1) for weighted sums of martingale differences. See Bružaitė and Vaičiulis (2005) for an extension of this scheme to multivariate processes and more general innovations satisfying the central limit theorem.

Let $MD(0,1)$ denote the class of all standardized stationary and ergodic martingale differences $\{\zeta_s, s \in \mathbb{Z}\}$, i.e., $\mathrm{E}\zeta_s^2 = 1$, $\mathrm{E}[\zeta_s|\mathcal{F}_{s-1}] = 0$ for any $s \in \mathbb{Z}$, where $\{\mathcal{F}_s, s \in \mathbb{Z}\}$ is a nondecreasing family of $\sigma-$fields. Let be given $\{0 < m_n \to \infty, n \in \mathbb{N}^*\}$, a sequence of positive numbers tending to infinity, and $\{p_n \in \mathbb{Z}, n \in \mathbb{N}^*\}$, an arbitrary sequence of integers.

Introduce a discrete stochastic measure η_n on $\mathbb{R}$: for any bounded Borel set $B \subset \mathbb{R}$, let

$$\eta_n(B) \;:=\; m_n^{-1/2} \sum_{s \in \mathbb{Z}:\, s/m_n \in B} \zeta_{s+p_n}. \tag{1.10}$$

Observe, for any $B = (b_1, b_2] \subset \mathbb{R}$, $\mathrm{E}\eta_n^2(B) \sim b_2 - b_1 = \mathrm{leb}(B)$. From the above mentioned central limit theorem in Billingsley (1968, Theorem 23.1) it follows that for any finite union B of disjoint finite intervals, $\eta_n(B) \to_{\rm law} \eta(B)$, where η is a Gaussian white noise with zero mean and variance $\mathrm{E}(\eta(\mathrm{d}u))^2 = \mathrm{d}u$.

Let $L^2_n(\mathbb{R})$ be the class of all piecewise-constant functions $g \in L^2(\mathbb{R})$ taking constant values g_s on intervals $\big(s/m_n, (s+1)/m_n\big] \subset \mathbb{R}$, $s \in \mathbb{Z}$. A discrete stochastic integral $\int g(u)\eta_n(\mathrm{d}u)$ is defined for such $g \in L^2_n(\mathbb{R})$ by

$$\int g(u)\eta_n(\mathrm{d}u) \ := \ m_n^{-1/2} \sum_{s\in\mathbb{Z}} g_s\, \zeta_{s+p_n}.$$

Proposition 1 *Let $\{\zeta_s\} \in MD(0,1)$ be a sequence of martingale differences and let $g^{(n)} \in L^2_n(\mathbb{R})$, $n = 1, 2, \dots$ be a sequence of functions convergent to $g \in L^2(\mathbb{R})$, viz.*

$$\lim_{n\to\infty} \|g^{(n)} - g\| \ = \ 0, \tag{1.11}$$

where $\|g\|^2 = \int g^2(u)\mathrm{d}u$. Then

$$\int g^{(n)}(u)\eta_n(\mathrm{d}u) \ \to_{\rm law} \ \int g(u)\eta(\mathrm{d}u) \ \sim \ \mathcal{N}(0, \|g\|^2).$$

Notation. In what follows, $\to_{D[0,1]}$, $\to_{D(\mathbb{R})}$, etc., denote the weak convergence of random elements in the Skorokhod spaces $D[0,1]$, $D(\mathbb{R})$, etc., of cadlag functions, with the sup-topology. The weak convergence of finite-dimensional distributions is denoted by $\to_{\rm f.d.d.}$. Notation C stands for a constant whose precise value is unimportant and which may change from line to line.

2 Partial sums limits under fast changing memory

Consider first the case of nonstationary long memory linear process in (1.3). Let $v_n := \log^{-1/2} n$, $n \ge 2$.

Assumption (A.1) There exists $0 < d_+ < \infty$ such that $d_t < d_+$ for any $t = 1, 2, \dots$ and relation (1.9) holds, viz., $d_+ = \limsup_{t\to\infty} d_t$. Moreover, there exist $\beta \in (0, 1/2)$ and $c_+ > 0$ such that

$$\sup_{x\in[0,1]} \big|\Phi_n(x)\big| \ = \ O(n^{-\beta}), \quad n \to \infty, \tag{2.1}$$

where

$$\Phi_n(x) := \frac{1}{n v_n} \sum_{t=1}^{n} \big(\mathbf{1}(d_+ - x v_n < d_t < d_+) - c_+ x v_n\big), \qquad x \in [0,1]. \tag{2.2}$$

Remark 1 As noted in the Introduction, a natural example of a sequence $\{d_t\}$ satisfying Assumption (A.1) is a typical realization of a sequence of stationary r.v.'s taking values in the interval $(0, d_+)$. In such case, assuming the uniform marginal distribution on $(0, d_+)$

and $c_+ = 1/d_+$, the process Φ_n in (2.2) with general $v_n \to 0$, $nv_n \to \infty$ has zero mean and is called the (centered) *tail empirical process.* In addition, if $\{d_t\}$ are i.i.d. and $v_n = \log^{-1/2} n$, as above, then (2.1) holds a.s. with any $0 < \beta < 1/2$ (Mason (1988)). The last result is also valid under certain dependence conditions on $\{d_t\}$. See Csörgő and Horváth (1993), Drees (2002), Kulik and Soulier (2011).

Theorem 2 *Let $\{X_t\}$ be the linear process in (1.3), where $\{\zeta_s\} \in MD(0,1)$, $\{d_t\}$ satisfy Assumption (A.1) and $\psi_j(d)$ satisfy the following conditions:*

$$\frac{\psi_j(d)}{\kappa(d) j^{d-1}} \longrightarrow 1, \qquad \text{as } j \to \infty \ \text{ and } \ d \to d_+, \tag{2.3}$$

and

$$|\psi_j(d)| \le C(j+1)^{d-1}, \qquad \forall\, d \in (0,\infty),\ \forall\, j \ge 0, \tag{2.4}$$

where $C > 0$ is independent of d, j. Then

$$\frac{\log n}{n^{d_+ + 1/2}}\, S_n(\tau) \ \longrightarrow_{D[0,1]} \ c_1\, J_{d_+}(\tau), \quad \text{where} \quad c_1 := \frac{c_+ \kappa(d_+)}{d_+} \tag{2.5}$$

and where J_{d_+} is a type II fBm in (1.8).

Proof of Theorem 2. We shall restrict the proof of finite dimensional convergence in (2.5) to one-dimensional convergence at $\tau = 1$ since the general case follows analogously. To this end, using Proposition 1, it suffices to prove the convergence

$$\begin{aligned} g^{(n)}(z) \ &:= \ \frac{\log n}{n^{d_+}} \sum_{t=\lfloor zn \rfloor}^{n} \psi_{t-\lfloor nz \rfloor}(d_t) \\ &\to \ \kappa(d_+) c_+ \int_z^1 (y-z)^{d_+ - 1} \mathrm{d}y \ = \ c_1 (1-z)^{d_+} =: g(z) \end{aligned} \tag{2.6}$$

in $L^2[0,1]$. Split $g^{(n)}(z) = g_1^{(n)}(z) + g_2^{(n)}(z)$, where

$$g_1^{(n)}(z) \ := \ \frac{\log n}{n^{d_+}} \sum_{t=\lfloor zn \rfloor}^{n} \psi_{t-\lfloor zn \rfloor}(d_t) \mathbf{1}(d_+ - v_n < d_t < d_+), \quad g_2^{(n)}(z) := g^{(n)}(z) - g_1^{(n)}(z).$$

From (2.4) we obtain

$$\begin{aligned} |g_2^{(n)}(z)| \ &\le \ \frac{C \log n}{n^{d_+}} \sum_{t=\lfloor zn \rfloor}^{n} (t - \lfloor zn \rfloor + 1)^{d_+ - v_n - 1} \\ &\le \ \frac{C \log n}{n^{d_+}}\, n^{d_+ - v_n} \\ &\le \ C \mathrm{e}^{\log\log n - v_n \log n} = o(1) \end{aligned} \tag{2.7}$$

uniformly in $z \in [0,1]$. Consequently, it remains to show (2.6) for $g_1^{(n)}(z)$ instead of $g^{(n)}(z)$.

