Portfolio Management With Benchmark Related Incentives Under Mean Reverting Processes*

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Abstract

We study the problem of a fund manager whose compensation depends on the relative performance with respect to a benchmark index. In particular, the fund manager's risk-taking incentives are induced by an increasing and convex relationship of fund flows to relative performance. We consider a dynamically complete market with N risky assets and the money market account, where the dynamics of the risky assets exhibit mean reversions, either in the drift or in the volatility. The manager optimizes the expected utility of the final wealth, with an objective function that is non-concave. The optimal solution is found by using the martingale approach and a concavification method. The optimal wealth and the optimal strategy are determined by solving a system of Riccati equations. We provide a semi-closed solution based on the Fourier transform.

Keywords: Investment Analysis, Portfolio Management, Optimal Control, Mean Reverting Processes, Fourier Transform

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1 Introduction

We study the optimal policy of a fund manager whose compensation depends on the relative performance with respect to a benchmark. In particular we consider the fund manager's risk-taking incentives induced by an increasing and convex relationship of fund flows to relative performance. Empirical studies relating fund managers strategies to their relative performances originated form the paper by Brown, Harlow and Starks (1996). They found some evidence of the so called *tournament effect* that is the fact that managers whose performances are lower after some evaluation period alter their portfolios by assuming more risk in the attempt to improve their relative position. This behavior may be explained by the competition to get new fund flows from investors entering the market. Several empirical papers, following Sirri and Tufano (1998) and Chevalier and Ellison (1997) studied the relation between fund flows and performances. Basak, Pavlova and Shapiro (2007) (henceforth BPS) determined the optimal strategy for a manager with incentives related to performances with respect to a benchmark, a stylized version of the relation presented by Chevalier and Ellison (1997). BPS considered power utility and a Black-Scholes economy, thus extending the results derived by Merton (1971) to a benchmark related objective function. Their solution shows that a manager of a trailing fund may either increase or decrease the exposure to the risky asset, depending on the risk aversion, a result that provides a theoretical justification to the empirical study of Busse (2001) who casted some doubts on the conclusions of Brown, Harlow and Starks (1996).

One notable feature of the optimization problem solved by BPS is the non-concavity of the objective function. Such a difficulty can be overcome by a concavification argument proposed by Carpenter (2000) who proved that in a complete market the optimal solution takes values on a set where the original non-concave objective function is equal to the minimal concave function dominating it.

A related problem was solved by Cuoco and Kaniel (2011), who studied a problem of delegated portfolio management where managers receive a direct compensation, related to the performance, from investors and determined the conditions for equilibrium. Different compensation schemes have been considered in Barucci and Marazzina (2016) who studied the case of a manager who is compensated through a high water mark incentive fee and in Barucci and Marazzina (2015) where a non convex remuneration fee has been considered under partial information.

Our contribution is mostly related to the field of stochastic optimization, as we provide a solution to the problem of a manager with the same objective function studied by BPS but under more general assumptions for the dynamics of the processes for the risky assets. While BPS consider a model where the market price of risk and the volatility are constant and deterministic, we consider two multi-dimensional models where either the market price of risk or the volatility are mean reverting and, more precisely, Ornstein-Uhlenbeck processes. Optimal asset allocation under mean reversion in the instantaneous short rate has been considered in Hainaut (2009) while Beltratti and Colla (2007) considered more general affine term structure models.

The importance of introducing mean reversion for different aspects of asset price dynamics is demonstrated by a vast empirical literature, starting from the seminal paper by Poterba and Summers (1988) who illustrated that for longer time horizons there is a reduction of the variance of stock returns. Our first model can be seen as an extension to multiple assets of the one proposed by Wachter (2002), who found the optimal portfolio choice for an investor with utility over consumption under mean reverting returns. Our second model is a multi-dimensional extension of Stein and Stein (1991), who computed the value of an option under mean reverting volatility.

To determine the optimal dynamic strategy we compute the Fourier transform of the optimal final wealth, following a methodology frequently adopted in mathematical finance to obtain the value of financial derivatives, see for instance Eberlein, Glau, and Papapantoleon (2010), and Fusai and Meucci (2007), or to compute the variance of the hedging error as in Hubalek, Kallsen, and Krawczyk (2006), Angelini and Nicolosi (2010) and Angelini and Herzel (2015). An application of Fubini's Theorem reduces the evaluation problem to that of computing the conditional characteristic function, with respect to an equivalent martingale measure, of a random variable whose distribution depends on the state variables of the model. The final step exploits the fact that the logarithm of the characteristic function is a linear-quadratic function of the state variables, whose coefficients satisfy a set of Riccati equations. This last argument traces back to the theory of affine processes, firstly proposed by Duffie and Kan (1996) and systematized by Duffie, Filipović, and Schachermayer (2003). The implementation of the optimal strategy is done via an application of the Fast Fourier Transform algorithm.

The rest of the paper is organized as follows. In Section 2 we set the general framework, introduce the optimization problem and obtain the Fourier transform of the optimal final wealth relative to the benchmark. Section 3 computes the optimal dynamic strategy for two specifications of the general model, the first one with a mean reverting market price of risk, the second one with a mean reverting volatility. Section 4 provides some details of the implementation of the algorithm via Fast Fourier Transform and shows a few examples of the optimal policies under different settings. Section 5 concludes. The explicit solutions of the Riccati equations are derived in the Appendix.

2 The portfolio optimization problem

We study the problem of determining the optimal strategy for a risk-averse fund manager whose compensation depends on the fund's value at the end of an evaluation period. The fund's value is determined by the portfolio strategy of the manager during the period and by capital inflows/outflows at the end of the period. The fund flows depend on the fund's performance relative to a benchmark, that is a fixed portfolio of stocks and money market.

Let us define a complete probability space $(\Omega, \mathcal{F}, P)$, with filtration $\{\mathcal{F}_t\}$ $t \geq 0$ generated by a standard N-dimensional Brownian motion Z. The market is composed by a money market account, providing constant interest rate r, and by N risky assets, traded continuously in time. The N-dimensional process of the risky assets' prices follows

$$dS_t = \mathrm{diag}(S_t)\,(\mu_t dt + \sigma_t dZ_t) \tag{1}$$

where $\mathrm{diag}(S)$ is a diagonal matrix with diagonal equal to the vector S, and where the vector μ_t and the matrix σ_t are $\{\mathcal{F}_t\}$-progressively measurable processes and satisfy standard regularity conditions (see, e.g., Karatzas and Shreve (1991)). In addition, to ensure market completeness, we assume that the matrix σ_t is full rank almost surely. The relation between the drift μ_t and the volatility σ_t is given by

$$\mu_t = r\mathbf{1} + \sigma_t X_t,$$

where X is the N-dimensional process of the market prices of risk and $\mathbf{1}$ is the N-dimensional vector whose components are all equal to 1. Hence, the model will be fully specified after we define the dynamics of the volatility matrix and of the market prices of risk which will be done in Section 3.

A fund manager dynamically allocates the fund's assets, initially valued at W_0, through a self-financing strategy. The value of the portfolio W follows

$$\frac{dW_t}{W_t} = (r + \theta_t' \sigma_t X_t)dt + \theta_t' \sigma_t dZ_t \tag{2}$$

where θ_t is the N-dimensional vector representing the fractions of the portfolio invested in the risky assets at time t. Completeness and no arbitrage condition of the market imply the existence and uniqueness of a state price density process ξ whose dynamics

$$\frac{d\xi_t}{\xi_t} = -rdt - X_t' dZ_t$$

is derived from the fact that any tradable asset re-scaled by the state price density is a martingale under the natural measure P. Hence, since the strategy is self financing, the process $\xi_t W_t$ is a martingale with respect to the measure P.

We remark that we are here considering a dynamically complete and arbitrage free market model and hence we can follow the martingale approach proposed by Cox and Huang (1989) and exploit the fact that the budget constraint can be expressed as

$$E\left[\frac{\xi_T}{\xi_0} W_T\right] = W_0.$$

Consistently with the leading practice, the manager's compensation, due at the end of the period T, is proportional to the terminal value of assets under management. Such compensation provides the manager with implicit incentives arising from the fund-flows to relative-performance relationship shown by Chevalier and Ellison (1997). If the manager does well relative to the benchmark Y, usually a market index, her assets under management multiply owing to the inflow of new investors' money; if she does poorly, a part of assets under her management gets withdrawn. The benchmark is a constant portfolio β whose component β^i, for $i = 1, \ldots, N$, represents the fraction invested in the i-th asset S^i while the remaining fraction $1 - \sum_{i=1}^N \beta^i$ is invested in the risk-free money market. Therefore, the dynamics of the benchmark is

$$\frac{dY_t}{Y_t} = (r + \beta' \sigma_t X_t)dt + \beta' \sigma_t dZ_t.$$

We denote the continuously compounded returns on the manager's portfolio and on the benchmark over the period $[0, t]$ by $R_t^W = \ln \frac{W_t}{W_0}$ and $R_t^Y = \ln \frac{Y_t}{Y_0}$, respectively, and normalize $Y_0 = W_0$, without loss of generality. At the terminal date, the fund receives flows at a rate

$$f_T(W_T, Y_T) = \begin{cases} f_L & \text{if} \quad R_T^W - R_T^Y < \eta_L \\ f_L + \psi(R_T^W - R_T^Y - \eta_L) & \text{if} \quad \eta_L \le R_T^W - R_T^Y < \eta_H \\ f_H := f_L + \psi(\eta_H - \eta_L) & \text{if} \quad R_T^W - R_T^Y \ge \eta_H \end{cases} \tag{3}$$

with $f_L > 0, \psi = (f_H - f_L)/(\eta_H - \eta_L) > 0$, and $\eta_L \le \eta_H$. An illustration of the fund flow rate is given by Figure 1. The particular shape of the

