

Ambiguity, Entrepreneurs, and Corporate Finance*

Preliminary and Incomplete

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Abstract

Entrepreneurs are often considered engines of growth due to their ability and/or willingness to delve into uncharted economic terrain. We follow a literature that captures this idea with a non-Bayesian multiprior approach to decision making and consider an entrepreneur that faces investment opportunities with ambiguous outcomes. When self-financed, we show that ambiguity may cause entrepreneurs to be cautious in exercising expansion options but reluctant in abandoning pre-existing assets, even when abandonment payoffs are relatively large. With external financing, the optimal financing arrangements critically depends on the way in which ambiguity-sensitive preferences are modelled as well as on differences in ambiguity attitudes of financiers and entrepreneurs. Ambiguity can therefore have significantly different, and potentially contradicting, implications on real and financial decisions, depending on the model a researcher uses to accommodate deviations from the Bayesian paradigm of decision making. These findings suggest caution when using ambiguity aversion to “explain” empirically observed phenomena and provide a base for future empirical work that will shed light on how new ambiguous opportunities are in fact handled.

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1 Introduction

Entrepreneurs have long been considered engines of growth due to their ability and/or willingness to delve into uncharted economic terrain. Knight (1921) introduced the idea that delving into uncharted economic terrain could be thought of as taking an action when one is unable to identify a unique distribution governing the action's outcomes. A situation where probabilities are unknown is usually referred to as *Knightian uncertainty* or *ambiguity*, in contrast to *risk*, in which a decision maker acts under the assumption of a unique, known probability distribution over random outcomes.¹ Knight, in his theory of profit and entrepreneurship, argues that entrepreneurs are special given their role in identifying ambiguous opportunities and/or in their ability and willingness to deal with them. As he put it:

“[...] the facts upon which the working-out of the organization depends can no longer be objectively determined with accuracy by experiment; all the data in the case must be *estimated*, subject to a larger or smaller margin of error [...] The function of making these estimates and of “guaranteeing” their value to the other participating members of the group falls to the responsible entrepreneur in each establishment, producing a new type of activity and a new type of income entirely unknown in a society where uncertainty is absent.” (Knight, 1921, p. 276, his emphasis)

In this paper, we examine a model of entrepreneurs who are assumed to have monopoly access to an ambiguous project. We follow a literature that has formalized the idea of ambiguity in terms of multiple prior distributions over random events. In this setting we examine the allocation of savings to investment projects when ambiguity is present. We consider a canonical corporate finance model: an entrepreneur (E) has a potentially economically valuable idea but lacks the required investment resources while a financier (F) has the required resources but does not have direct access to investment opportunities. We examine how investment and financing decisions are affected by ambiguity and ambiguity attitudes of the agents involved.

We contrast our analysis to standard finance theory that is built on the Bayesian paradigm of subjective expected utility (SEU) axiomatized by of Savage (1954). The fundamental implication of the SEU paradigm is that individuals select among actions with risky outcomes by attaching

¹The terms “ambiguity”, “uncertainty” and “Knightian uncertainty” are commonly used interchangeably to describe situations with unknown probabilities.

a utility index to each outcome and a unique probability to the likelihood that the outcome will obtain. The decision maker then choose actions that maximize expected utility.

Despite being the dominant paradigm for decision making in the vast majority of finance applications, it is well-known that the SEU paradigm is not equipped to deal with several phenomena observed in experimental studies. Seminal among these studies are the thoughts experiments of Ellsberg (1961) that highlight how ambiguity² affects individuals' willingness to bet.³ Although Ellsberg's experiments and several other experimental papers emphasize the subjects' *aversion* to ambiguity, there are also studies that have shown situations in which subjects are ambiguity *loving*, as in Heath and Tversky (1991) "competence hypothesis" (see Luce (2000) for an extensive survey of the experimental evidence). It is however undeniable that the existing evidence in several fields of studies suggests that ambiguity and ambiguity attitudes are important for choice.