Consider first the above convergence for $\psi_j(d) \equiv \tilde\psi_j(d) := \kappa(d_+)(j+1)^{d-1}$, i.e.,

$$\tilde g_1^{(n)}(z) \quad := \quad \frac{\kappa(d_+)\log n}{n^{d_+}} \sum_{t=\lfloor zn\rfloor}^{n} (t-\lfloor zn\rfloor+1)^{d_t-1}\mathbf{1}(d_+-v_n<d_t<d_+), \tag{2.8}$$

or the main term according to the asymptotics in (1.2). To this end, rewrite $\tilde g_1^{(n)}(z) = h_n(z) + e_n(z)$, where

$$h_n(z) \quad := \quad \frac{c_+\kappa(d_+)\log n}{n^{d_+}} \sum_{t=\lfloor zn\rfloor}^{n} \int_0^{v_n} (t-\lfloor zn\rfloor+1)^{d_+-x-1}\mathrm{d}x, \quad e_n(z) \quad := \quad \tilde g_{n1}(z) - h_n(z).$$

We shall prove that

$$e_n(z) \quad \to \quad 0 \quad \text{and} \quad h_n(z) \quad \to \quad g(z) \quad \text{in } L^2[0,1]. \tag{2.9}$$

Let us check first that $h_n(z)$ is bounded on $(0,1)$ uniformly in n. Indeed, let $k = n - \lfloor nz\rfloor + 1$ and let z be such that $k \ge 2$ (the case $k = 1$ is obvious). Then

$$\begin{aligned} h_n(z) \quad &\le \quad \frac{C\log n}{n^{d_+}} \int_0^{v_n} k^{d_+-x}\mathrm{d}x \\ &= \quad C\Big(\frac{k}{n}\Big)^{d_+} \frac{\log n}{\log k}\,\big(1-k^{-v_n}\big) \\ &\le \quad C\Big(\frac{k}{n}\Big)^{d_+} \frac{\log n}{\log k} \ \le \ C \end{aligned}$$

since $d_+ > 0$ and $x \mapsto x^{d_+}/\log x$ is monotone increasing on $[x_0,\infty)$, $x_0 = \mathrm{e}^{1/d_+}$. Next, for any $0<z<1$ fixed, using the fact that $(sn-\lfloor zn\rfloor+1)^{-v_n} \to 0$ for any fixed $s \in (z,1)$, we obtain

$$\begin{aligned} h_n(z) \quad &= \quad o(1) + \frac{c_+\kappa(d_+)\log n}{n^{d_+}} \sum_{t=\lfloor zn\rfloor+1}^{n} (t-\lfloor zn\rfloor+1)^{d_+-1} \int_0^{v_n} (t-\lfloor zn\rfloor+1)^{-x}\mathrm{d}x \\ &= \quad o(1) + c_+\kappa(d_+) \sum_{t=\lfloor zn\rfloor+1}^{n} \frac{\log n}{\log(t-\lfloor zn\rfloor+1)} \frac{(t-\lfloor zn\rfloor+1)^{d_+-1}}{n^{d_+}} \big(1-(t-\lfloor zn\rfloor+1)^{-v_n}\big) \\ &\sim \quad c_+\kappa(d_+) \int_z^1 \frac{\log n}{\log(sn-\lfloor zn\rfloor+1)} \frac{(sn-\lfloor zn\rfloor+1)^{d_+-1}}{n^{d_+-1}} \big(1-(sn-\lfloor zn\rfloor+1)^{-v_n}\big)\mathrm{d}s \\ &\sim \quad c_+\kappa(d_+) \int_0^{1-z} \frac{\log n}{\log n + \log x}\, x^{d_+-1}\,\mathrm{d}x \\ &\sim \quad \frac{c_+\kappa(d_+)}{d_+}\,(1-z)^{d_+} \ = \ g(z). \end{aligned}$$

This proves the second relation in (2.9).

Next, let us estimate the remainder term, $e_n(z)$, using condition (2.1). To simplify the notation, assume that $\kappa(d_+) = 1$. We first rewrite $e_n(z)$ as

$$
\begin{aligned}
e_n(z) &= \frac{\log n}{n^{d_+}} \sum_{t=\lfloor zn \rfloor}^{n} \theta_{n,t}(z), \quad \text{where} \\
\theta_{n,t}(z) &:= (t - \lfloor zn \rfloor + 1)^{d_t - 1} \mathbf{1}(d_+ - v_n < d_t < d_+) - c_+ v_n \int_0^1 (t - \lfloor zn \rfloor + 1)^{d_+ - y v_n - 1} \mathrm{d}y.
\end{aligned}
$$

Introduce

$$
\Phi_n(x;t) := \frac{1}{n v_n} \sum_{k=1}^{t} (\mathbf{1}(d_+ - v_n x < d_k < d_+) - c_+ v_n x), \quad x \in [0,1], \quad t = 1, \ldots, n.
$$

Note that $\Phi_n(x;t) = \frac{t v_t}{n v_n} \Phi_t\big(\frac{x v_n}{v_t}\big)$ and therefore uniformly in $n \ge 2$

$$
\sup_{x \in [0,1]} \big| \Phi_n(x;t) \big| \; \le \; C t^{-\beta}, \tag{2.10}
$$

according to (2.1) and the fact that $n v_n$ increases with n. Moreover,

$$
\sup_{x \in [0,1]} \big| \Phi_n(x;t) - \Phi_n(x;t+1) \big| \le \frac{C}{n v_n}, \qquad t = 1, \ldots, n-1. \tag{2.11}
$$

Observe $(t - \lfloor zn \rfloor + 1)^{d_t - 1} \mathbf{1}(d_+ - v_n < d_t < d_+) = \int_0^1 (t - \lfloor zn \rfloor + 1)^{d_+ - y v_n - 1} \mathrm{d}_y \mathbf{1}(d_+ - y v_n < d_t < d_+)$. Therefore integrating by parts we obtain

$$
\theta_{n,t}(z) \; = \; n v_n \int_0^1 (t - \lfloor zn \rfloor + 1)^{d_+ - y v_n - 1} \mathrm{d}_y \big[\Phi_n(y;t) - \Phi_n(y;t-1) \big] = \theta'_{n,t}(z) + \theta''_{n,t}(z),
$$

where

$$
\begin{aligned}
\theta'_{n,t}(z) &:= n v_n (t - \lfloor zn \rfloor + 1)^{d_+ - v_n - 1} \big[\Phi_n(1;t) - \Phi_n(1;t-1) \big], \\
\theta''_{n,t}(z) &:= v_n \log(t - \lfloor zn \rfloor + 1) \int_0^1 (t - \lfloor zn \rfloor + 1)^{d_+ - y v_n - 1} [\Phi_n(y;t) - \Phi_n(y;t-1)] \mathrm{d}y.
\end{aligned}
$$

Using (2.11) and (2.7), we obtain

$$
\begin{aligned}
|e'_n(z)| &:= \frac{\log n}{n^{d_+}} \Big| \sum_{t=\lfloor zn \rfloor}^{n} \theta'_{n,t}(z) \Big| \; \le \; \frac{C \log n}{n^{d_+}} \sum_{t=\lfloor zn \rfloor}^{n} (t - \lfloor zn \rfloor + 1)^{d_+ - v_n - 1} \\
&\le \; \frac{C \log n}{n^{d_+}} n^{d_+ - v_n} \; = \; O\Big(\frac{\log n}{n^{v_n}} \Big) \; = \; o(1),
\end{aligned} \tag{2.12}
$$

uniformly in $0 < z < 1$.

Next, we estimate $e''_n(z) := \frac{\log n}{n^{d_+}} \sum_{t=\lfloor zn \rfloor}^{n} \theta''_{n,t}(z)$. Split $e''_n(z) = e''_{n1}(z) + e''_{n2}(z)$, where

$$
e''_{n1}(z) := \frac{\log n}{n^{d_+}} \sum_{t=\lfloor zn \rfloor}^{\lfloor zn \rfloor + \lfloor n^{1-\beta} v_n \rfloor} \theta''_{n,t}(z), \qquad e''_{n2}(z) := \frac{\log n}{n^{d_+}} \sum_{t=\lfloor zn \rfloor + \lfloor n^{1-\beta} v_n \rfloor + 1}^{n} \theta''_{n,t}(z).
$$

Using (2.11), we estimate

$$
\begin{aligned}
|e''_{n1}(z)| &\leq \frac{C\log n}{n^{d_+}} \sum_{t=\lfloor zn\rfloor}^{\lfloor zn\rfloor+\lfloor n^{1-\beta}v_n\rfloor} v_n \log(t-\lfloor zn\rfloor+1)\int_0^1 (t-\lfloor zn\rfloor+1)^{\bar{d}_+-yv_n-1}\mathrm{d}y \\
&\leq \frac{C\log n}{n^{d_+}} \sum_{s=1}^{\lfloor n^{1-\beta}v_n\rfloor} v_n \log s \int_0^1 s^{d_+-yv_n-1}\mathrm{d}y \\
&\leq \frac{C\log n}{n^{d_+}}\,(n^{1-\beta}v_n)^{d_+} = O\big((v_n n^{-\beta})^{d_+}\log n\big) \;=\; o(1) \qquad (2.13)
\end{aligned}
$$

uniformly in $z \in (0,1)$, since $\beta > 0$, $d_+ > 0$ and $v_n = \log^{-1/2} n$.

To estimate $e''_{n2}(z)$, we sum by parts over t and then write

$$
e''_{n2}(z) = \frac{\log n}{n^{d_+}}\, n v_n\big(\imath_{n1}(z) + \imath_{n2}(z)\big), \qquad (2.14)
$$

where

$$
\begin{aligned}
\imath_{n1}(z) &:= v_n \log(t-\lfloor zn\rfloor+1)\int_0^1 (t-\lfloor zn\rfloor+1)^{d_+-yv_n-1}\Phi_n(y;t)\mathrm{d}y\Big|_{t=\lfloor zn\rfloor+\lfloor n^{1-\beta}v_n\rfloor}^{t=n}, \\
\imath_{n2}(z) &:= \sum_{t=\lfloor zn\rfloor+\lfloor n^{1-\beta}v_n\rfloor}^{n-1} v_n \int_0^1 \big[(t-\lfloor zn\rfloor+1)^{d_+-yv_n-1}\log(t-\lfloor zn\rfloor+1) \\
&\qquad -(t+\lfloor zn\rfloor+2)^{d_+-yv_n-1}\log(t-\lfloor zn\rfloor+2)\big]\,\Phi_n(y;t)\mathrm{d}y
\end{aligned}
$$

and where we use the notation $g(t)|_{t=a}^{t=b} := g(b) - g(a)$.