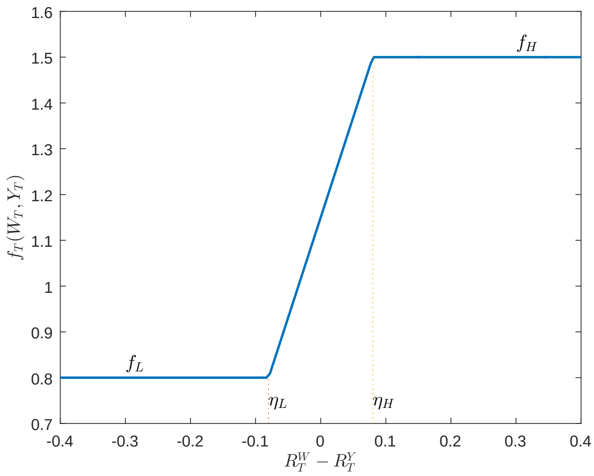

Figure 1: The fund flow rate $f_T(W_T, Y_T)$ as a function of relative performance $R_T^W - R_T^Y$.

function $f_T(W_T, Y_T)$ was proposed by BPS to model the empirical estimates of Chevalier and Ellison (1997) who observed a flat flow rate for managers who performed well below the market (returns smaller than $\eta_L = -8\%$), followed by a linear, upward-sloping segment for returns below $\eta_H = 8\%$ and again by a flat flow rate afterwards. According to BPS, this simple way of modeling fund flows is able to capture most of the insights pertaining risk-taking incentives of a risk averse manager.

The manager, endowed with a constant relative risk aversion utility function

$$u(x) = \frac{x^{1-\gamma}}{1-\gamma}, \quad \gamma > 0,$$

solves the problem

$$\max_{W_T} E[u(W_T f_T(W_T, Y_T))], \tag{4}$$

$$\text{s.t. } E\left[\frac{\xi_T}{\xi_0} W_T\right] = W_0. \tag{5}$$

2.1 Optimal final wealth

We start by computing the optimal final wealth W_T^* using results of BPS. Under the implicit incentives given by (3), the objective function (4) is non-concave with respect to W_T. To find W_T^*, a concavification argument, proposed by Carpenter (2000), can be applied. The general expression for the optimal final wealth relative to the benchmark has been determined by BPS and can be written as

$$V_T = \frac{W_T^*}{Y_T} = \varphi_1(\zeta_T;\zeta^1) \ + \ \varphi_2(\zeta_T;\zeta^1,\zeta^2) \ + \ \varphi_3(\zeta_T;\zeta^2,\zeta^3) \ + \ \varphi_4(\zeta_T;\zeta^3) \tag{6}$$

where $\zeta_T = \xi_T Y_T^\gamma$ and

$$\begin{aligned} \varphi_1(\zeta;\zeta^1) &= f_H^{1/\gamma-1} y^{-1/\gamma} \zeta^{-1/\gamma} \mathbf{1}_{\zeta<\zeta^1} \\ \varphi_2(\zeta;\zeta^1,\zeta^2) &= e^{\eta_H} \mathbf{1}_{\zeta^1\leq\zeta<\zeta^2} \\ \varphi_3(\zeta;\zeta^2,\zeta^3) &= h(\zeta) \mathbf{1}_{\zeta^2\leq\zeta<\zeta^3} \\ \varphi_4(\zeta;\zeta^3) &= f_L^{1/\gamma-1} y^{-1/\gamma} \zeta^{-1/\gamma} \mathbf{1}_{\zeta\geq\zeta^3} \ , \end{aligned} \tag{7}$$

where y is the Lagrange multiplier that ensures that the optimal wealth satisfies $W_0 = E_0\left[\xi_T W_T^*\right]$ and $h(\zeta)$ is the solution of equation

$$\frac{d}{dV} u(V f_L + V\psi(\ln V - \eta_L)) = y\zeta. \tag{8}$$

The expression of the parameters ζ^1, ζ^2 and ζ^3, and hence the value of V_T, depends on the following relation, which we call Condition A:

$$\frac{\gamma}{1-\gamma}\left(\frac{f_H+\psi}{f_L}\right)^{1-1/\gamma} + \left(\frac{f_H+\psi}{f_H}\right) - \frac{1}{1-\gamma} \geq 0. \tag{9}$$

Proposition 2 of BPS shows that, when Condition A holds, then in (6) $\zeta^1 = f_H^{1-\gamma} e^{-\gamma\eta_H}/y$, and $\zeta^3 = \zeta^2 > \zeta^1$ satisfying $g(\zeta) = 0$, with

$$g(\zeta) = \left(\gamma\left(\frac{y}{f_L}\zeta\right)^{1-1/\gamma} - (f_H e^{\eta_H})^{1-\gamma}\right)/(1-\gamma) + e^{\eta_H} y\zeta.$$

Hence, in this case, $\varphi_3(\zeta;\zeta^2,\zeta^3)$ is the indicator function of the empty set and hence it is zero.

When Condition A is not satisfied, Appendix C of BPS, shows that $\zeta^1 = f_H^{1-\gamma}e^{-\gamma\eta_H}/y$, $\zeta^2 = (e^{\eta_H}f_H)^{-\gamma}(f_H+\psi)/y$ and $\zeta^3 = (f_L\underline{V})^{-\gamma}f_L/y$ where the point $\underline{V}$ is the left boundary of the region where the objective function is not concave. Denoting with $\overline{V}$ the right boundary of the non concave region, $\underline{V}$ and $\overline{V}$ can be computed as the points where the straight line between these two points is tangent to the objective function.

Let us notice that the choice of a power-law utility function, that is a homogeneous function, allows us to express the optimal terminal relative wealth as a function of a unique state variable, that is the state price density re-scaled with a power-law function of the benchmark. Without the homogeneity property one would have considered also the benchmark as a state variable, making the problem much more hard to be solved even in the Black-Scholes case.

2.2 Optimal running wealth

To compute the optimal relative wealth $V_t = \frac{W_t^*}{Y_t}$ at any time t, our idea is to apply Fourier transform techniques. We first compute the Fourier transforms $\hat{\varphi}_j$ of the functions φ_j's in (6) with respect to the variable ζ. To simplify the notation, we will not indicate explicitly the dependence of the Fourier transforms on the parameters ζ^1,ζ^2,ζ^3. For $j = 1,2,4$, they come from straightforward calculations:

$$\begin{aligned}
\hat{\varphi}_1(z) &= f_H^{1/\gamma-1}y^{-1/\gamma}\frac{(\zeta^1)^{-1/\gamma+iz}}{-1/\gamma+iz} \qquad (10)\\
\hat{\varphi}_2(z) &= e^{\eta_H}\frac{(\zeta^2)^{iz}-(\zeta^1)^{iz}}{iz}\\
\hat{\varphi}_4(z) &= f_L^{1/\gamma-1}y^{-1/\gamma}\frac{(\zeta^3)^{-1/\gamma+iz}}{-1/\gamma+iz}.
\end{aligned}$$

Section 4.1 shows how to compute numerically the Fourier transform $\hat{\varphi}_3(z)$, which is needed only when Condition A is not satisfied.

Let us denote by Q the measure equivalent to P under which the price process of any traded asset, expressed in units of the numeraire Y, is a martingale (see Björk (2009), chapter 26). Let us consider the conditional characteristic function of $\ln(\zeta_T)$ under Q:

$$H_t(z) = E^Q\left[e^{z\ln(\zeta_T)}|\mathcal{F}_t\right], \tag{11}$$

where z is a complex number. The following proposition shows that the value at time t of the optimal relative value $V_t = \frac{W_t^*}{Y_t}$, is related to the characteristic function H_t.

Proposition 2.1 *Suppose that, for some real numbers $R_j, j = 1, 2, 3, 4$,*

$$H_t(R_j) = E^Q\left[e^{R_j\ln(\zeta_T)}|\mathcal{F}_t\right] < \infty, \tag{12}$$

with $R_1 < -1/\gamma$ and $R_4 > -1/\gamma$. Then

$$V_t = \frac{1}{2\pi}\sum_{j=1}^{4}\int_{-\infty}^{+\infty}\hat{\varphi}_j(u + iR_j)H_t(R_j - iu)du \tag{13}$$

where the integrals in (13) are principal value integrals.