Decision theorists have proposed several extensions of the SEU paradigm to accommodate choices that are sensitive to ambiguity and ambiguity attitude of the decision maker. These generalizations typically involve representations of preferences in which a *set* of probabilities (instead of a unique probability) is involved.⁴ Perhaps the most widely cited generalization of SEU is the "Maxmin expected utility with multiple priors" (MEU) of Gilboa and Schmeidler (1989), in which beliefs are characterized by the prior that delivers the lowest expected utility. These preferences can "explain" the decisions observed in the Ellsberg's experiments.

An alternative decision rule based on multiple priors is the "Inertia Based Expected Utility" (IBEU) proposed by Bewley (2002) as a formalization of Knight's idea of uncertainty. This preference representation can be obtained from SEU by removing the assumption that preferences are complete (i.e., any two gambles can always be ranked) and replace it with an assumption of *inertia* (i.e., a gamble is never accepted unless acceptance is preferred to rejection). Bewley

²Ellsberg refers to ambiguity as "a quality depending on the amount, type, reliability and 'unanimity' of information, and giving rise to one's 'degree of confidence' in an estimate of relative likelihoods." (Ellsberg, 1961, p. 657).

³One of Ellsberg's experiments involves two urns with 100 balls each. In the first urn, the unambiguous urn, the subject is told that there are 50 white and 50 blue balls. In the second urn, the ambiguous urn, no information is given on the proportion of white and blue balls. The subject has to choose an urn and a color. After that, a ball will be drawn from the chosen urn and a prize will be awarded if the drawn ball is of the chosen color. The vast majority of subjects chooses to place either of the bets (blue or white ball) on the unambiguous urn. This behavior cannot be justified by any probability distribution since it implies that the subject believes that the probability of a specific outcome (drawing a blue ball from the ambiguous urn) is both less than and greater than 50%.

⁴An alternative representation characterizes agents' beliefs through expected utility in which expectations are computed with respect to a non-additive probability (capacity), as in the "Choquet Expected Utility" of Schmeidler (1989).

(2002) shows that by removing the assumption of completeness from SEU one obtains a *set* of probability distributions that allows to characterize the choice of an agent according to a “unanimity rule”: a gamble is preferred to another if and only if its expected value is higher under *all* possible probability distributions.⁵ Because these unanimity preferences are incomplete, there does not exist a numerical index that represents them, making them unsuitable for optimization problems on which large part of economic decision making is based. To complete the model, Bewley (2002) imposes the assumption of inertia under which a person remains with the status quo unless an alternative is deemed unanimously better.⁶ It is important to note that in Bewley’s characterization, the status quo is not considered a behavioral *bias* but a device to complete the preferences. The treatment of the status quo is conceptually different in the behavioral economic literature that relies on “reference-dependent” preferences to analyze the implications of biases such as the endowment effect, loss aversion or framing.⁷

Several other approaches to decision making in the presence of ambiguity have been examined in more general settings, as Gilboa and Marinacci (2011) discuss in their excellent survey. It is, however, beyond the scope of our paper to explore all of these approaches as we are primarily concerned with basic investment and financing issues in the presence of ambiguity. As a result, we examine the entrepreneur’s problem in the context of SEU, MEU and IBEU only.

Our analysis delivers a number of novel interesting findings. In the context of real investment decisions, we show that MEU and IBEU are observationally equivalent when an entrepreneur faces the decision to expand an existing venture but have opposite predictions when he is faced with the decision to shut down operations. Specifically, while MEU implies a “pessimist” decision rule in both expansion and shut-down decisions, IBEU entrepreneurs are pessimist in expansions but *optimist* in contraction. Because of the unanimity nature of IBEU preferences, entrepreneurs expand only if the scaled up venture is better under the worst possible scenario (similar to MEU).

⁵Incomplete preferences were studied originally by Aumann (1962). Dubra, Maccheroni, and Ok (2004) show that an incomplete order over (unambiguous) lotteries can be represented via a multi-utility representation. Kraus and Sagi (2006) extend their analysis to the case of temporal lotteries and provide a model of inter-temporal choice with incomplete preference that generalize Kreps and Porteus (1978) recursive utility. Ghirardato, Maccheroni, and Marinacci (2004) and Gilboa, Maccheroni, Marinacci, and Schmeidler (2010) provide a general form of the Bewley’s representation theorem. Ortoleva (2010) provides a different perspective on Bewley’s inertia assumption and shows how by starting with a *complete* set of preferences that exhibit *status quo bias* one can obtain incomplete preferences as in Bewley.