Using (2.10) we obtain

$$
\begin{aligned}
|\imath_{n1}(z)| &\leq C(n-\lfloor zn\rfloor+1)^{d_+-1}(1-(n-\lfloor zn\rfloor+1)^{-v_n})n^{-\beta} \\
&\quad + C(\lfloor n^{1-\beta}v_n\rfloor+1)^{d_+-1}(1-(n-\lfloor n^{1-\beta}v_n\rfloor+1)^{-v_n})(\lfloor zn\rfloor+\lfloor n^{1-\beta}v_n\rfloor)^{-\beta} \\
&\leq C(nz)^{-\beta}(n^{1-\beta}v_n)^{d_+-1}. \qquad (2.15)
\end{aligned}
$$

Next, using (2.10) and $x^{d-1} - (x+1)^{d-1} \leq x^{d-2}$, $|x^{d-1}\log x - (x+1)^{d-1}\log(x+1)| \leq Cx^{d-2}\log x$, for $x \geq 2$, we obtain

$$
\begin{aligned}
|\imath_{n2}(z)| &\leq C \sum_{t=\lfloor zn\rfloor+\lfloor n^{1-\beta}v_n\rfloor}^{n-1} t^{-\beta} v_n \log(t-\lfloor zn\rfloor+1)\int_0^1 (t-\lfloor zn\rfloor+1)^{d_+-yv_n-2}\mathrm{d}y \\
&\leq C(\lfloor zn\rfloor+1)^{-\beta} \sum_{t=\lfloor zn\rfloor+\lfloor n^{1-\beta}v_n\rfloor}^{n-1} (t-\lfloor zn\rfloor+1)^{d_+-2} \\
&\leq C(nz)^{-\beta}\begin{cases} (n^{1-\beta}v_n)^{d_+-1}, & \text{if } \; 0 < d_+ < 1, \\ \log n & \text{if } \; d_+ = 1, \\ n^{d_+-1}, & \text{if } \; d_+ > 1. \end{cases} \qquad (2.16)
\end{aligned}
$$

From (2.13), (2.14), (2.15), (2.16) we obtain

$$|e''_n(z)| \quad \le \quad C\epsilon(n) z^{-\beta}, \tag{2.17}$$

where $\epsilon(n) := (\frac{v_n}{n^\beta})^{d_+\wedge 1} \log^2 n \to 0$ does not depend on $z \in (0,1)$. Clearly, (2.12) and (2.17) imply the first convergence (2.9) in view of the fact that $0 < \beta < 1/2$. This proves (2.9) and the convergence in (2.6) for $g^{(n)}(z)$ replaced by $\tilde{g}_1^{(n)}(z)$ in (2.8).

To complete the proof of (2.6), it remains to prove that $u_n(z) := g_1^{(n)}(z) - \tilde{g}_1^{(n)}(z) \to 0$ in $L^2[0,1]$. We have

$$u_n(z) \quad := \quad \frac{\log n}{n^{\bar{d}_+}} \sum_{t=\lfloor zn \rfloor}^{n} \{\psi_{t-\lfloor zn \rfloor}(d_t) - \kappa(\bar{d}_+)(t - \lfloor zn \rfloor + 1)^{d_t - 1}\} \mathbf{1}(\bar{d}_+ - v_n < d_t < \bar{d}_+).$$

From the discussion above, it follows the dominating bound $|\tilde{g}_1^{(n)}(z)| \le \bar{g}(z)$, where $\bar{g}(z) := Cz^{-\beta}$ belongs to $L^2[0,1]$. In view of (2.4), a similar bound holds for $|g_1^{(n)}(z)|$ and hence for $|u_n(z)|$ as well. Consequently, it suffices to show that $u_n(z) \to 0$ for a.e. $z \in (0,1)$. The last fact can be proved using condition (2.3) and a standard argument using the dominated convergence theorem. This ends the proof of Theorem 2. □

Next, we discuss the case of time-varying fractionally integrated processes in (1.5). Following Bružaitė et al. (2007) we introduce the following definitions.

Definition 3 *A bounded sequence $\{d_t, t = 1, 2, \dots\}$ of real numbers will be called:*

(i) Averageable at $+\infty$ if the following limit exists

$$\bar{d} := \lim_{n\to\infty} n^{-1} \sum_{k=s}^{s+n} d_k \quad \textit{uniformly in } s \ge 1. \tag{2.18}$$

(ii) Almost periodic at $+\infty$ if for each $\epsilon > 0$ there exist $k_\epsilon > 0$ and a periodic sequence $\{d_t^\epsilon, t \in \mathbb{Z}\}$ such that $\sup_{t > k_\epsilon} |d_t - d_t^\epsilon| < \epsilon$.

The limit $\bar{d}$ in (2.18) will be called the mean value of $\{d_t\}$. Denote $\mathcal{A}(+\infty)$ and $\mathcal{AP}(+\infty)$ the classes of all sequences $\{d_t\}$ that are averageable at $+\infty$ and almost periodic at $+\infty$, respectively. Then $\mathcal{AP}(+\infty) \subset \mathcal{A}(+\infty)$ but the converse implication is not true (see Bružaitė et al. (2007)). Clearly any asymptotic sequence (i.e. having a finite limit $d_\infty := \lim_{t\to\infty} d_t$) belongs to the class $\mathcal{AP}(+\infty)$ and hence to $\mathcal{A}(+\infty)$, with $\bar{d} = d_\infty$. As noted in Bružaitė et al. (2007, Remark 2.6), the class $\mathcal{AP}(+\infty)$ is closed under algebraic operations, shifts and uniform limits. By easy observation, the last fact holds for the class of asymptotic sequences, too.

Assumption (A.2) $\mathcal{M} \subset \mathcal{A}(+\infty)$ is a class which is closed under algebraic operations, shifts, and uniform limits.

Assumption (A.3) Assume $\{d_t\} \in \mathcal{M}$, $d_t \notin \{0, -1, -2, \dots\}$ for any $t = 1, 2, \dots$. Moreover, let there exist $C, \delta > 0$ such that for any $s \geq 1$

$$\Big| n^{-1} \sum_{k=s}^{s+n} (d_k - \bar{d}) \Big| \;\leq\; C n^{-\delta}.$$

With a given sequence $\{d_t\} \in \mathcal{M}$ we associate sequences $\{q_a(t)\} \in \mathcal{M},\ \{q_b(t)\} \in \mathcal{M}$ by

$$q_a(t) := \prod_{k=1}^{t} \Big(1 + \frac{d_k - \bar{d}}{\bar{d} + t - k - 1}\Big), \quad q_b(t) := \prod_{k=t}^{\infty} \Big(1 + \frac{d_k - \bar{d}}{\bar{d} + k - t + 1}\Big).$$

Denote

$$S_n^a(\tau) = \sum_{t=1}^{\lfloor n\tau \rfloor} X_t^a, \quad S_n^b(\tau) = \sum_{t=1}^{\lfloor n\tau \rfloor} X_t^b, \quad \tau \in [0,1].$$

Theorem 4 *Let $\{X_t^a\}, \{X_t^b\}$ be time-varying fractionally integrated processes in (1.5) with innovations $\{\zeta_s\} \in MD(0,1)$. Assume that $\mathcal{M}$ and $\{d_t\} \in \mathcal{M}$ satisfy Assumptions (A.2) and (A.3), and that $\bar{d} \in (0, \infty)$. Then*

$$\begin{aligned} n^{-\bar{d}-1/2} S_n^a(\tau) &\longrightarrow_{D[0,1]} \; c_a\, J_{\bar{d}}(\tau), \\ n^{-\bar{d}-1/2} S_n^b(\tau) &\longrightarrow_{D[0,1]} \; c_b\, J_{\bar{d}}(\tau), \end{aligned}$$

where $J_{\bar{d}}$ is a type II fBm in (1.8), and the asymptotic constants $c_a := \overline{q_a}/\Gamma(\bar{d})$, $c_b := \big(\overline{q_b^2}\big)^{1/2}/\Gamma(\bar{d})$ are written in terms of the mean values $\overline{q_a}, \overline{q_b^2}, \bar{d}$ of the averageable at $+\infty$ sequences $\{q_a(t)\}, \{q_b^2(t)\}, \{d_t\}$, respectively.

The proof of Theorem 4 follows from Bružaitė et al. (2007, proofs of Theorems 3.3, 3.4). Let us note that the restriction $\bar{d} \in (0, 1/2)$ in the above mentioned paper is due to the fact that Bružaitė et al. (2007) discuss the case of infinite (nontruncated) moving averages of the type (1.5), in which case stronger conditions on $\{d_t\}$ are needed.

Remark 2 The averaging property in (2.18) is quite strong and is not satisfied e.g. by a typical realization of i.i.d. sequence $\{d_t\}$ (with $\bar{d} = \mathrm{E}d_1$), a.s., unless this sequence is constant. On the other hand, Doukhan et al. (2007) proved *unconditional* convergence to a type I fBm of partial sums of (nontruncated) moving averages of (1.5) with i.i.d. $\{d_t\}$. The question remains open if it is possible to weaken the uniformity condition in (2.18) and replace it by the existence of a (simple) Cesaro mean in (1.7) so that Theorem 4 can be extended to a typical realization of an i.i.d. sequence $\{d_t\}$, independent of innovations $\{\zeta_s\}$, and the a.s. conditional convergence of the partial sums in $D[0,1]$.