Proof. From (6) and the definition of the martingale measure Q

$$V_t = E^Q\left[\varphi_1(\zeta_T;\zeta^1) + \varphi_2(\zeta_T;\zeta^1,\zeta^2) + \varphi_3(\zeta_T;\zeta^2,\zeta^3) + \varphi_4(\zeta_T;\zeta^3)|\mathcal{F}_t\right].$$

The idea now would be to apply the standard Fourier transform technique for derivative pricing, namely to use Fourier inversion formula to write V_t as

$$V_t = \frac{1}{2\pi}\sum_{j=1}^{4}E^Q\left[\int_{-\infty}^{+\infty}\hat{\varphi}_j(u + iR_j)e^{-i(u+iR_j)\ln(\zeta_T)}du|\mathcal{F}_t\right]. \tag{14}$$

Equation (13) would then be obtained by using Fubini's theorem and exchanging the order of integration in (14). However, the above argument does not directly apply since the functions φ_j's are discontinuous and their Fourier transforms are not integrable. We will then use Theorem 2.7 of Eberlein et al. (2010). For this, we have to check conditions D1 and D2 of that theorem. For Condition D1 we need to find, for each $j = 1, \ldots, 4$, R_j such that the dampened terminal function $e^{-R_j\ln(\zeta)}\varphi_j(\zeta;\ldots)$ is in $\mathbb{L}^1(\mathbb{R})$. This is easily verified when $R_1 < -1/\gamma$, $R_4 > -1/\gamma$ and for any real number R_2 and R_3. Condition D2 holds because of (12). Finally, the prerequisites of Theorem 2.7 on continuity and bounded variation are satisfied in our model as the process $\zeta_t = \xi_t Y_t^\gamma$ is a diffusion. $\square$

Let us remark that the Fourier transform approach proposed in Proposition 2.1 provides us with a semi-closed solution also when Condition A is not satisfied, whereas the method proposed by BPS requires, even for the simple Black-Scholes setting, the use of Monte Carlo simulations.

When there are not implicit incentives, that is when the function f_T is a positive constant, the optimal final wealth can be found directly from the first order conditions for problem (4), (5), and the corresponding relative wealth is

$$V_T^{\text{no incentives}} = (y\zeta_T)^{-1/\gamma}$$

where $y^{1/\gamma} = E^Q\left[\zeta_T^{-1/\gamma}|\mathcal{F}_0\right]$ (since $V_0 = 1$). Hence

$$V_t^{\text{no incentives}} = y^{-1/\gamma} E^Q\left[\zeta_T^{-1/\gamma}|\mathcal{F}_t\right],$$

that is

$$V_t^{\text{no incentives}} = \frac{H_t(-1/\gamma)}{H_0(-1/\gamma)}. \tag{15}$$

We note that Equation (15) provides the optimal relative wealth under implicit incentives in the regions where the return of the managed portfolio is far (above or below) the return of the benchmark, since in these cases the flow function f_T is constant.

3 The optimal strategies for two mean reverting models

We define two models by providing two specifications for the processes μ_t and σ_t in Equation (1) for which we compute the conditional characteristic function (11). From Proposition 2.1 we will then compute the processes for optimal relative wealth and then we will recover the optimal strategies.

We will consider two Markovian models with state variables $\zeta_t = \xi_t Y_t^\gamma$ and ν_t. The N-dimensional process ν_t represents, for the first model, the market price of risk and, for the second one, the volatility of the asset prices. For both models we define the function

$$H(t, \zeta, \nu; z) = E^Q\left[\zeta_T^z|\zeta_t = \zeta, \nu_t = \nu\right],$$

and, using the Markov property,

$$H(t, \zeta, \nu; z) = H_t(z).$$

When the assumptions of Proposition 2.1 hold, it then follows that the optimal relative value given by Equation (13) is the function $V(t, \zeta_t, \nu_t)$. To identify the optimal trading strategy we will compute the dynamics of the optimal wealth $W_t^* = Y_t V_t$ by applying Ito's Lemma and equate its diffusion coefficient to that of (2).

Before we specify the models, we remind that to compute $H_t(z)$ we need to determine the dynamics of the relevant processes under the martingale measure Q. To this end, we use the fact that the process

$$Z_t^Q = Z_t - \int_0^t (\sigma_s' \beta - X_s)\, ds \tag{16}$$

is a Q-Brownian motion (Björk (2009), chapter 26).

3.1 Mean reverting market prices of risk

In this subsection we assume that the volatility process σ_t in Equation (1) is a constant, full rank $N \times N$ matrix which we denote by σ, and that the expected returns μ_t are given by

$$\mu_t = r\mathbf{1} + \sigma X_t$$

where the vector of market prices of risk X_t satisfies

$$dX_t = \lambda_X(\bar{X} - X_t)dt + \sigma_X dZ_t$$

where λ_X is a strictly positive $N \times N$ diagonal matrix, $\bar{X}$ a constant $N \times 1$ vector and σ_X is a constant $N \times N$ full rank matrix. Note that both X and μ are mean reverting, gaussian processes. The constant vector $\bar{X}$ represents the long run expected value of the market price of risk, the matrix λ_X drives the strength of attraction towards $\bar{X}$ and σ_X the uncertainty on the evolution. A specification of this model to the case $N = 1$ and perfect negative correlation between log-returns and the market price of risk has been proposed by Wachter (2002) and applied to the different problem of an agent who optimizes intertemporal consumption without convex incentives.

The dynamics of X under the measure Q is

$$dX_t = \left(\lambda_X(\bar{X} - X_t) + \sigma_X(\sigma'\beta - X_t)\right) dt + \sigma_X dZ_t^Q$$

where the Brownian motion Z^Q is given by (16).

The dynamics of the process $\zeta_t = \xi_t Y_t^{\gamma}$ under Q are obtained by Ito's Lemma

$$\begin{aligned} \frac{d\zeta_t}{\zeta_t} &= \left(r(\gamma-1) + \frac{1}{2}\gamma(\gamma+1)||\beta'\sigma||^2 - (\gamma+1)\beta'\sigma X_t + X_t' X_t \right) dt \\ &+ (\gamma\beta'\sigma - X_t') dZ_t^Q. \end{aligned}$$

The $N+1$ dimensional process (ζ, X) is Markovian, hence the function

$$H(t,\zeta,X;z) = E^Q\left[\zeta_T^z | \zeta_t = \zeta, X_t = X\right]$$

is the conditional characteristic function (11). From the fact that the process $H(t,\zeta_t,X_t;z)$ is a Q-martingale and assuming the usual regularity conditions, we get the following Partial Differential Equation for the function H (for simplicity we drop the arguments of the functions)

$$\begin{aligned} 0 &= \frac{\partial H}{\partial t} + \frac{\partial H}{\partial \zeta}\zeta \left(r(\gamma-1) + \frac{1}{2}\gamma(\gamma+1)||\beta'\sigma||^2 - (\gamma+1)\beta'\sigma X + X'X \right) \\ &+ \frac{1}{2}\frac{\partial^2 H}{\partial \zeta^2}\zeta^2 (\gamma\beta'\sigma - X')(\gamma\sigma'\beta - X) \\ &+ \frac{\partial H}{\partial X}\left(\lambda_X(\bar{X} - X) + \sigma_X(\sigma'\beta - X)\right) + \frac{1}{2}\mathrm{tr}\left(\sigma_X' \frac{\partial^2 H}{\partial X^2}\sigma_X\right) \\ &+ \frac{\partial^2 H}{\partial \zeta \partial X}\zeta\sigma_X(\gamma\sigma'\beta - X) \end{aligned} \tag{17}$$

where $\frac{\partial H}{\partial X}$ and $\frac{\partial^2 H}{\partial \zeta \partial X}$ are $1 \times N$ vectors and $\frac{\partial^2 H}{\partial X^2}$ is the Hessian matrix of H with respect to X, and $\mathrm{tr}(\cdot)$ is the trace operator. The boundary condition at time T is

$$H(T,\zeta,X;z) = \zeta^z, \quad \zeta \in \mathbb{R}^+, X \in \mathbb{R}^N, z \in \mathbb{C} \tag{18}$$

Following Wachter (2002), we guess a solution of the type

$$H(t,\zeta,X;z) = \zeta^z \widetilde{H}(t,X;z) \tag{19}$$

with

$$\widetilde{H}(t,X;z) = e^{A(T-t;z) + B(T-t;z)'X + \frac{1}{2}X'C(T-t;z)X_t}$$

where $A(T-t;z)$ is a scalar function, $B(T-t;z)$ is a $N \times 1$ vector and $C(T-t;z)$ is a symmetric matrix. From (18) we get

$$A(0;z) = 0, \quad B(0;z) = 0, \quad C(0;z) = 0. \tag{20}$$

For ease of notation, in the partial derivatives of the ansatz, we drop the dependence of the functions A, B and C on the complex variable z,

$$\begin{aligned}
\frac{\partial H}{\partial t} &= H\left(\frac{\partial A}{\partial t}+\frac{\partial B'}{\partial t}X+\frac{1}{2}X'\frac{\partial C}{\partial t}X\right) \\
\frac{\partial H}{\partial \zeta} &= H\zeta^{-1}z \qquad (21) \\
\frac{\partial H}{\partial X} &= H(B'+X'C) \qquad (22) \\
\frac{\partial^2 H}{\partial \zeta^2} &= H\zeta^{-2}(z^2-z) \\
\frac{\partial^2 H}{\partial X^2} &= H\left(C+(B+CX)(B'+X'C)\right) \\
\frac{\partial^2 H}{\partial \zeta \partial X} &= H\zeta^{-1}z(B'+X'C)
\end{aligned}$$

Substituting the partial derivatives into (17) and imposing that the coefficients of X^iX^j, for $i,j=1,\dots,N$, and of X^i, for $i=1,\dots,N$, as well as the constant terms, must be zero, we get a system of equations for $A(\tau;z)$, $B(\tau;z)$ and $C(\tau;z)$, where $\tau = T-t$, which we write in matrix form as

$$\frac{\partial C}{\partial \tau} = C\sigma_X\sigma_X'C-(1+z)(C\sigma_X+\sigma_XC)-(C\lambda_X+\lambda_XC)+z(z+1)I \qquad (23)$$

$$\begin{aligned}
\frac{\partial B}{\partial \tau} &= \left(-(z+1)\sigma_X'+C\sigma_X\sigma_X'-\lambda_X\right)B \\
&+ C\left(\lambda_X\bar{X}+(1+z\gamma)\sigma_X\sigma'\beta\right)-z(1+z\gamma)\sigma'\beta \qquad (24) \\
\frac{\partial A}{\partial \tau} &= zr(\gamma-1)+\frac{1}{2}z\gamma(1+z\gamma)||\beta'\sigma||^2+ \qquad (25) \\
&+ B'(\lambda_X\bar{X}+(1+z\gamma)\sigma_X\sigma'\beta)+\frac{1}{2}\mathrm{tr}\left(\sigma_X'BB'\sigma_X\right)+\frac{1}{2}\mathrm{tr}\left(\sigma_X'C\sigma_X\right)
\end{aligned}$$

where I is the $N\times N$ identity matrix.