⁶Gilboa, Maccheroni, Marinacci, and Schmeidler (2010) suggest another approach to obtain complete preferences that relies on imposing axioms for both the unanimous and the MEU preferences and show that MEU can be seen as a completion of unanimous incomplete preferences.

⁷See, for example, Kahneman and Tversky (1979), Kahneman, Knetsch, and Thaler (1991), and Tversky and Kahneman (1991). Sagi (2006) studies the implications of imposing a “no-regret” axiom on a family of complete preferences “anchored” at the status quo.

However, in contractions, for a IBEU entrepreneur the worst case scenario is getting rid of a venture that is very profitable, i.e., IBEU takes into account the opportunity cost or potential regret of an action. This implies that, contrary to MEU entrepreneurs, an IBEU entrepreneur may continue operating a project even when they believe the project may be worth less than the scrap value of the asset. This finding is in sharp contrast with Miao and Wang (2011) who apply MEU to study expansion and contraction decisions. In essence, ambiguity has a symmetric effect on expansion and contraction options in the MEU setting while it has asymmetric effect in an IBEU setting: MEU entrepreneurs are quick to expand and quick to contract, IBEU entrepreneurs are quick to expand and *slow* to contract.

A direct implication of the previous finding is that the effect of ambiguity aversion in a MEU real option model cannot be distinguished by a volatility or risk aversion effect in an equivalent SEU setting. A lower volatility (higher risk aversion) in this case reduces both the option to expand and contract, which, therefore will be exercised earlier. On the other hand, the asymmetric effect we observed in the IBEU setting cannot be obtained by a change in volatility (or risk aversion) in a SEU setting. This is a potentially important channel that can help to empirically identify which approach best describes investment decisions.

When studying the optimal contract in a static financing setting between a risk-neutral, multi-prior financier and a risk neutral, single-prior entrepreneur, we find that MEU and IBEU are observationally equivalent. Both models predict a preference for “debt financing”. Intuitively this happens because, under both preferences, the cost of capital for the E is minimized when F is offered a contract with common payoff in all possible states and hence “immune” to F ’s ambiguity aversion. The intuition for this finding is however similar to what one would find in the case of SEU with a risk averse F and so, observationally, MEU, IBEU and SEU with a risk averse F do not seem to be distinguishable. Interestingly though, the optimal contract under SEU when both E and F are risk neutral but have different beliefs (e.g., E optimist and F pessimist) *never* involves debt as an optimal contract. In this case the optimist E wants to offer a contract that pays a lot in the state he feels least likely to occur.

When studying the optimal contract in a static financing setting between a risk-neutral, multi-prior entrepreneur and a risk-neutral, single-prior financier, we find that the set of feasible contract under MEU is always a strict subset of those feasible under IBEU. In particular, because of E ’s ambiguity aversion under both MEU and IBEU, equity is always an optimal contract.

However, while equity is the *unique* optimal contract for MEU, several other non-equity-like contract will be accepted as a financing arrangement by an ambiguity averse entrepreneur with IBEU preferences.

Finally, we make a first attempt at studying the problem of financing in a dynamic setting. Because of the complexity of the issues that emerge when dealing with ambiguity in a dynamic setting, we limit our analysis to the special cases of a multi-prior financier who is offered option-like instruments and study his optimal exercise decisions under both MEU and IBEU. Specifically, we suppose F is offered a choice between two hybrid securities A and B , where A (B) offers ownership of a security X (Y) with the option to convert to security Y (X) a period later. We find that, while under SEU and MEU the proceeds from issuing A and B are identical, they can be different under IBEU, due to the inertia effect on the exercise choice when X and Y are not comparable. This suggests that under IBEU, contract design has to carefully consider the *sequencing* of payoffs that a security offers. Furthermore, we find that when F can commit to a particular exercise policy, the value of a security can be different from its value in the absence of commitment. This also point to an important issue about dynamic consistency that needs to be further investigated in the solution of an optimal contract in a dynamic setting.