3 Partial sums limits under slowly changing memory

In this section we discuss the partial sums process of linear sequences in (1.3) and (1.5) with slowly changing memory parameter $d_t = d(t/n)$, where $d(\tau)$, $\tau \in [0,1]$ is a given function having a unique supremum $d_{\max} := d(\tau_{\max}) \in (0,1/2)$ at some point $\tau_{\max} \in (0,1)$. This, clearly, implies that for any $\epsilon > 0$ satisfying $(\tau_{\max} - \epsilon, \tau_{\max} + \epsilon) \subset [0,1]$ there exists $\delta = \delta(\epsilon) > 0$ such that

$$d(\tau) \le d_{\max} - \delta, \quad \tau \notin (\tau_{\max} - \epsilon, \tau_{\max} + \epsilon). \tag{3.1}$$

Precise conditions on $d(\cdot)$ are given in Assumption (A.4) below.

Assumption (A.4) The function $d\colon [0,1] \mapsto (0,1/2)$ is a measurable function having a unique supremum $d(\tau_{\max}) =: d_{\max} \in (0,1/2)$ at some point $\tau_{\max} \in (0,1)$. Moreover, for some $\gamma > 0$, there exist the limits

$$\lim_{u \downarrow 0} u^{-\gamma}(d(\tau_{\max}) - d(\tau_{\max} \pm u)) \ =: \ \Delta_{\pm} > 0. \tag{3.2}$$

Theorem 5 *Let $d(\cdot)$ satisfy Assumption (A.4) and the innovations $\{\zeta_s\} \in MD(0,1)$. Let $\{X_t\}$ be the linear process in (1.3), where $\psi_j(d)$ satisfy conditions (2.4) and (2.3) of Theorem 2, with d_+ replaced by $d_{\max}$. Then*

$$\Big(\frac{\log^{1/\gamma} n}{n}\Big)^{d_{\max}+1/2} S_n\Big(\tau_{\max} + \frac{\tau}{\log^{1/\gamma} n}\Big) \ \longrightarrow_{D(\mathbb{R})} \ U(\tau), \tag{3.3}$$

where the limit process

$$U(\tau) \ \ := \ \ \kappa(d_{\max}) \int_{-\infty}^{\tau} \eta(\mathrm{d}u) \int_u^{\tau} (v-u)^{d_{\max}-1} \mathrm{e}^{-\Delta_{\mathrm{sgn}(v)}|v|^{\gamma}} \mathrm{d}v, \qquad \tau \in \mathbb{R} \tag{3.4}$$

is well-defined as a stochastic integral with respect to a Gaussian white noise $\eta(\mathrm{d}u)$ on the real line, with zero mean and variance $\mathrm{E}(\eta(\mathrm{d}u))^2 = \mathrm{d}u$.

Note that, if $\Delta_+ = \Delta_- = 0$, then the increment process

$$U(\tau) - U(0) = \frac{\kappa(d_{\max})}{d_{\max}} \int_{-\infty}^{\infty} \big((\tau-u)_+^{d_{\max}} - (-u)_+^{d_{\max}}\big)\eta(\mathrm{d}u), \quad \tau \in \mathbb{R}$$

is well-defined as well and coincides with a type I fBm with Hurst parameter $H = d_{\max} + 1/2$ and stationary increments.

Remark 3 Condition $d_{\max} < 1/2$ is necessary for the existence of (3.4). Indeed,

$$\begin{aligned} \mathrm{E}U^2(0) \ &= \ \kappa^2(d_{\max}) \int_0^{\infty} \Big(\int_0^u (u-v)^{d_{\max}-1} \mathrm{e}^{-\Delta_- v^{\gamma}} \mathrm{d}v\Big)^2 \mathrm{d}u \\ &\ge \ C \int_1^{\infty} \Big(\int_0^1 (u-v)^{d_{\max}-1} \mathrm{d}v\Big)^2 \mathrm{d}u \\ &= \ C \int_1^{\infty} \big((u-1)^{d_{\max}} - u^{d_{\max}}\big)^2 \mathrm{d}u \ = \ \infty \end{aligned}$$

for $d_{\max} \ge 1/2$.

Proposition 6 *Let* $\gamma > 0$, $\Delta_+ > 0$, $\Delta_- > 0$. *Then* $\{U(\tau), \tau \in \mathbb{R}\}$ *in (3.4) has a sample continuous version and finite a.s. limits* $\lim_{\tau\to\pm\infty} U(\tau) =: U(\pm\infty), U(-\infty) = 0$.

Proof. Let $c := \min(\Delta_+, \Delta_-) > 0$ and let

$$\mu(B) := \int_B \Big(\mathbf{1}(|x| \le 1) + \mathrm{e}^{-c|x|^{\gamma\wedge 1}}\mathbf{1}(|x| > 1)\Big)\mathrm{d}x$$

be a finite measure on the real line. Denote $\mu(B) =: \mu(a, b)$ in the case $B = (a, b]$. Note that for any $c, \gamma > 0$ and any $0 \le a < b$

$$\int_a^b \mathrm{e}^{-cx^\gamma}\mathrm{d}x \le \mu(a, b), \qquad \mathrm{e}^{-ca^\gamma}[(b-a)\wedge 1] \le \mathrm{e}^{2c}\mu(a, b). \tag{3.5}$$

The first inequality in (3.5) is obvious. Let us check the second one. It suffices to consider the case $a < b \le a + 1$ only since the l.h.s. of this inequality does not depend on b for $b \ge a + 1$. Let $v := b - a \in [0, 1]$. Then the second inequality in (3.5) becomes

$$v\mathrm{e}^{-ca^\gamma} \le \mathrm{e}^{2c}\int_a^{a+v} \Big(\mathbf{1}(0 < x \le 1) + \mathrm{e}^{-cx^{\gamma\wedge 1}}\mathbf{1}(x > 1)\Big)\mathrm{d}x. \tag{3.6}$$

The above inequality is obvious for $a + v \le 1$. Let $a + v > 1$. Then (3.6) follows from $\mathrm{e}^{-ca^\gamma} \le \mathrm{e}^{2c}\mathrm{e}^{-c(a+v)^{\gamma\wedge 1}}$, or $(a + v)^{\gamma\wedge 1} \le 2 + a^\gamma$. Let $\gamma \le 1$, then the last inequality holds by $(a+v)^\gamma \le a^\gamma + v^\gamma \le a^\gamma + 1$, since $v \in [0, 1]$. Next, let $\gamma > 1$, then $(a+v)^{\gamma\wedge 1} = a+v \le a^\gamma + 2$ if $a > 1$ and $a + v \le 2 \le 2 + a^\gamma$ for $a \in [0, 1]$. This proves (3.6) and (3.5), too.

According to the Kolmogorov moment criterion, it suffices to show that there exists a constant $C < \infty$ such that for any $-\infty < \tau_1 < \tau_2 < \infty$

$$\mathrm{E}(U(\tau_2) - U(\tau_1))^2 \ \le\ C(\mu(\tau_1, \tau_2))^{1+2d_{\max}}. \tag{3.7}$$

It suffices to show (3.7) for $\tau_i \ge 0$ $(i = 1, 2)$ and $\tau_i \le 0$ $(i = 1, 2)$ separately. Let us assume $0 \le \tau_1 < \tau_2$ in the rest of the proof. According to the definition in (3.4), the left-hand side of (3.7) does not exceed $CI(\tau_1, \tau_2)$, where

$$\begin{aligned}
I(\tau_1, \tau_2) &:= \int_{\mathbb{R}} \Big\{ \int_{\tau_1}^{\tau_2} (v - u)_+^{d_{\max}-1}\mathrm{e}^{-cv^\gamma}\mathrm{d}v \Big\}^2 \mathrm{d}u \\
&= \int_{\tau_1}^{\tau_2}\int_{\tau_1}^{\tau_2} \mathrm{e}^{-cv_1^\gamma}\mathrm{e}^{-cv_2^\gamma}\mathrm{d}v_1\mathrm{d}v_2 \int_{\mathbb{R}} (v_1 - u)_+^{d_{\max}-1}(v_2 - u)_+^{d_{\max}-1}\mathrm{d}u \\
&= C_{d_{\max}} \int_{\tau_1}^{\tau_2}\int_{\tau_1}^{\tau_2} \mathrm{e}^{-cv_1^\gamma}\mathrm{e}^{-cv_2^\gamma}|v_1 - v_2|^{2d_{\max}-1}\mathrm{d}v_1\mathrm{d}v_2 \\
&= C_{d_{\max}}(I_1 + I_2),
\end{aligned}$$

where $C_{d_{\max}} = B(d_{\max}, 1 - 2d_{\max})$ and

$$\begin{aligned}
I_1 &:= \int_{\tau_1}^{\tau_2}\int_{\tau_1}^{\tau_2} \mathrm{e}^{-cv_1^\gamma}\mathrm{e}^{-cv_2^\gamma}|v_1 - v_2|^{2d_{\max}-1}\mathbf{1}(|v_1 - v_2| > 1)\mathrm{d}v_1\mathrm{d}v_2 \\
&\le (\mu(\tau_1, \tau_2))^2 \le C(\mu(\tau_1, \tau_2))^{1+2d_{\max}}
\end{aligned}$$