The explicit solutions to the system of Riccati equations (23), (24), (25) with boundary conditions (20) for the case $N=1$ is provided in Appendix A.1. The existence of the solution in the N-dimensional case is a non trivial issue, that depends on the choices of the coefficients of the model, which is beyond the scope of the paper. When a solution exists, it can be computed numerically, for example by applying the fourth-order Runge-Kutta method.

To compute the optimal strategy we recall that $W_t^* = Y_t V_t$ and that $V_t = V(t, \zeta_t, X_t)$. From Ito's lemma:

$$\begin{aligned} dW_t^* &= Y_t dV_t + V_t dY_t + < dV_t, dY_t >= (\text{drift})dt + \\ &+ Y_t \left(\frac{\partial V}{\partial \zeta} \zeta_t (\gamma \beta' \sigma - X_t') + \frac{\partial V}{\partial X} \sigma_X + V_t \beta' \sigma \right) dZ_t \end{aligned} \quad (26)$$

Equating the coefficients of the Brownian innovations in (2) and (26), we get

$$\theta_t^* = \beta + \frac{1}{V_t} \left(\frac{\partial V}{\partial \zeta} \zeta_t (\gamma \beta - (\sigma^{-1})' X_t) + (\sigma^{-1})' \sigma_X' \left(\frac{\partial V}{\partial X} \right)' \right). \quad (27)$$

The partial derivatives of the function V can be readily computed from (13) by taking the derivative under the integral sign and using (21) and (22). Examples of applications of this formula are provided in Section 4.2.

When there are not implicit incentives, the optimal relative wealth is given by (15). Inserting that into (27) we obtain the optimal strategy in this case:

$$\theta_t^{*\text{no incentives}} = \frac{1}{\gamma} (\sigma^{-1})' X_t + (\sigma^{-1})' \sigma_X' \left(C(T-t; -1/\gamma) X_t + B(T-t; -1/\gamma) \right). \quad (28)$$

When the market prices of risk are deterministic, that is when $\sigma_X = 0$, Equation (28) gives the classical result by Merton (1971), when it is a mean reverting stochastic process, it provides the N-dimensional version of a result by Wachter (2002).

3.2 Mean reverting volatilities

In this subsection we assume that σ_t in Equation (1) is given by

$$\sigma_t = \text{diag}(v_t) L$$

where L is a constant N-dimensional square matrix and v_t, an N-dimensional process representing the volatilities of the assets at time t, satisfies

$$dv_t = -\lambda_v (v_t - \bar{v}) dt + \sigma_v dZ_t, \quad (29)$$

where λ_v is strictly positive N-dimensional diagonal matrix, $\bar{v}$ is an N-dimensional vector, representing the long term expected volatilities, and σ_v

is an N-dimensional square matrix that parameterizes the covariance of the volatility process. Equation (29) for $N = 1$ and a zero correlation between volatility and asset return is the dynamics used in the Stein and Stein (1991) stochastic volatility model.

The instantaneous covariance between assets' returns is

$$\sigma_t \sigma_t' = \text{diag}(v_t) L L' \text{diag}(v_t)$$

therefore the matrix

$$\rho = LL'$$

represents the local correlation of asset returns. For simplicity, we assume a constant, N-dimensional market price of risk X, and hence the expected returns μ_t are given by

$$\mu_t = r\mathbf{1} + \sigma_t X.$$

The dynamics of v_t and $\zeta_t = \xi_t Y_t^\gamma$ under the martingale measure Q are

$$\begin{aligned}
\frac{d\zeta_t}{\zeta_t} &= \Big(r(\gamma-1) + X'X - (\gamma+1) v_t' \text{diag}(\beta) L X + \\
&+ \frac{1}{2}\gamma(\gamma+1) v_t' \text{diag}(\beta)\, \rho\, \text{diag}(\beta) v_t \Big)\, dt + (\gamma v_t' \text{diag}(\beta) L - X') dZ_t^Q \\
dv_t &= \left(-\lambda_v (v_t - \bar{v}) + \sigma_v L' \text{diag}(\beta) v_t - \sigma_v X\right) dt + \sigma_v dZ_t^Q
\end{aligned}$$

The conditional characteristic function of ζ_T under Q is

$$H(t, \zeta, v; z) = E_t^Q \left[\zeta_T^z | \zeta_t = \zeta, v_t = v\right] \; .$$

Assuming that H is a sufficiently smooth function of t, ζ and v to apply Ito's lemma, it must satisfy the partial differential equation (suppressing the arguments of the function)

$$\begin{aligned}
0 &= \frac{\partial H}{\partial t} + \frac{\partial H}{\partial \zeta}\zeta \Big(r(\gamma-1) + X'X - (\gamma+1) v' \text{diag}(\beta) L X \\
&+ \frac{1}{2}\gamma(\gamma+1) v' \text{diag}(\beta)\, \rho\, \text{diag}(\beta) v \Big) \\
&+ \frac{1}{2}\frac{\partial^2 H}{\partial \zeta^2}\zeta^2 (\gamma v' \text{diag}(\beta) L - X')(\gamma L' \text{diag}(\beta) v - X) \\
&+ \frac{\partial H}{\partial v}\left(-\lambda_v (v - \bar{v}) + \sigma_v L' \text{diag}(\beta) v - \sigma_v X\right) + \frac{1}{2}\text{tr}\left(\frac{\partial^2 H}{\partial v^2}\sigma_v {\sigma_v}'\right) \\
&+ \frac{\partial^2 H}{\partial \zeta \partial v}\zeta \sigma_v (\gamma L' \text{diag}(\beta) v - X)
\end{aligned} \tag{30}$$

with boundary condition

$$H(T, \zeta, v; z) = \zeta^z. \tag{31}$$

In Equation (30) we assume that $\frac{\partial H}{\partial v}$ and $\frac{\partial^2 H}{\partial \zeta \partial v}$ are N-dimensional row vectors and that $\frac{\partial^2 H}{\partial v^2}$ is the N-dimensional square matrix, that is the Hessian of H with respect to v.

To solve the problem (30), (31) we make the ansatz

$$H(t, \zeta, v; z) = \zeta^z \widetilde{H}(t, v; z) \tag{32}$$

with

$$\widetilde{H}(t, v; z) = e^{A(T-t;z)+B(T-t;z)'v+\frac{1}{2}v'C(T-t;z)v}$$

where $A(T-t; z)$ is a scalar function, $B(T-t; z)$ is a $N \times 1$ vector and $C(T-t; z)$ is a symmetric matrix. From the boundary condition (31) we get

$$A(0; z) = 0, \quad B(0; z) = 0, \quad C(0; z) = 0. \tag{33}$$

The partial derivatives of the function H are

$$\begin{aligned}
\frac{\partial H}{\partial t} &= H\left(\frac{\partial A}{\partial t} + \frac{\partial B'}{\partial t}v + \frac{1}{2}v'\frac{\partial C}{\partial t}v\right) \\
\frac{\partial H}{\partial \zeta} &= H\zeta^{-1}z \\
\frac{\partial H}{\partial v} &= H(B' + v'C) \\
\frac{\partial^2 H}{\partial \zeta^2} &= H\zeta^{-2}(z^2 - z) \\
\frac{\partial^2 H}{\partial v^2} &= H\left(C + (B + Cv)(B' + v'C)\right) \\
\frac{\partial^2 H}{\partial \zeta \partial v} &= H\zeta^{-1}z(B' + v'C)
\end{aligned}$$