In summary, our findings indicate that ambiguity and ambiguity aversion can have significantly different implication on real and financial decision, depending on the model a researcher uses to accommodate these deviations from the Bayesian paradigm. These results therefore suggest caution when relying on ambiguity aversion to “explain” empirically observed phenomena.

Recent studies have applied the multiple prior approach to finance problems. For the most part, these applications have been in the asset pricing and portfolio choice areas. Epstein and Schneider (2010) provide an excellent survey. Fewer studies however have considered how ambiguity aversion could affect corporate decisions. For example, Miao and Wang (2011), Nishimura and Ozaki (2007), Riis Flor and Hesel (2011), and Riedel (2009) examine the exercise decision in a real option setting but do so only in an MEU framework and do not consider financing issues. Our contribution is to study the effect of different modelling choices for ambiguity aversion (MEU vs. IBEU) on real and financial decisions.

Rigotti (2004) is, to the best of our knowledge, the first paper that addresses financing issues in the context of the incomplete Bewley-type preferences.⁸ We complete and extend his analysis by considering both static and dynamic settings as well as real decisions. Moreover, by drawing out investment and financing implications under both MEU and IBEU we open the possibility of using actual investment and financing decisions to address the empirical question of which approach best describes managerial decision making.

In the next section we set out some preliminary elements of our analysis. We do this in the simple setting of a single decision maker selecting between a safe asset and an asset that is ambiguous. We also discuss our approach to modelling entrepreneurs. In section 3 we consider dynamic investment problem with expansion and contraction options. Section 3.3 characterizes the solution of the investment problem and show that different representations of ambiguity lead to stark difference in observed investment behavior. Section 4 introduces financing of a project in a static setting and Section 5 considers the problem of financing a multi-stage project. Section 6 concludes. Appendix A contains proofs for all propositions.

2 Ambiguity and decision making: preliminaries and examples

We illustrate our approach to decision making under ambiguity through two decisions being contemplated by a decision maker (DM). Both decisions involve a certain cash amount I and a gamble \tilde{C} that will produce an unknown future cash flow. The realization of \tilde{C} will depend on the state of the world drawn from the set $\Omega = \{U, D\}$ and for now assume the cash flow is simply the state value, U or D . Both decision problems require that the DM choose between the *status quo*, represented by the existing asset position, and an alternative for which the status quo can be exchanged. The decision problems we consider are:

1. *Investment*: The status quo is I and the alternative is \tilde{C} ;
2. *Contraction*: The status quo is \tilde{C} and the alternative is I .

In Section 3 we consider a two-period model where an investment is made that can subsequently be expanded or contracted. Hence the decision problem and status quo for the same DM will change over time. In evaluating a choice, the value the DM derives from each outcome

⁸See also Lopomo, Rigotti, and Shannon (2009) for an application of Knightian uncertainty to contract design and Lopomo, Rigotti, and Shannon (2011) for an application to moral hazard.

is captured by a utility function which we assume to be linear in the outcome, implying a risk neutral *DM*.

We refer to an *unambiguous* gamble \tilde{C} as on in which the *DM* believes that there exists a unique subjective probability governing the outcomes. This is the situation in the SEU benchmark. Let p denote the probability attached to the outcome U and $1 - p$ that attached to the outcome D . We denote by $E_p(\tilde{C})$ the expected value of a gamble under the distribution p , i.e., $E_p(\tilde{C}) = D + p(U - D)$.

An *ambiguous* gamble is one where the *DM* feels that the outcome of a particular choice is governed by one distribution, denoted π , taken from a set of several possible distributions, denoted Π . The *DM* is, however, unable to quantify the likelihood that any particular distribution governs the outcome of the gamble. Specifically for our example, we assume that Π contains two distributions,

$$\Pi = \{\pi : \pi = p - \varepsilon, \pi = p + \varepsilon\}, \quad (1)$$

where π represent the probability of U which takes value $p - \varepsilon$ under the first distribution and $p + \varepsilon$ under the the second, with $p \in (0, 1)$, $0 \leq \varepsilon < p$, and $0 \leq \varepsilon < 1 - p$. We take the quantity ε to be a measure of the ambiguity of the cash flow. The unambiguous gamble corresponds to the special case in which $\varepsilon = 0$. A *DM* is said to be *ambiguity averse* if, all else equal, he prefers less ambiguous gambles, i.e., gambles described by smaller set of priors. If the set of priors is a singleton ($\varepsilon = 0$) ambiguity vanishes and hence the *DM* is necessarily *ambiguity neutral* with respect to such a gamble.