according to the first inequality in (3.5). Next,

$$\begin{aligned} I_2 &:= \int_{\tau_1}^{\tau_2}\int_{\tau_1}^{\tau_2} \mathrm{e}^{-cv_1^\gamma}\mathrm{e}^{-cv_2^\gamma}|v_1-v_2|^{2d_{\max}-1}\mathbf{1}(|v_1-v_2|\le 1)\mathrm{d}v_1\mathrm{d}v_2 \\ &\le \mathrm{e}^{-c\tau_1^\gamma}\int_{\tau_1}^{\tau_2}\int_{\tau_1}^{\tau_2} \mathrm{e}^{-cv_1^\gamma}|v_1-v_2|^{2d_{\max}-1}\mathbf{1}(|v_1-v_2|\le 1)\mathrm{d}v_1\mathrm{d}v_2 \\ &= \frac{\mathrm{e}^{-c\tau_1^\gamma}}{d_{\max}}\int_{\tau_1}^{\tau_2} \mathrm{e}^{-cv^\gamma}(\tau_2\wedge(1+v)-v)^{2d_{\max}}\mathrm{d}v \\ &\le \frac{1}{d_{\max}}\,[(\tau_2-\tau_1)\wedge 1]^{2d_{\max}}\mathrm{e}^{-c\tau_1^\gamma}\int_{\tau_1}^{\tau_2}\mathrm{e}^{-cv^\gamma}\mathrm{d}v \\ &\le \frac{1}{d_{\max}}\,(\mu(\tau_1,\tau_2))^{2d_{\max}+1}, \end{aligned}$$

where we used (3.5). This proves (3.7) and the proposition. □

Proof of Theorem 5. We shall restrict the proof of finite-dimensional convergence in (3.3) to one-dimensional convergence at $\tau > 0$. The proof uses Proposition 1 with

$$m_n := \frac{n}{\log^{1/\gamma} n}, \qquad p_n := \lfloor n\tau_{\max}\rfloor. \tag{3.8}$$

Accordingly, we need, firstly, to write the l.h.s. of (3.3) as a discrete stochastic integral $\int h_n(u;\tau)\eta_n(\mathrm{d}u)$ with $\eta_n(\mathrm{d}u)$ given in (1.10) and some integrand $h_n(\cdot;\tau) \in L^2_n(\mathbb{R})$ and, secondly, to verify (1.11) for $g^{(n)} = h_n(\cdot;\tau)$ and $g = h(\cdot;\tau)$ as defined in (3.4), viz.

$$h(u;\tau) := \kappa(d_{\max})\int_u^\tau (v-u)^{d_{\max}-1}\mathrm{e}^{-\Delta_{\mathrm{sgn}(v)}|v|^\gamma}\mathrm{d}v.$$

It is easy to see that for sufficiently large n

$$\Big(\frac{\log^{1/\gamma} n}{n}\Big)^{d_{\max}+1/2} S_n\Big(\tau_{\max}+\frac{\tau}{\log^{1/\gamma} n}\Big) = \int h_n(u;\tau)\eta_n(\mathrm{d}u)$$

with

$$\begin{aligned} h_n(u;\tau) &:= m_n^{-d_{\max}}\sum_{t=s}^{\lfloor n(\tau_{\max}+\tau\frac{m_n}{n})\rfloor-p_n}\psi_{t-s}\Big(d\Big(\frac{t+p_n}{n}\Big)\Big), \quad \text{if} \quad u\in\Big(\frac{s}{m_n},\frac{s+1}{m_n}\Big] \\ &\qquad \text{and} \quad s = 1-p_n, 2-p_n, \ldots, \Big\lfloor n\Big(\tau_{\max}+\tau\frac{m_n}{n}\Big)\Big\rfloor - p_n, \end{aligned}$$

$h_n(u;\tau) := 0$ otherwise and $\eta_n\big((\frac{s}{m_n},\frac{s+1}{m_n}]\big) = m_n^{-1/2}\zeta_{s+p_n}$.

It is convenient to split $S_n\big(\tau_{\max}+\frac{\tau}{\log^{1/\gamma} n}\big) = S_n(\tau_{\max}) + \big[S_n\big(\tau_{\max}+\frac{\tau}{\log^{1/\gamma} n}\big) - S_n(\tau_{\max})\big]$ and, correspondingly,

$$\int h_n(u;\tau)\eta_n(\mathrm{d}u) = \int h_n^-(u;0)\eta_n(\mathrm{d}u) + \int h_n^+(u;\tau)\eta_n(\mathrm{d}u),$$

where $h_n^-(u;0) := h_n(u;0)$, $h_n^+(u;\tau) := h_n(u;\tau) - h_n^-(u;0)$. Hence, (1.11) follows from

$$\lim_{n\to\infty}\|h_n^-(\cdot;0)-h^-(\cdot;0)\| = 0, \qquad \lim_{n\to\infty}\|h_n^+(\cdot;\tau)-h^+(\cdot;\tau)\| = 0, \tag{3.9}$$

where $h^-(u;0) := h(u;0),\ h^+(u;\tau) := h(u;\tau) - h(u;0)$. We have

$$\Big(\frac{\log^{1/\gamma} n}{n}\Big)^{d_{\max}+1/2}\Big[S_n\Big(\tau_{\max}+\frac{\tau}{\log^{1/\gamma} n}\Big)-S_n(\tau_{\max})\Big]$$
$$= m_n^{-d_{\max}-1/2}\sum_{s=1}^{\lfloor n(\tau_{\max}+\tau\frac{m_n}{n})\rfloor}\Bigg\{\sum_{t=(\lfloor n\tau_{\max}\rfloor+1)\vee s}^{\lfloor n(\tau_{\max}+\tau\frac{m_n}{n})\rfloor}\psi_{t-s}(d(t/n))\Bigg\}\zeta_s = \int h_n^+(u;\tau)\eta_n(\mathrm{d}u)$$

with

$$h_n^+(u;\tau) := m_n^{-d_{\max}}\sum_{t=1\vee s}^{\lfloor n(\tau_{\max}+\tau\frac{m_n}{n})\rfloor-p_n}\psi_{t-s}\Big(d\Big(\frac{t+p_n}{n}\Big)\Big),\quad \text{if}\quad u\in\Big(\frac{s}{m_n},\frac{s+1}{m_n}\Big]$$
$$\text{and}\quad s = 1-p_n, 2-p_n,\dots,\Big\lfloor n\Big(\tau_{\max}+\tau\frac{m_n}{n}\Big)\Big\rfloor-p_n,$$

and $h_n^+(u;\tau) := 0$ otherwise. To prove the second relation in (3.9), note that by (3.2)

$$d\big(\tau_{\max}+u\big) = d_{\max}-\Delta_+u^\gamma(1+\epsilon_1(u)),$$

where $\epsilon_1(u)$ vanishes as $u\downarrow 0$, so that

$$d\Big(\frac{t+p_n}{n}\Big) = d\Big(\tau_{\max}+\frac{t+\lfloor n\tau_{\max}\rfloor-n\tau_{\max}}{n}\Big) = d_{\max}-\Delta_+\Big(\frac{t}{n}\Big)^\gamma\big(1+\epsilon_2(t,n)\big), \qquad (3.10)$$

where $\epsilon_2(t,n)$ is some vanishing function as $t/n\downarrow 0,\ n\to 0$. Since $s = \lceil um_n\rceil - 1$ we can rewrite $h_n^+(u;\tau)$ as

$$h_n^+(u;\tau) = m_n^{-d_{\max}}\sum_{t=1\vee(\lceil um_n\rceil-1)}^{\lfloor n(\tau_{\max}+\tau\frac{m_n}{n})\rfloor-p_n}\psi_{t-\lceil um_n\rceil+1}\Big(d\Big(\frac{t+p_n}{n}\Big)\Big)$$
$$\times\ \mathbf{1}\Big(-p_n < um_n \le \Big\lfloor n\Big(\tau_{\max}+\tau\frac{m_n}{n}\Big)\Big\rfloor-p_n\Big)$$
$$= \widetilde{h}_n^+(u;\tau)+(h_n^+(u;\tau)-\widetilde{h}_n^+(u;\tau)), \qquad (3.11)$$

where

$$\tilde{h}_n^+(u;\tau) := m_n^{-d_{\max}}\sum_{t=1\vee(\lceil um_n\rceil-1)}^{\lfloor n(\tau_{\max}+\tau\frac{m_n}{n})\rfloor-p_n}\kappa\Big(d\Big(\frac{t+p_n}{n}\Big)\Big)(t-\lceil um_n\rceil+1)^{d(\frac{t+p_n}{n})}$$
$$\times\ \mathbf{1}\Big(-p_n < um_n \le \Big\lfloor n\Big(\tau_{\max}+\tau\frac{m_n}{n}\Big)\Big\rfloor-p_n\Big).$$

We will show next that for any fixed $u\le\tau$

$$\tilde{h}_n^+(u;\tau) \to \kappa(d_{\max})\int_{u\vee 0}^{\tau}(v-u)^{d_{\max}-1}\,\mathrm{e}^{-\Delta_+v^\gamma}\mathrm{d}v \qquad (3.12)$$

and

$$\|h_n^+(\cdot;\tau)-\widetilde{h}_n^+(\cdot;\tau)\|\to 0. \qquad (3.13)$$

The convergence (3.13) follows from assumptions (2.3)–(2.4) and a standard argument using the dominated convergence theorem. To prove (3.12), rewrite

$$\widetilde{h}_n^+(u;\tau) \quad = \sum_{t=1\vee(\lceil um_n\rceil-1)}^{\lfloor n(\tau_{\max}+\tau\frac{m_n}{n})\rfloor-p_n} \psi_n^{(1)}(t)\psi_n^{(2)}(t;u)\psi_n^{(3)}(t;u),$$

where

$$\begin{aligned}
\psi_n^{(1)}(t) &:= \kappa\Big(d\Big(\frac{t+p_n}{n}\Big)\Big),\\
\psi_n^{(2)}(t;u) &:= \frac{(t-\lceil um_n\rceil+1)^{d_{\max}-1}}{m_n^{d_{\max}}},\\
\psi_n^{(3)}(t;u) &:= \frac{(t-\lceil um_n\rceil+1)^{d(\frac{t+p_n}{n})}}{(t-\lceil um_n\rceil+1)^{d_{\max}}}\,\mathbf{1}\Big(-p_n < um_n \le \Big\lfloor n\Big(\tau_{\max}+\tau\frac{m_n}{n}\Big)\Big\rfloor - p_n\Big).
\end{aligned}$$