Substituting the partial derivatives in (30) and matching the coefficients of the terms v^i, for $i = 1, \ldots, N$, of the terms $v^i v^j$, for $i, j = 1, \ldots, N$, and the constant, leads to the following system of equations for $A(\tau; z)$, $B(\tau; z)$

and $C(\tau; z)$, with $\tau = T - t$:

$$\begin{aligned}
\frac{\partial C}{\partial \tau} &= +\gamma z(\gamma z+1)\mathrm{diag}(\beta)\,\rho\,\mathrm{diag}(\beta) - (\lambda_v C + C\lambda_v) \\
&\quad +(\gamma z+1)\left(C\sigma_v L'\mathrm{diag}(\beta) + \mathrm{diag}(\beta)L{\sigma_v}'C\right) + C\sigma_v{\sigma_v}'C \quad (34) \\
\frac{\partial B}{\partial \tau} &= -z(\gamma z+1)\mathrm{diag}(\beta)LX - (1+z)C\sigma_v X + C\lambda_v\bar{v} - \lambda_v B \\
&\quad +(\gamma z+1)\mathrm{diag}(\beta)L{\sigma_v}'B + C\sigma_v{\sigma_v}'B \quad (35) \\
\frac{\partial A}{\partial \tau} &= +zr(\gamma-1) + \frac{1}{2}z(z+1)X'X - (1+z)B'\sigma_v X \\
&\quad +B'\lambda_v\bar{v} + \frac{1}{2}\mathrm{tr}\left((BB'+C)\sigma_v{\sigma_v}'\right). \quad (36)
\end{aligned}$$

The problem (34), (35), (36) with the initial conditions (33) is a squared system of $N(N+1)/2 + N + 1$ independent Riccati equations of the same type of those of Section 3.1. As before, the analytical solution for the case $N = 1$ is provided in Appendix A.1 while we obtained numerical solutions to the general case by applying the fourth-order Runge-Kutta method.

To compute the optimal strategy we write the dynamics of the optimal wealth process $W_t^* = Y_t V(t, \zeta_t, v_t)$

$$\begin{aligned}
dW_t^* &= Y_t dV_t + V_t dY_t + < dV_t, dY_t >= (\mathrm{drift})dt + \\
&+ \; Y_t\left(\frac{\partial V}{\partial \zeta}\zeta_t(\gamma\beta'\sigma_t - X') + \frac{\partial V}{\partial v}\sigma_v + V_t\beta'\sigma_t\right)dZ_t
\end{aligned}$$

and equating the diffusive term to the diffusive term of the wealth process in (2) we get:

$$\theta_t^* = \beta + \frac{1}{V_t}\frac{\partial V}{\partial \zeta}\zeta_t\left(\gamma\beta - (\sigma_t^{-1})'X\right) + \frac{1}{V_t}(\sigma_t^{-1})'{\sigma_v}'\left(\frac{\partial V}{\partial v_t}\right)' \quad (37)$$

almost surely.

As in section 3.1, we can compute the optimal strategy in case there are not incentives as

$$\theta_t^{*\text{no incentives}} = \frac{1}{\gamma}(\sigma_t^{-1})'X + (\sigma_t^{-1})'\sigma_v'\left(C(T-t; -1/\gamma)v_t + B(T-t; -1/\gamma)\right).$$

When the volatilities are deterministic, that is when $\sigma_v = 0$, we obtain the classical result by Merton (1971).

4 Implementation and examples

In this section we illustrate the implementation of the algorithm for the numerical integration of the value function in Equation (13) and we show a few examples of optimal strategies for different models.

4.1 The Fast Fourier Transform computation

We implemented the formulas for the optimal strategy in both the models proposed using the Fast Fourier Transform (FFT). The integrals defining the optimal relative wealth in (13) and those that define the optimal strategy either in (27) or in (37) through the partial derivatives of V are of the form:

$$I(x) = \frac{e^{xR}}{2\pi}\int_{-\infty}^{\infty}\chi(u+iR)e^{-iux}du = \frac{e^{xR}}{\pi}\int_{0}^{\infty}\chi(u+iR)e^{-iux}du$$

where $x = \ln\zeta_t$. This is because the conditional characteristic functions (19) and (32) in our models are written as $H(t,\zeta,\cdot;z) = e^{z\ln\zeta}\tilde{H}(t,\cdot;z)$. The last equality in the above expression comes from the fact that the result has to be real and from the symmetry property of the real and imaginary part of the kernel of integration χ.

Using the Simpson's rule with N nodes $u_j = \eta(j-1)$ for $j = 1,\ldots,N$, and truncating the integral at $N\eta$, the integral reads

$$I(x) \approx \frac{\eta e^{xR}}{\pi}\sum_{j=1}^{N}\chi(u_j+iR)e^{-iu_jx}w_j$$

where $w_j = \frac{1}{3}(3+(-1)^j-\delta_{j-1})$ are the Simpson weights and the Kronecker delta δ_j is different from zero only for $j=0$.

The FFT representation of the integral, see Carr and Madan (1999), is obtained by a discretization of x as follows: $x_k = -b+v_k$ where $v_k = \lambda(k-1)$, for $k = 1,\ldots,N$, $b = \frac{N\lambda}{2} - \frac{1}{2}(\ln\zeta^1+\ln\zeta^3)$, λ is the size spacing, and $\eta\lambda = \frac{2\pi}{N}$

$$\begin{aligned} I(x_k) &\approx \frac{\eta e^{x_kR}}{\pi}\sum_{j=1}^{N}e^{-i\frac{2\pi}{N}(j-1)(k-1)}e^{ibu_j}\chi(u_j+iR)w_j \\ &= \frac{\eta e^{x_kR}}{\pi}FFT\left(e^{ibu_j}\chi(u_j+iR)w_j\right). \end{aligned} \quad (38)$$

We compute numerically, using FFT, the Fourier transform of $\varphi_3(\zeta_T)$ in (13). The Fourier transform of φ_3 is defined as

$$\hat{\varphi}_3(z) = \int_{-\infty}^{\infty} \varphi_3(e^x) e^{izx} dx \tag{39}$$

where in this case $x = \ln \zeta_T$. We know from (13) and (38) that $\hat{\varphi}_3(z)$ has to be computed at points $z_j = u_j + iR_3$ for $j = 1, \ldots, N$ where u_j is defined as above. Hence to express (39) in terms of an inverse FFT we need to set a suitable grid for the variable of integration: $x_k = -b + v_k$, for $k = 1, \ldots, N$ where b and v_k are defined as above. Therefore

$$\begin{aligned} \hat{\varphi}_3(u_j + iR_3) &= \lambda e^{bR_3 - ibu_j} \sum_{k=1}^{N} e^{i\frac{2\pi}{N}(j-1)(k-1)} \varphi_3(e^{x_k}) e^{-R_3 v_k} w_k \\ &= N\lambda e^{bR_3 - ibu_j} IFFT\left(\varphi_3(e^{x_k}) e^{-R_3 v_k} w_k\right). \end{aligned} \tag{40}$$

To compute (40) we need to know $\varphi_3(e^x)$ at points $x_k = -b + \lambda(k-1)$, for $k = 1, \ldots, N$. This is done by solving numerically equation (8) for $\zeta = e^{x_k}$.

In the examples presented in Section 4.2, the parameters used to implement the FFT algorithm are $N = 2^{12}$, $\eta = 0.125$, $R_1 = -0.4 - 1/\gamma$, $R_3 = R_2 = R_1$ and $R_4 = 0.4 - 1/\gamma$, when Condition A is satisfied. When such a condition does not hold, a finer grid is needed and we set $N = 2^{14}$, $\eta = 0.125/4$.

4.2 Examples

In the following examples the contract is specified as in BPS, that is: $f_L = 0.8$, $f_H = 1.5$, $\eta_L = -0.08$, $\eta_H = 0.08$ and maturity $T = 1$.

The first example presents the one-dimensional case of the model with mean reverting market price of risk. Both panels of Figure 2 show the optimal strategy at time $t = 0.25$, as a function of the relative return with respect to the benchmark. On the left panel we see the influence of σ_X, the volatility of the market price of risk, on the right one that of its long term expected value $\bar{X}$. For both panels we assumed a current value of the market price of risk $X_t = 0.5$ and a coefficient $\lambda_X = \ln(2)/0.5$ so that the expected half-life to decay to the long term level is six months. For the left panel we assumed a long term mean equal to the current level of the market price of risk, and set σ_X to 0, to -0.10 and to -0.20 (we chose negative values to get a negative correlation between asset returns and risk premium variations as it is usually observed, see e.g. Wachter (2002)). The continuous line, corresponds to the deterministic case of BPS, represented in Figure 1 (a) of their paper. For the right panel we set $\sigma_X = -0.10$ and $\bar{X}$ equal to X_t, $0.5X_t$ and $2X_t$. As for the remaining parameters, we set $\gamma = 1.5, \beta = 1, r = 0$ and $\sigma = 0.2$. With such choices of the parameters, Condition A is seen to be satisfied.

In both the panels of Figure 2 we see that the optimal strategy without incentives, that is the asymptotic value (28) corresponding to the extreme levels of relative performances, is more leveraged than the benchmark, since the share of wealth invested in the risky asset is higher than the value of β. Therefore we may say that in this case the natural attitude of the manager would be to bet on the risky asset. For less extreme, but still negative performances, we see a hump in the optimal strategy, which shows that the manager increases her bet to beat the benchmark. When the performances reach more positive levels, the manager decreases the leverage at levels closer to the value of β and, when safely ahead of the benchmark, she returns to her asymptotic policy.