In the Introduction we described two specific approaches to decision making in the presence of ambiguity: Inertia Based Expected Utility (IBEU) and Maxmin Expected Utility (MEU). To illustrate these approaches and their relationship to Savage's SEU let us assume $p = 0.5$, $U = 10$, $D = 0$, and define I^* as the largest value of I at which the *DM* is willing to leave the status quo for the alternative. In what follows we use the symbol \succ to denote preferences over alternative actions available to the *DM*.

No ambiguity: Subjective Expected Utility (SEU)

A SEU decision maker will use the following decision rule

$$\tilde{C} \succ I \iff E_p(\tilde{C}) > E_p(I), \quad \text{i.e.,} \quad \tilde{C} \succ I \iff 5 > I \quad (2)$$

$$I \succ \tilde{C} \iff E_p(I) > E_p(\tilde{C}), \quad \text{i.e.,} \quad I \succ \tilde{C} \iff I > 5. \quad (3)$$

The largest value of I at which the DM is willing to leave the status quo is hence $I^* = 5$. Suppose $I \leq I^*$. If I is the status quo the DM will choose the alternative of \tilde{C} while if \tilde{C} is the status quo then the DM will stay with \tilde{C} rather than selling it for I . Therefore, when $I \leq I^*$ the DM selects \tilde{C} over I *regardless of the status quo*.

In the case of SEU, we can define the net present value of the investment, NPV_{inv} and contraction, NPV_{con} , decisions as follows:

$$\begin{aligned} NPV_{\text{inv}} &= E_p(\tilde{C}) - I, \\ NPV_{\text{con}} &= -E_p(\tilde{C}) + I. \end{aligned}$$

Hence the decision rules (2) and (3) are nothing more than an application of the familiar NPV rule, $NPV_{\text{inv}} > 0$ and $NPV_{\text{con}} > 0$.

For the rest of this section we consider the case in which the gamble \tilde{C} is ambiguous. We characterize this case via a set of prior (1) with $p = 0.5$ and $\varepsilon = 0.3$. Since there are two distributions in Π , the DM faces two possible “expected values” of the gamble: $E_{p-\varepsilon}(\tilde{C}) = 2$ and $E_{p+\varepsilon}(\tilde{C}) = 8$. In what follows, we analyze how ambiguity is dealt with by a DM with IBEU and MEU preference.

Ambiguity: Inertia Based Expected Utility (IBEU)

Bewley (2002) elegantly presents and motivates Inertia Based Decision Making. His approach to modelling ambiguity consists in starting from the traditional Savage (1954) SEU axioms but dropping the completeness one. Once this axiom is removed, he shows that the DM preferences cannot be represented via a unique probability distribution. In particular, the decisions among alternative gambles are characterized by a “unanimity rule”, i.e., one lottery is preferred over another if its expected value is higher under *all* possible distributions. Formally, according to

this rule, the preference ordering between two gambles \tilde{A} and \tilde{B} is given by the rule

$$\tilde{A} \succ \tilde{B} \iff E_\pi(\tilde{A}) > E_\pi(\tilde{B}), \forall \pi \in \Pi, \quad (4)$$

or, equivalently,

$$\tilde{A} \succ \tilde{B} \iff \min_{\pi \in \Pi} E_\pi(\tilde{A} - \tilde{B}) > 0. \quad (5)$$

It is clear that this decision rule leads to an incomplete preference ordering. Unlike SEU, the model cannot assign a “value” to a gamble and therefore cannot always specify what the *DM* will do when faced with a choice. To resolve the indeterminacy in the presence of incomparable gambles, Bewley introduces the following *status quo* or *inertia* assumption:

Inertia Assumption. *A multi-prior DM will only accept a gamble if its expected value is strictly better than that of the status quo under all possible priors.*

We refer to the unanimity rule with the addition of the inertia assumption as the *Inertia Based Expected Utility* (IBEU) decision rule. In (4), assuming that the status quo is the gamble \tilde{B} , IBEU implies that if there exists a prior $\pi \in \Pi$ such that $E_\pi(\tilde{A}) \not> E_\pi(\tilde{B})$, then the *DM* chooses to remain with the status quo \tilde{B} .