Below we show that for any fixed $u>0$, uniformly in $t\in\{\lceil um_n\rceil,\dots,\lfloor n(\tau_{\max}+\tau\frac{m_n}{n})\rfloor - p_n\}$, the following relations hold:

$$\psi_n^{(1)}(t)\to\kappa(d_{\max}), \tag{3.14}$$

and

$$\psi_n^{(2)}(t;u)\sim\Big(\frac{t}{m_n}-u\Big)^{d_{\max}-1}\frac{1}{m_n},\quad \psi_n^{(3)}(t;u)\sim \mathrm{e}^{-\Delta_+(\frac{t}{m_n})^\gamma}\,\mathbf{1}(-\infty<u\le\tau). \tag{3.15}$$

Relations (3.14)–(3.15) imply that for any $u>0$

$$\begin{aligned}
\widetilde{h}_n^+(u;\tau) &\sim \kappa(d_{\max})\sum_{t=\lceil um_n\rceil-1}^{\lfloor \tau m_n\rfloor}\Big(\frac{t}{m_n}-u\Big)^{d_{\max}-1}\frac{1}{m_n}\,\mathrm{e}^{-\Delta_+(\frac{t}{m_n})^\gamma}\,\mathbf{1}(u\le\tau)\\
&\to \kappa(d_{\max})\int_u^\tau (v-u)^{d_{\max}-1}\,\mathrm{e}^{-\Delta_+v^\gamma}\mathrm{d}v,
\end{aligned}$$

i.e. (3.12) holds.

Relation in (3.14) follows from (3.10), by noting that

$$\begin{aligned}
\kappa\Big(d_{\max}-\Delta_+\Big(\frac{t}{n}\Big)^\gamma\big(1+\epsilon_2(t,n)\big)\Big) &= \kappa\Big(d_{\max}-\Delta_+\frac{1}{\log n}\Big(\frac{t}{m_n}\Big)^\gamma\big(1+\epsilon_2(t,n)\big)\Big)\\
&\to \kappa(d_{\max})
\end{aligned}$$

uniformly for $t\in\{\lceil um_n\rceil,\dots,\lfloor n(\tau_{\max}+\tau\frac{m_n}{n})\rfloor-p_n\}$.

The first relation in (3.15) is obvious. To prove the second relation, rewrite

$$\begin{aligned}
\frac{(t-\lceil um_n\rceil+1)^{d(\frac{t+p_n}{n})}}{(t-\lceil um_n\rceil+1)^{d_{\max}}} &= \exp\Big\{\Big(d\Big(\frac{t+p_n}{n}\Big)-d_{\max}\Big)\log(t-\lceil um_n\rceil+1)\Big\}\\
&= \exp\Big\{-\Delta_+\Big(\frac{t}{m_n}\Big)^\gamma(1+\epsilon_2(t,n))\frac{\log m_n+\log\frac{t-\lceil um_n\rceil+1}{m_n}}{\log n}\Big\}\\
&= \exp\Big\{-\Delta_+\Big(\frac{t}{m_n}\Big)^\gamma(1+\epsilon_2(t,n))(1+\epsilon_3(t,n))\Big\},
\end{aligned}$$

where both $\epsilon_2(t,n)$, $\epsilon_3(t,n)$ vanish uniformly for $t \in \{\lceil um_n \rceil, \dots, \lfloor n(\tau_{\max} + \tau \frac{m_n}{n}) \rfloor - p_n\}$.

The convergence in $L^2(\mathbb{R})$, or the second relation in (3.9), follows easily from (3.11), relations (3.12)–(3.13) and the boundedness of $\kappa(\cdot)$.

Let us prove the first relation in (3.9). We have

$$m_n^{-d_{\max}-1/2} S_n(\tau_{\max}) = m_n^{-d_{\max}-1/2} \sum_{s=1}^{\lfloor n\tau_{\max} \rfloor} \sum_{t=s}^{\lfloor n\tau_{\max} \rfloor} \psi_{t-s}(d(t/n))\zeta_s = \int h_n^-(u;0)\eta_n(\mathrm{d}u),$$

where

$$h_n^-(u;0) \;:=\; m_n^{-d_{\max}} \sum_{t=s}^{0} \psi_{t-s}\Big(d\Big(\frac{t+p_n}{n}\Big)\Big), \quad \text{if} \quad u \in \Big(\frac{s}{m_n}, \frac{s+1}{m_n}\Big] \quad \text{and} \quad s = 1-p_n, 2-p_n, \dots, 0$$

and $h_n^-(u;0) := 0$ otherwise. Similarly to (3.10) we have

$$d\Big(\frac{t+p_n}{n}\Big) \;=\; d_{\max} - \Delta_-\Big(\frac{t}{n}\Big)^{\gamma}\big(1+\epsilon_4(t,n)\big) \quad \text{for} \quad t = 0, -1, \dots$$

where $\epsilon_4(t,n)$ vanishes as $|t|/n \downarrow 0$, $n \to 0$. Hence, similarly to (3.16) we obtain the pointwise convergence

$$h_n^-(u;0) \;\to\; \kappa(d_{\max}) \int_u^0 (v-u)^{d_{\max}-1} \mathrm{e}^{-\Delta_-|v|^{\gamma}} \mathrm{d}v \;=\; h^-(u;0)$$

for any $u < 0$. Moreover, the above mentioned convergence is uniform on any compact interval $u \in [-K, 0]$. To prove the first relation in (3.9), it suffices to show that for any $\tilde\epsilon > 0$ one can find $K > 0$ and n_0 such that

$$J_{K,n} \;:=\; \int_{-\infty}^{-K} (h_n^-(u;0))^2 \mathrm{d}u \;\le\; \tilde\epsilon, \qquad \forall n > n_0. \tag{3.16}$$

Using (2.4) and the definition of $h_n^-(u)$, we have that for all sufficiently large n

$$\begin{aligned} J_{K,n} \;&=\; m_n^{-2d_{\max}} \sum_{s=1-p_n}^{0} \Big(\sum_{t=s}^{0} \psi_{t-s}\Big(d\Big(\frac{t+p_n}{n}\Big)\Big)\Big)^2 \int_{\infty}^{-K} \mathbf{1}\Big(\frac{s}{m_n} < u \le \frac{s+1}{m_n}\Big)\mathrm{d}u \\ &\le\; C m_n^{-2d_{\max}-1} \sum_{-p_n \le s \le -Km_n} \Big(\sum_{t=s}^{0} (t-s+1)^{d(\frac{t+p_n}{n})-1}\Big)^2. \end{aligned}$$

According to Assumption (A.4),

$$d\Big(\frac{t+p_n}{n}\Big) \le \begin{cases} d_{\max} - \Delta\big(\frac{|t|}{n}\big)^{\gamma}, & -\bar\epsilon n < t \le 0, \\ d_{\max} - \delta(\bar\epsilon), & -p_n \le t \le -\bar\epsilon n, \end{cases}$$

for some $0 < \Delta < \Delta_-$, sufficiently small $\bar{\epsilon} > 0$ and sufficiently large n. Therefore,

$$
\begin{aligned}
J_{K,n} &= C m_n^{-2d_{\max}-1} \sum_{-p_n \le s \le -Km_n} \Big\{ \Big(\sum_{-p_n \le t \le -\bar{\epsilon}n} + \sum_{-\bar{\epsilon}n < t \le 0} \Big) (t-s+1)^{d(\frac{t+p_n}{n})-1} \mathbf{1}(t \ge s) \Big\}^2 \\
&\le C(J'_{K,n} + J''_{K,n}),
\end{aligned}
$$

where

$$
\begin{aligned}
J'_{K,n} &:= m_n^{-2d_{\max}-1} \sum_{-p_n \le s \le -Km_n} \Big(\sum_{-p_n \le t \le -\bar{\epsilon}n} (t-s+1)^{d(\frac{t+p_n}{n})-1} \mathbf{1}(t \ge s) \Big)^2, \\
J''_{K,n} &:= m_n^{-2d_{\max}-1} \sum_{-p_n \le s \le -Km_n} \Big(\sum_{-\bar{\epsilon}n < t \le 0} (t-s+1)^{d_{\max} - \Delta(\frac{|t|}{n})^{\gamma} - 1} \mathbf{1}(t \ge s) \Big)^2.
\end{aligned}
$$

Clearly, since $0 < \delta(\bar{\epsilon}) < d_{\max}$,

$$
\begin{aligned}
J'_{K,n} &\le m_n^{-2d_{\max}-1} \sum_{-n \le s \le 0} \Big(\sum_{s \le t \le 0} (t-s+1)^{d_{\max} - \delta(\bar{\epsilon}) - 1} \Big)^2, \\
&\le C m_n^{-2d_{\max}-1} \sum_{s=0}^{n} (s+1)^{2(d_{\max} - \delta(\bar{\epsilon}))} \\
&\le C(n/m_n)^{2d_{\max}+1} n^{-2\delta(\bar{\epsilon})} = o(1).
\end{aligned}
$$