In the left panel of Figure 2 we note that the fund manager pursues a more aggressive strategy for more negative values of σ_X. The reason is that she is betting on a positive return of the stock, which, because of the negative correlation with the market price of risk, will produce a reduction of the expected rate of return of the stock. In the right panel of Figure 2 we see that, when the market price of risk is expected to double in the future, the risk shifting range gets wider, since the manager believes that it is still possible to improve the performance and beat the benchmark even when the current return of her fund is much lower than that of the benchmark. Moreover, we see that the manager, waiting for more favorable times, takes more prudent positions for higher values of the long run market price of risk. Figure 3 replicates the analysis of Figure 2 for a choice of the parameter set that induces an optimal exposure without incentives less leveraged than the benchmark. In particular, we set the parameters as $\gamma = 5, \beta = 1, r = 0, \sigma = 0.2, t = 0.25, T = 1, X_t = 0.5, \lambda_X = \ln(2)/0.5$ We remark that in this case Condition A is not satisfied. Now to beat the benchmark, the manager prefers to place her bets on a negative performance of the stock. In the left panel we represent the optimal strategies for different values of the parameter σ_X. The continuous line, corresponds to the deterministic case of BPS, represented in Figure 1 (b) of their paper. In the figure we observe a new phenomenon, a kind of step for values of the performance

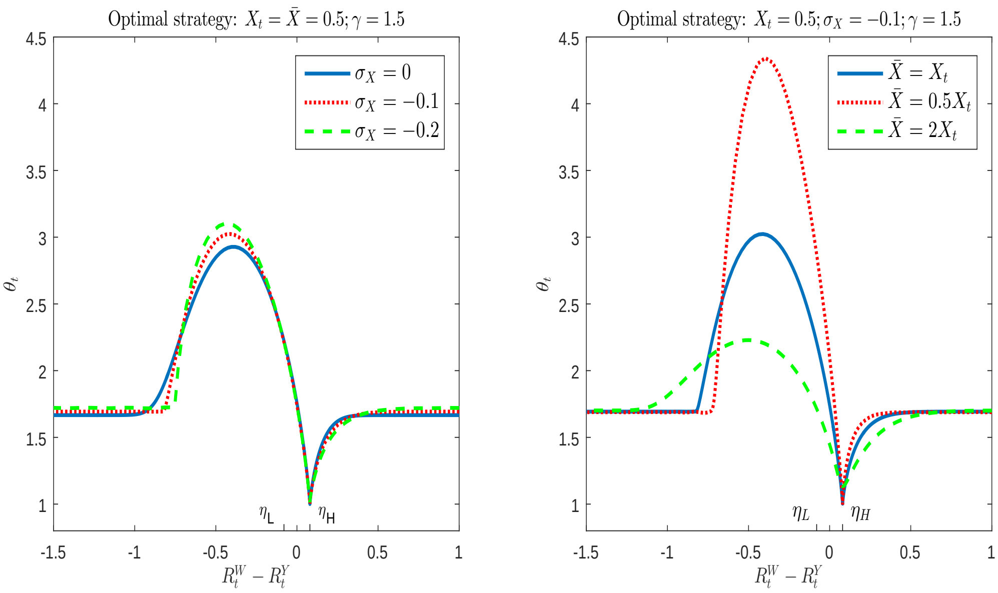

Figure 2: The optimal strategy as a function of the relative return of the fund with respect to the benchmark for a mean reverting market price of risk. The left panel shows different levels of the uncertainty on the market price of risk, the right one different levels of long run expected level of the market price of risk. For both panels we used the parameters $\gamma = 1.5, \beta = 1, r = 0, \sigma = 0.2, t = 0.25, T = 1, X_t = 0.5, \lambda_X = \ln(2)/0.5$. In this case Condition A is satisfied.

around zero. For such a range of values it is high the probability that at the horizon T the manager ends in the region between the upper bound $\ln \overline{V}$ of the non concavity region and the upper threshold η_H of the fund flow to relative performance relationship [1]. When the probability of ending in the sub-optimal non concavity region is high, the manager deviates from her constant asymptotic strategy in order to end up outside that region. This gives the huge hump of the strategy at negative values of the performance. Then for higher values of the performance, the manager feels already safe, outside the sub-optimal non concavity region, and hence she takes a roughly constant strategy. This gives the step in Figure 3. When she is close to η_H she takes an additional risk in order to obtain the maximum compensation. Because of the negative correlation between stock returns and market price of risk, a negative return of the stock increases its expected return. Consequently, the left panel is the mirror image of the left panel in Figure 2. Interestingly, this is not the case for the right panel. In fact, if the return of the risky asset is expected to increase (i.e. when $\bar{X} = 2X_t$), the manager prefers to take a more extreme bet today rather than wait for tomorrow when, most likely, the expected return for her bearish strategy will be less favorable.

Figure 4 represents the optimal strategies for different parameterizations of the one dimensional model with mean reverting volatility. The left panel shows the effect of the parameter σ_v, the volatility of the asset volatility, and the right one that of the long term volatility $\bar{v}$. We see that the effects are reversed with respect to the mean reverting risk premium, in fact while in the previous case an increase (in absolute value) of the parameter driving the diffusive factor of the risk premium produced a stronger position in the stock in the risk-shifting range, the corresponding increase for the present case produces a weaker position in the stock. The same observation applies to the long term expected value of the mean reverting factor. The intuitive reason is that, while an increase in the risk premium represents an improvement of the opportunity set of the manager, who will defer the investment to the more favorable period expected, a corresponding increase in volatility will only increase the risk of the investment and hence the manager does not wait for better times to come.

In Figure 5 we analyze a market with two assets. We assume that the benchmark coincides with the first asset and that the risk premium on the

[1] In Figure 1 (b) of BPS, such a step is not observed as $\ln \overline{V}$ is closer to η_H with respect to the case we analyze here in Figure 3.

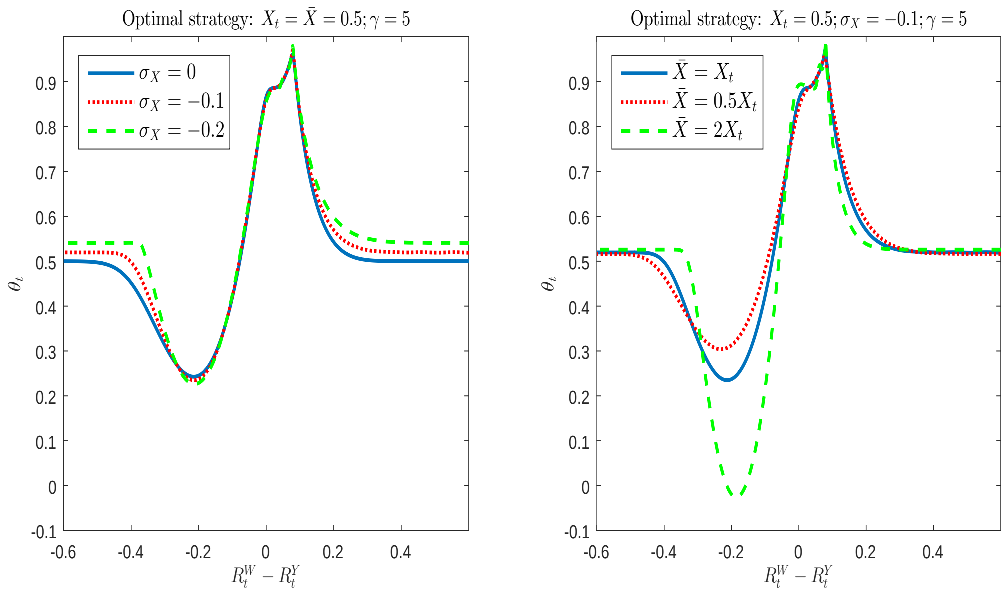

Figure 3: The optimal strategy as a function of the relative return of the fund with respect to the benchmark for a mean reverting market price of risk. The left panel shows different levels of the uncertainty on the market price of risk, the right one different levels of long run expected level of the market price of risk. For both panels we used the parameters: $\gamma = 5, f_L = 0.8, f_H = 1.5, \eta_L = -0.08, \eta_H = 0.08, \beta = 1, r = 0, \sigma = 0.2, t = 0.25, T = 1, X_t = 0.5, \lambda_X = \ln(2)/0.5$. In this case Condition A is not satisfied.

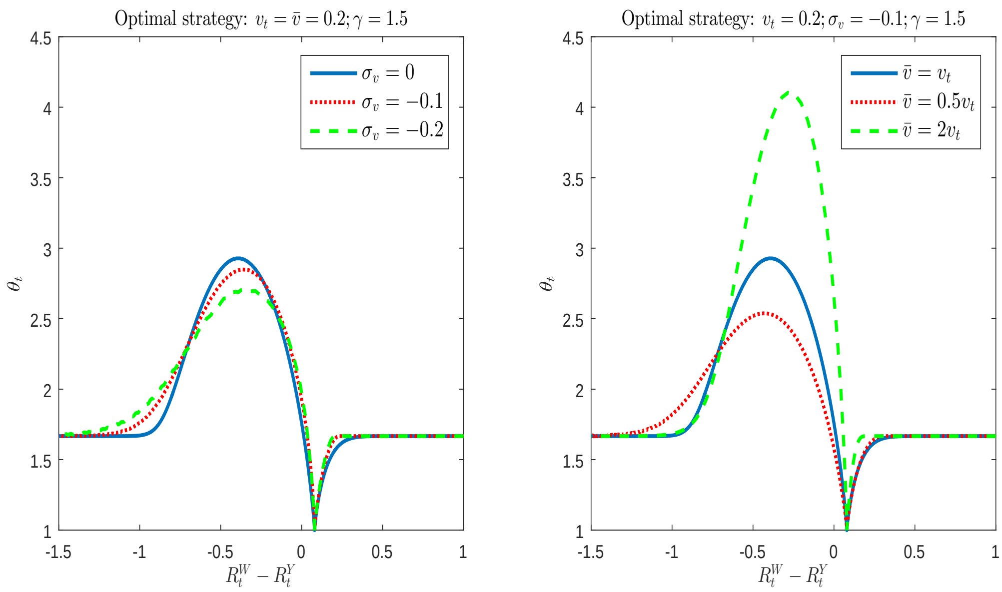

Figure 4: The optimal strategy as a function of the relative return of the fund with respect to the benchmark for a mean reverting volatility. The left panel shows different levels of the uncertainty on the volatility the right one different levels of long run expected level of the volatility. For both panels we used the parameters $\gamma = 1.5, f_L = 0.8, f_H = 1.5, \eta_L = -0.08, \eta_H = 0.08, \beta = 1, r = 0, X = 0.5, t = 0.25, T = 1, v_t = 0.2, \lambda_v = \ln(2)/0.5$

second asset is zero, that is, it produces only idiosyncratic risk. The figure represents the exposures to the two assets for the case of a mean reverting volatility and of a deterministic one. When the volatility is deterministic (solid line), we replicate the case examined by BPS in their Figure 6, and we see that, as expected, the manager does not assume any position in the second asset. Interestingly, however, for a mean reverting volatility (dotted line), the manager does take a position also in the second asset. The reason is that, since the second asset is correlated to the stochastic volatility, it is used by the manager to hedge the volatility risk. Therefore it is kept in the optimal portfolio strategy for purely hedging purposes, even if it does not bring any premium.

5 Conclusion

We presented an algorithm to compute the optimal strategy of a portfolio manager with power utility who optimizes the final wealth by taking into account the implicit incentives coming from fund flow relative to a benchmark, for a class of multi-dimensional asset processes. Our work, based on a fundamental result by Basak, Pavlova and Shapiro (2007), who solved the problem when the driving process is a Geometric Brownian Motion, considers models with mean reversion either in the market price of risk or in the volatility of the stock.

The algorithm, based on Fourier Transforms, reduces the problem of finding the optimal value function to that of computing the characteristic function of a random variable that depends on the state price density, the benchmark and the risk aversion of the fund manager. In fact, such a characteristic function provides a different way to write the solution to the simpler problem of a manager optimizing her own wealth, that is the same objective function studied by Merton (1971). When either the market price of risk or the volatility of the assets are multi-dimensional Ohrstein-Ulenbeck processes, the characteristic function is an exponential affine-quadratic function in the state variables, whose coefficients are the solutions of a system of Riccati equations.

We propose an implementation of the semi-closed formula for the optimal strategy with Fast Fourier Transform. A nice feature of our algorithm is that it works even in the case when the solution proposed by Basak, Pavlova and Shapiro (2007) needs to adopt Monte Carlo simulations.

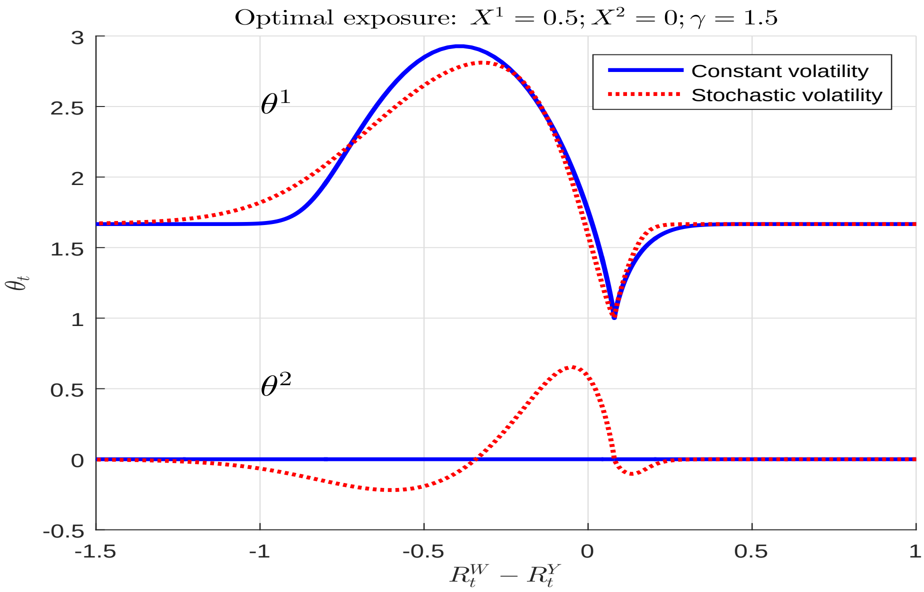

Figure 5: Optimal strategy as a function of the relative return of the fund with respect to the benchmark for mean reverting volatilities (dotted lines) and constant volatility (continuous lines). The parameters common to both the cases are: $\gamma = 1.5, f_L = 0.8, f_H = 1.5, \eta_L = -0.08, \eta_H = 0.08, \beta = (1,0)', r = 0, X = (0.5,0)', t = 0.25, T = 1, v_t = (0.2,0.2)'$. The correlation between the two assets is zero. For the stochastic volatility case we set also $\bar{v} = (0.2,0.2)'$, $\lambda_v = \mathrm{diag}(\ln(2)/0.5, \ln(2)/0.5)$ and matrix σ_v is set in such a way that for both the assets, the volatility of volatility is 0.2 while the correlation between the volatility and the asset return is -0.8.

Some examples illustrate the impact on the optimal strategies of mean reversion in market price of risk, showing that it may induce the manager to adopt either a more aggressive or a more prudent strategy, depending on which strategy she would choose when investing her own wealth. A case with two assets and stochastic volatility shows that it may be optimal to invest in a stock with zero risk-premium when it can be used to hedge the volatility risk.

Even though our findings may be sensitive to the particular choice of the utility function, we guess that the risk shifting behavior of the manager should still be observable for different kinds of utility. Indeed such a behavior is a consequence of the sub-optimality of the non-concave region of the objective function and of the convex incentive scheme, and not of the particular shape of the utility.

An econometric analysis on mutual fund returns to check if the optimal strategies for the model are supported by real data would be both challenging and very interesting.

A Appendix

A.1 Solutions of the Riccati equations for the one-dimensional models

We show the solutions $A(\tau; z)$, $B(\tau; z)$ and $C(\tau; z)$, with $\tau = T - t$, of the Riccati equations arising from our models, when the number of assets is $N = 1$, both in the case of stochastic market price of risk of Section 3.1 and in the case of stochastic volatility of Section 3.2.

The basic result is that the solution of the equation

$$\begin{aligned} \frac{\partial C}{\partial \tau} &= aC^2 + bC + c \\ C(0) &= 0 \end{aligned} \tag{41}$$

where $a, b, c \in \mathbb{C}$, with $a \neq 0$, is given by

$$C(\tau) = \frac{\alpha_+ \alpha_- (e^{\alpha\tau} - 1)}{\alpha_+ e^{\alpha\tau} - \alpha_-}$$

where

$$\alpha_\pm = \frac{-b \pm \alpha}{2a}$$

and

$$\alpha = \sqrt{b^2 - 4ac}$$

where $b^2 - 4ac \in \mathbb{C} - \mathbb{R}_-$ and $\sqrt{\cdot}$ denotes the analytic extension of the real square root to $\mathbb{C} - \mathbb{R}_-$. Moreover

$$\int_0^s C(u)du = \alpha_+ s + \frac{\alpha_- - \alpha_+}{\alpha} \ln \left(\frac{\alpha_+ e^{\alpha s} - \alpha_-}{\alpha_+ - \alpha_-} \right).$$

Such a result is a particular case of Lemma 10.12 in Filipović (2009) and it can be obtained by standard methods for differential equations. For convenience of the reader, we show how to integrate $C(u)$ through the change of variable $x = e^{\alpha u}$:

$$\begin{aligned}
\int_0^s C(u)du &= \frac{\alpha_+\alpha_-}{\alpha} \int_1^{e^{\alpha s}} \frac{x-1}{x(\alpha_+ x - \alpha_-)} dx \\
&= \frac{\alpha_+\alpha_-}{\alpha} \left(\int_1^{e^{\alpha s}} \frac{1}{\alpha_- x} dx + \int_1^{e^{\alpha s}} \frac{\alpha_- - \alpha_+}{\alpha_-(\alpha_+ x - \alpha_-)} dx \right) \\
&= \alpha_+ s + \frac{\alpha_- - \alpha_+}{\alpha} \ln \left(\frac{\alpha_+ e^{\alpha s} - \alpha_-}{\alpha_+ - \alpha_-} \right).
\end{aligned}$$

Equation (41) is an equation for $C(\tau)$ obtained when setting to zero the quadratic terms of the partial differential equation for $H_t(z)$ for both models. In the model with stochastic market price of risk this is Equation (23), hence in this case the coefficients of Equation (41) are

$$\begin{aligned}
a &= \sigma_X^2, \\
b &= -2(\lambda_X + (1+z)\sigma_X), \\
c &= z(z+1).
\end{aligned}$$