Next,

$$
\begin{aligned}
J''_{K,n} &\le m_n^{-2d_{\max}-1} \sum_{-n \le s \le -Km_n} \Big(\sum_{s \le t \le 0} (t-s+1)^{d_{\max} - \Delta(\frac{|t|}{n})^{\gamma} - 1} \Big)^2 \\
&= \sum_{Km_n+1 \le s \le n+1} \frac{1}{m_n^3} \Big(\sum_{0 \le t < s} \Big(\frac{s-t}{m_n} \Big)^{d_{\max}-1} \exp\Big\{ -\Delta \Big(\frac{t}{m_n} \Big)^{\gamma} \frac{\log m_n + \log(\frac{s-t}{m_n})}{\log n} \Big\} \Big)^2.
\end{aligned}
$$

Split the last inner sum into two sums according to whether $s - t \ge m_n$ or $1 \le s - t < m_n$ holds and denote the corresponding terms by $I'_{K,n}$ and $I''_{K,n}$ so that $J''_{K,n} \le C(I'_{K,n} + I''_{K,n})$. Since $\frac{\log m_n}{\log n} > 1/2$ for all n large enough, we obtain that

$$
\begin{aligned}
I'_{K,n} &\le \int_K^{\infty} \mathrm{d}s \Big(\int_0^s (s-t)^{d_{\max}-1} \mathrm{e}^{-\frac{1}{2}\Delta t^{\gamma}} \mathrm{d}t \Big)^2 \\
&= \int_0^{\infty} \int_0^{\infty} \mathrm{e}^{-\frac{1}{2}\Delta t_1^{\gamma}} \mathrm{e}^{-\frac{1}{2}\Delta t_2^{\gamma}} g_K(t_1, t_2) \mathrm{d}t_1 \mathrm{d}t_2,
\end{aligned}
$$

where

$$
\begin{aligned}
g_K(t_1, t_2) &:= \int_{K \vee t_1 \vee t_2}^{\infty} (s-t_1)^{d_{\max}-1} (s-t_2)^{d_{\max}-1} \mathrm{d}s \\
&\le \int_{K \vee t_1 \vee t_2}^{\infty} (s - t_1 \vee t_2)^{2d_{\max}-2} \mathrm{d}s \to 0 \quad \text{as} \quad K \to \infty
\end{aligned}
$$

for any $t_1, t_2 \geq 0$ fixed, and $g_K(t_1,t_2) \leq C|t_1-t_2|^{2d_{\max}-1} =: g(t_1,t_2)$, $t_1 \neq t_2$, where $\int_0^\infty \int_0^\infty \mathrm{e}^{-\frac{1}{2}\Delta t_1^\gamma} \mathrm{e}^{-\frac{1}{2}\Delta t_2^\gamma}\, g(t_1,t_2) \mathrm{d}t_1 \mathrm{d}t_2 < \infty$. Therefore, by the dominated convergence theorem, $I'_{K,n} \to 0$ $(K \to \infty)$ uniformly in n.

Consider $I''_{K,n}$. Since $t^\gamma > \frac{s^\gamma}{2}$ for $s-t < m_n$ and $s > Km_n$, with K large enough, so

$$
\begin{aligned}
I''_{K,n} &\leq m_n^{-2d_{\max}-1} \sum_{s=Km_n}^{n+1} \Bigg(\sum_{s-m_n \leq t < s} (s-t)^{d_{\max}-1-\frac{1}{2}\Delta(\frac{s}{n})^\gamma} \Bigg)^2 \\
&\leq C m_n^{-2d_{\max}-1} \sum_{s=Km_n}^{n+1} m_n^{2d_{\max}-\Delta(\frac{s}{n})^\gamma} \\
&= C m_n^{-1} \sum_{s=Km_n}^{n+1} \mathrm{e}^{-\Delta(\frac{s}{n})^\gamma \log m_n} \\
&= C m_n^{-1} \sum_{s=Km_n}^{n+1} \mathrm{e}^{-\Delta(\frac{s}{m_n})^\gamma \frac{\log m_n}{\log n}} \\
&\leq C \int_K^\infty \mathrm{e}^{-\frac{1}{2}\Delta u^\gamma} \mathrm{d}u \;\to\; 0 \quad (K \to \infty)
\end{aligned}
$$

uniformly in n for all n large enough. This proves (3.16), (3.9) and the finite-dimensional convergence in (3.3).

To prove the tightness in (3.3), we shall verify the Kolmogorov criterion: there exists a constant $C > 0$ such that for any $-\infty < \tau_1 < \tau_2 < \infty$

$$
\mathrm{E}\Big[S_n\Big(\tau_{\max} + \frac{\tau_2}{\log^{1/\gamma} n}\Big) - S_n\Big(\tau_{\max} + \frac{\tau_1}{\log^{1/\gamma} n}\Big)\Big]^2 \;\leq\; C m_n^{1+2d_{\max}} (\tau_2-\tau_1)^{1+2d_{\max}}, \tag{3.17}
$$

cf. (3.7).

Let $q_n(\tau_1,\tau_2)$ denote the left hand side of (3.17). Then $q_n(\tau_1,\tau_2) = \sum_{t_1,t_2=\lfloor n\tau_{\max}+m_n\tau_1\rfloor+1}^{\lfloor n\tau_{\max}+m_n\tau_2\rfloor} \rho(t_1,t_2)$, where $\rho(t_1,t_2) := \sum_{s=1}^\infty \psi_{t_1-s}(d_{t_1})\psi_{t_2-s}(d_{t_2})$ is the covariance function of $\{X_t\}$ in (1.3), with the convention $\psi_{t-s}(d_t) := 0$ $(s > t)$. Using (2.4), we obtain $|\rho(t_1,t_2)| \leq C(|t_1-t_2|+1)^{2d_{\max}-1}$ and therefore (3.17) easily follows. This ends the proof of Theorem 5. $\square$

Next, we discuss the case of time-varying fractionally integrated process $\{X_t^a\}$ in (1.5). We shall need to strengthen Assumption (A.4) with the following condition: there exist a constant $C < \infty$ and $\epsilon > 0$ such that for any $0 < u_1, u_2 < \epsilon$

$$
|d(\tau_{\max} \pm u_1) - d(\tau_{\max} \pm u_2)| \leq C|u_1-u_2|(u_1^{\gamma-1} + u_2^{\gamma-1}). \tag{3.18}
$$

Note that condition (3.18) is satisfied if $d(\cdot)$ is differentiable in a neighborhood of $\tau_{\max}$, with exception of the point $\tau = \tau_{\max}$ itself, and the derivative satisfies $|d'(\tau)| \leq C|\tau-\tau_{\max}|^{\gamma-1}$. In particular, condition (3.18) is satisfied by the function $d(\tau \pm u) = d_{\max} - \Delta_\pm u^\gamma$, $u, \Delta_\pm > 0$.

Theorem 7 *Let $d(\tau), \tau \in [0,1]$ satisfy Assumption (A.4) and condition (3.18). Let $\{X_t^a\}$ be a time-varying fractionally integrated processes in (1.5) corresponding to the slowly changing memory parameter $d_t = d(t/n)$ and martingale difference innovations as in Theorem 5. Then the statement (3.3) of Theorem 5 holds with S_n replaced by S_n^a and $\kappa(d_{\max}) = 1/\Gamma(d_{\max})$.*

Proof. For $d_t = d(t/n)$, the coefficients $a_{t-s}(t), b_{t-s}(t)$ in (1.6) can be rewritten as

$$a_{t-s}(t) \quad := \quad \frac{d(\frac{t-1}{n})}{1} \cdot \frac{d(\frac{t-2}{n})+1}{2} \cdot \frac{d(\frac{t-3}{n})+2}{3} \cdots \frac{d(\frac{s}{n})-1+t-s}{t-s}.$$

Let $\psi_{t-s}(d)$ be FARIMA$(0,d,0)$ coefficients in (1.4). Then

$$\psi_{t-s}\Big(d\Big(\frac{t}{n}\Big)\Big) \quad = \quad \frac{d(\frac{t}{n})}{1} \cdot \frac{d(\frac{t}{n})+1}{2} \cdots \frac{d(\frac{t}{n})-1+t-s}{t-s}, \quad s < t.$$

Consider the ratio

$$\Theta^a(t,s) := \frac{a_{t-s}(t)}{\psi_{t-s}\big(d(\frac{t}{n})\big)}, \quad s \le t.$$

Let m_n, p_n be defined as in (3.8). We shall prove that for any t, s in a $O(m_n)-$"neighborhood" of $p_n = \lfloor n\tau_{\max} \rfloor$ the above ratio tends to 1: for any $K < \infty$

$$\sup_{t,s \in M_{n,K}} |\Theta^a(t,s) - 1| \to 0, \tag{3.19}$$

where $M_{n,K} := \{(t,s) : |t-p_n| \le Km_n, |s-p_n| \le Km_n, s \le t\}$. We shall also need the following dominating bound:

$$\sup_{|t-p_n| \le Km_n, 1 \le s \le t} |\Theta^a(t,s)| \ \le \ C. \tag{3.20}$$

Consider (3.19). We have

$$\Theta^a(t,s) \quad = \quad \prod_{k=s}^{t-1} \big(1 + \lambda_{t,k}^a\big), \quad \lambda_{t,k}^a \ := \ \frac{d(\frac{k}{n}) - d(\frac{t}{n})}{d(\frac{t}{n}) + t - k - 1}.$$