In the model with stochastic volatility this is Equation (34), hence the coefficients are

$$\begin{aligned}
a &= \sigma_v^2, \\
b &= -2(\lambda_v - (\gamma z + 1)\sigma_v \beta), \\
c &= \gamma z(\gamma z + 1)\beta^2.
\end{aligned}$$

The linear terms give a linear differential equation of the first order for $B(\tau)$ in both models:

$$\begin{aligned}
\frac{\partial B}{\partial \tau} &= Bg(\tau) + f(\tau) \qquad (42) \\
B(0) &= 0,
\end{aligned}$$

where

$$g(\tau) = \frac{b}{2} + aC(\tau).$$

Also, we can write

$$f(\tau) = a_f + b_f C(\tau)$$

with coefficients $a_f, b_f \in \mathbb{C}$ that depend on the model. Namely, from Equation (24), in the case of mean reverting market price of risk, we have

$$\begin{aligned} a_f &= -z(1 + z\gamma)\sigma\beta \\ b_f &= \lambda_X \bar{X} + (1 + z\gamma)\sigma_X \sigma\beta \end{aligned}$$

and from Equation (35) for the model with mean reverting volatility

$$\begin{aligned} a_f &= -z(1 + z\gamma)\beta X \\ b_f &= \lambda_v \bar{v} - (1 + z)\sigma_v X. \end{aligned}$$

We can now proceed to the computation of $B(\tau)$ for both models at once. The solution of (42) is

$$B(\tau) = e^{\int_0^\tau g(u)du} \int_0^\tau f(s) e^{-\int_0^s g(u)du} ds. \tag{43}$$

We have

$$\begin{aligned} \int_0^s g(u)du &= \frac{b}{2}s + a\int_0^s C(u)du \\ &= (\frac{b}{2} + a\alpha_+)s + a\frac{\alpha_- - \alpha_+}{\alpha} \ln\left(\frac{\alpha_+ e^{\alpha s} - \alpha_-}{\alpha_+ - \alpha_-}\right) \\ &= \frac{\alpha}{2}s - \ln\left(\frac{\alpha_+ e^{\alpha s} - \alpha_-}{\alpha_+ - \alpha_-}\right). \end{aligned}$$

Therefore

$$e^{-\int_0^s g(u)du} = \frac{\alpha_+ e^{\frac{\alpha}{2}s} - \alpha_- e^{-\frac{\alpha}{2}s}}{\alpha_+ - \alpha_-}.$$

Next we compute

$$\begin{aligned} \int_0^\tau f(s)e^{-\int_0^s g(u)du} ds &= \int_0^\tau \left(a_f + b_f \frac{\alpha_+\alpha_-(e^{\alpha s} - 1)}{\alpha_+ e^{\alpha s} - \alpha_-}\right) \frac{\alpha_+ e^{\frac{\alpha}{2}s} - \alpha_- e^{-\frac{\alpha}{2}s}}{\alpha_+ - \alpha_-} ds \\ &= \frac{2a_f}{\alpha(\alpha_+ - \alpha_-)}\left(\alpha_+(e^{\frac{\alpha\tau}{2}} - 1) + \alpha_-(e^{\frac{-\alpha\tau}{2}} - 1)\right) \\ &+ \frac{2b_f\alpha_+\alpha_-}{\alpha(\alpha_+ - \alpha_-)}\left(e^{\frac{\alpha\tau}{2}} + e^{\frac{-\alpha\tau}{2}} - 2\right). \end{aligned}$$

Substituting in Equation (43), we obtain:

$$\begin{aligned} B(\tau) &= \frac{2a_f}{\alpha}\left(\frac{\alpha_+(e^{\frac{\alpha\tau}{2}}-1)+\alpha_-(e^{\frac{-\alpha\tau}{2}}-1)}{\alpha_+ e^{\frac{\alpha}{2}\tau}-\alpha_- e^{-\frac{\alpha}{2}\tau}}\right) \\ &+ \frac{2b_f\alpha_+\alpha_-}{\alpha}\left(\frac{e^{\frac{\alpha\tau}{2}}+e^{\frac{-\alpha\tau}{2}}-2}{\alpha_+ e^{\frac{\alpha}{2}\tau}-\alpha_- e^{-\frac{\alpha}{2}\tau}}\right). \end{aligned}$$

Finally, we get $A(\tau)$ in both models by direct integration.

References

[1] Angelini, F., Herzel, S. (2015). Evaluating discrete dynamic strategies in affine models. *Quantitative Finance*, 15(2), 313-326.

[2] Angelini, F., Nicolosi, M. (2010). On the Effect of Skewness and Kurtosis Misspecification on the Hedging Error. *Economic Notes*, 39(3), 203-226.

[3] Barucci, E., and Marazzina, D. (2016). Asset management, High Water Mark and flow of funds. *Operations Research Letters* 44(5), 607-611.

[4] Barucci, E., and Marazzina, D. (2015). Risk seeking, non-convex remuneration and regime switching. *International Journal of Theoretical and Applied Finance* 18(02), DOI: http://dx.doi.org/10.1142/S0219024915500090.

[5] Basak, S., Pavlova, A., and Shapiro, A. (2007). Optimal asset allocation and risk shifting in money management. *The Review of Financial Studies* 20(5), 1583-1621.

[6] Beltratti, A., Colla, P. (2007). A portfolio-based evaluation of affine term structure models. *Annals of Operations Research*, 151(1), 193-222.

[7] Björk T. (2009). *Arbitrage theory in continuous time.* Oxford university press.

[8] Brown, K. C., Van Harlow, W. and Starks, L.T. (1996). Of tournaments and temptations: An analysis of managerial incentives in the mutual fund industry. *The Journal of Finance* 51(1), 85-110.

[9] Busse, J. A. (2001). Another look at mutual fund tournaments. *Journal of financial and quantitative Analysis*, 36, 53-73.

[10] Campbell, J. Y., and Shiller, R. (1988). The Dividend-Price Ratio and Expectations of Future Dividends and Discount Factors. *Review of Financial Studies*, 1, 195-227.

[11] Carpenter, J. N. (2000). Does Option Compensation Increase Managerial Risk Appetite? *Journal of Finance*, 55, 2311 - 31.

Carr, P., and Madan, D. (1999). Option valuation using the fast Fourier transform. *Journal of computational finance* 2(4), 61-73.

[12] Chevalier, J., and Ellison, G. (1997). Risk Taking by Mutual Funds as a Response to Incentives. *The Journal of Political Economy*, 105, 1167-1200.

[13] Cox, J.C., and Huang, C.F. (1989). Optimal Consumptions and Portfolio Policies when Asset Prices Follow a Diffusion Process. *Journal of Economic Theory*, 49, 33 - 83.

[14] Cuoco, D., and Kaniel, R. (2011). Equilibrium prices in the presence of delegated portfolio management. *Journal of Financial Economics*, 101, 264-296.

[15] Duffie, D., Filipović,D. and Schachermayer, W. (2003). Affine processes and applications in finance. *Annals of applied probability*, 984-1053.

[16] Duffie, D., and Kan, R. (1996). A yield-factor model of interest rates. *Mathematical finance*, 6, 379-406.

[17] Eberlein, E., Glau, K., and Papapantoleon, A. (2010). Analysis of Fourier Transform Valuation Formulas and Applications, *Applied Mathematical Finance*, 17 (3), 211-240.

[18] Fama, E. F., and French, K. R. (1989). Business Conditions and Expected Returns on Stocks and Bonds. *Journal of Financial Economics*, 29, 23-49.

[19] Filipović, D. (2009). *Term Structure Models*, Springer-Verlag, Berlin.

[20] Fusai, G., Meucci, A. (2007). Pricing discretely monitored Asian options under Lévy processes. *Journal of Banking & Finance*, 32(10), 2076-2088

[21] Karatzas, I., and Shreve, S. E. (1991). *Brownian motion and stochastic calculus.* Springer-Verlag, Graduate Texts in Mathematics 113 .

[22] Hainaut, D. (2009). Dynamic asset allocation under VaR constraint with stochastic interest rates. *Annals of Operations Research*, 172(1), 97-117

[23] Hubalek, F., Kallsen, J., and Krawczyk, L. (2006). Variance-Optimal Hedging for Processes with Stationary Independent Increments. *The Annals of Applied Probability*, 16(2), 853-885

[24] Merton, R. (1971). Optimum Consumption and Portfolio Rules in a Continuous-Time Model. *Journal of Economic Theory*, 3, 373-413.

[25] Poterba, J. M., and Summers, L. H. (1988). Mean Reversion in Stock Prices: Evidence and Implications. *Journal of Financial Economics*, 22, 27-59.

[26] Sirri, E. R., and Tufano, P. (1998). Costly search and mutual fund flows. *The Journal of Finance*, 53, 1589-1622.

[27] Stein, E. M., and Stein, J. C. (1991). Stock Price Distributions with Stochastic Volatility: An Analytic Approach. *Review of Financial Studies*, 4, 727-752

[28] Wachter, J. A. (2002). Portfolio and Consumption Decisions under Mean-Reverting Returns: An Exact Solution for Complete Markets. *Journal of financial and quantitative analysis*, 37, 63-91.