From Assumption (A.4) and condition (3.18) we have that for all sufficiently large n and any $p_n < s < t \le p_n + Km_n$ it holds

$$\begin{aligned}
\sum_{k=s}^{t-1} |\lambda_{t,k}^a| \ &\le \ C \sum_{k=s}^{t-1} \frac{\big|d\big(\frac{k}{n}\big) - d\big(\frac{t}{n}\big)\big|}{1 + |t-k-1|} \\
&\le \ Cn^{-\gamma} \sum_{k=s}^{t-1} \frac{((k-p_n)^{\gamma-1} + (t-p_n)^{\gamma-1})(t-k)}{1+|t-k-1|} \\
&\le \ Cn^{-\gamma} \sum_{k=p_n+1}^{p_n+Km_n} ((k-p_n)^{\gamma-1} + (t-p_n)^{\gamma-1}) \\
&\le \ CK^\gamma \Big(\frac{m_n}{n}\Big)^\gamma \ = \ o(1).
\end{aligned} \tag{3.21}$$

Note the last bound applies for any $\gamma > 0$. Similar argument leads to the same bound for the sum $\sum_{k=s}^{t-1} |\lambda^a_{t,k}|$ in the case $p_n - Km_n \le s < t < p_n$. In the case $p_n - Km_n \le s \le p_n < t \le p_n + Km_n$, split $\sum_{k=s}^{t-1} |\lambda^a_{t,k}| = \sum_{k=s}^{p_n} |\lambda^a_{t,k}| + \sum_{k=p_n+1}^{t-1} |\lambda^a_{t,k}| =: \Lambda_1 + \Lambda_2$, where $\Lambda_2 \le CK^\gamma \big(\frac{m_n}{n}\big)^\gamma = o(1)$ as above, while

$$\Lambda_1 \ \le \ Cn^{-\gamma} \sum_{k=s}^{p_n} \frac{(p_n - k)^\gamma + (t - p_n)^\gamma}{1 + |t - k - 1|} =: \Lambda_{11} + \Lambda_{12}$$

according to Assumption (A.4). Here, $\Lambda_{11} \le CK^\gamma \big(\frac{m_n}{n}\big)^\gamma$ and the same bound follows easily for Λ_{12} in the case $\gamma \ge 1$. Consider Λ_{12} for $0 < \gamma < 1$. Then

$$\begin{aligned}
\Lambda_{12} \ &\le \ C\Big(\frac{t - p_n}{n}\Big)^\gamma \sum_{k=s}^{p_n - 1} \frac{1}{t - k - 1} \\
&\le \ C\Big(\frac{t - p_n}{n}\Big)^\gamma \log\Big(1 + \frac{p_n - s - 1}{t - p_n}\Big) \\
&\le \ C\Big(\frac{Km_n}{n}\Big)^\gamma \Big(\frac{t - p_n}{Km_n}\Big)^\gamma \log\Big(1 + \frac{Km_n}{t - p_n}\Big) \le C\Big(\frac{Km_n}{n}\Big)^\gamma
\end{aligned}$$

since $\sup_{x>0} x^\gamma \log(1 + \frac{1}{x}) \le C$. This proves the bound in (3.21) uniformly in $|t - p_n| \le Km_n, |s - p_n| \le Km_n$. Now, using (3.21) and the telescoping identity $\prod_k a_k - \prod_k b_k = \sum_k (a_k - b_k) \prod_{j<k} a_j \prod_{j>k} b_k$, we obtain

$$\begin{aligned}
|\Theta^a(t, s) - 1| \ &\le \ \sum_{k=s}^{t-1} |\lambda^a_{t,k}| \prod_{j=s}^{k-1} (1 + |\lambda^a_{t,j}|) \\
&\le \ \sum_{k=s}^{t-1} |\lambda^a_{t,k}| \exp\Big\{\sum_{j=s}^{k-1} |\lambda^a_{t,j}|\Big\} \ \le \ C \sum_{k=s}^{t-1} |\lambda^a_{t,k}| \ = \ o(1),
\end{aligned}$$

proving (3.19).

Consider (3.20). From assumptions (3.1) and (3.2) we infer that for any $K > 0$ there exists $K' > 0$ such that the following inequalities

$$d\Big(\frac{k}{n}\Big) \ < \ d\Big(\frac{t}{n}\Big), \qquad \forall\, |t - p_n| \le Km_n, \quad \forall\, 1 \le k \le p_n - K'm_n \tag{3.22}$$

are satisfied. (Indeed, the required K' can be easily determined by $K' = 2K(\Delta_+/\Delta_-)^{1/\gamma}$, for all n large enough.) Next, write

$$|\Theta^a(t, s)| \ = \ \prod_{s \le k < p_n - K'm_n} \big|1 + \lambda^a_{t,k}\big| \prod_{p_n - K'm_n \le j < t} \big|1 + \lambda^a_{t,j}\big|.$$

Here, the first product on the r.h.s. is bounded by 1 since $-1 < \lambda^a_{t,k} < 0$ in view of (3.22) and the definition of $\lambda^a_{t,k}$. The second product tends to 1 as $n \to \infty$ for each $K' < \infty$ according to (3.19). This proves (3.20).

With (3.19)–(3.20) in mind, the proof of Theorem 7 follows from the proof of Theorem 5, with minor modifications. $\square$

Remark 4 The proof of Theorem 7 reduces the study of S_n^a to that of S_n in Theorem 5 by showing that the ratio $\Theta^a(t,s)$ of the corresponding coefficients tends to 1 (see (3.19)). The last fact, as well as the statement of Theorem 7, is not true for the process $\{X_t^b\}$ (under the same assumptions on $d(\cdot)$) since the corresponding ratio $\Theta^b(t,s) := b_{t-s}(t)/\psi_{t-s}\big(d(\frac{t}{n})\big)$ tends to a different limit:

$$\Theta^b(t,s) \quad \to \quad \mathrm{e}^{\Delta_{\mathrm{sgn}(v)}|v|^\gamma - \Delta_{\mathrm{sgn}(u)}|u|^\gamma}, \quad \text{as} \quad (t-p_n)/m_n \to v, \ (s-p_n)/m_n \to u. \tag{3.23}$$

The limit in (3.23) can be shown for any $u < v$ fixed, by writing $\Theta^b(t,s) = \big(d(\frac{t-1}{n})/d(\frac{t}{n})\big) \prod_{k=s}^{t-2}\big(1+\lambda_{t,s,k}^b\big)$ and decomposing the $\lambda_{t,s,k}^b$'s as follows:

$$\lambda_{t,s,k}^b \ := \ \frac{d(\frac{k}{n})-d(\frac{t}{n})}{d(\frac{t}{n})+k-s+1} = \frac{d(\frac{k}{n})-d(\frac{s}{n})}{d(\frac{t}{n})+k-s+1} + \frac{d(\frac{s}{n})-d(\frac{t}{n})}{d(\frac{t}{n})+k-s+1} \ =: \ \lambda'_{t,s,k}+\lambda''_{t,s,k}.$$

Here, $\sum_{k=s}^{t-2}|\lambda'_{t,s,k}| = o(1)$ similarly to (3.21) but $\sum_{k=s}^{t-2}\lambda''_{t,s,k} \sim \big(d(\frac{s}{n})-d(\frac{t}{n})\big)\log(t-s)$ tends to the r.h.s. of (3.23). The above discussion suggests that the partial sums limit of $\big(\frac{\log^{1/\gamma} n}{n}\big)^{d_{\max}+1/2} S_n^b\big(\tau_{\max}+\frac{\tau}{\log^{1/\gamma} n}\big)$ might be written as

$$U^b(\tau) \quad := \quad \kappa(d_{\max})\int_{-\infty}^{\tau} \mathrm{e}^{-\Delta_{\mathrm{sgn}(u)}|u|^\gamma}\eta(\mathrm{d}u)\int_u^\tau (v-u)^{d_{\max}-1}\mathrm{d}v.$$

Note that, contrary to the process U in (3.4), the process U^b is well-defined for any $d_{\max} > 0$ provided $\Delta_- > 0$.

The proofs of Theorems 5 and 7 yield the following corollary.

Corollary 8 *(i) Let the conditions of Theorem 5 be satisfied. Then*

$$\Big(\frac{\log^{1/\gamma} n}{n}\Big)^{d_{\max}+1/2} S_n(\tau) \ \to_{\text{f.d.d.}} \ \begin{cases} 0, & \tau < \tau_{\max}, \\ U(0), & \tau = \tau_{\max}, \\ U(+\infty), & \tau > \tau_{\max}, \end{cases} \tag{3.24}$$

where $\{U(\tau)\}$ is defined in (3.4). Moreover,

$$\Big(\frac{\log^{1/\gamma} n}{n}\Big)^{d_{\max}+1/2} \int_0^1 S_n(\tau)\mathrm{d}\tau \quad \to_{\text{law}} \quad (1-\tau_{\max})U(+\infty), \tag{3.25}$$

$$\Big(\frac{\log^{1/\gamma} n}{n}\Big)^{2d_{\max}+1} \int_0^1 S_n^2(\tau)\mathrm{d}\tau \quad \to_{\text{law}} \quad (1-\tau_{\max})U(+\infty)^2. \tag{3.26}$$

(ii) Let the conditions of Theorem 7 be satisfied. Then relations (3.24), (3.25), (3.26) hold with S_n replaced by S_n^a.

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