In the case of our simple numerical example about investment and contraction, IBEU implies that, when I is the status quo

$$\tilde{C} \succ I \iff E_\pi(\tilde{C}) > E_\pi(I), \text{ for all } \pi \in \Pi, \text{ i.e., } \tilde{C} \succ I \iff E_{p-\varepsilon}(\tilde{C}) > I, \text{ or } 2 > I. \quad (6)$$

When \tilde{C} is the the status quo, IBEU implies

$$I \succ \tilde{C} \iff E_\pi(I) > E_\pi(\tilde{C}), \text{ for all } \pi \in \Pi, \text{ i.e., } I \succ \tilde{C} \iff I > 8, \text{ or } I > 8. \quad (7)$$

In words, (6) states that if the status quo is the certain amount of cash I , the *DM* will only leave the status quo if the expected value of the gamble under the most pessimistic prior in Π is greater than I . On the other hand, if the *DM* possesses the gamble \tilde{C} , (7) states that he will only give up the gamble if the payment received, I , is larger than the expected value of the gamble under the most optimistic distribution in Π . Essentially, IBEU accounts for the potential regret of an action. Note further that, unlike SEU, from (6) and (7) the largest value of I at which the *DM* leaves the status quo for the alternative depends on the status quo. This

value is $I^* = 2$ when the status quo is I and it is $I^* = 8$ when the status quo is \tilde{C} . This is reminiscent of a “Bid-Ask” spread in the investment and contraction decisions. Indeed, the decision rules in (6) and (7) is equivalent to SEU if we assume the existence of a transaction cost or both investment and contraction. Specifically, assuming a transaction cost equal to 3, and SEU preferences we will have

$$\tilde{C} \succ I \iff E_p(\tilde{C}) > E_p(I) + 3, \quad \text{i.e.,} \quad \tilde{C} \succ I \iff 2 > I \quad (8)$$

$$I \succ \tilde{C} \iff E_p(I) - 3 > E_p(\tilde{C}), \quad \text{i.e.,} \quad I \succ \tilde{C} \iff I > 8. \quad (9)$$

Figure 1 shows how the relative value of the gamble depends on the action being considered and the degree of ambiguity ε of the *DM*. For an investment the status quo is I , so the choice is one of *buying* the gamble. When buying, the subjective value of the gamble in (6) is $E_{p-\varepsilon}(\tilde{C})$, which decreases in the ambiguity measure ε . For a contraction the status quo is the gamble \tilde{C} that the *DM* is *selling* and its subjective value in (7) is $E_{p+\varepsilon}(\tilde{C})$, which increases in the ambiguity measure ε .

Ambiguity: Maxmin Expected Utility (MEU)

If the *DM* has *Maxmin Expected Utility* (MEU) the preference relation between two gambles \tilde{A} or \tilde{B} , is given by

$$\tilde{A} \succ \tilde{B} \iff \min_{\pi \in \Pi} E_{\pi}(\tilde{A}) > \min_{\pi \in \Pi} E_{\pi}(\tilde{B}). \quad (10)$$

Note that these preferences are complete and we can think of the quantities $\min_{\pi \in \Pi} E_{\pi}(\tilde{A})$ and $\min_{\pi \in \Pi} E_{\pi}(\tilde{B})$ as the subjective value of the gambles \tilde{A} and \tilde{B} , respectively. There is no special role for the status quo in this framework.

For our simple example, a MEU decision maker will have the following ordering.

$$\tilde{C} \succ I \iff \min_{\pi \in \Pi} E_{\pi}(\tilde{C}) > \min_{\pi \in \Pi} E_{\pi}(I), \quad \text{i.e.,} \quad \tilde{C} \succ I \iff E_{p-\varepsilon}(\tilde{C}) > I, \quad \text{or} \quad 2 > I \quad (11)$$

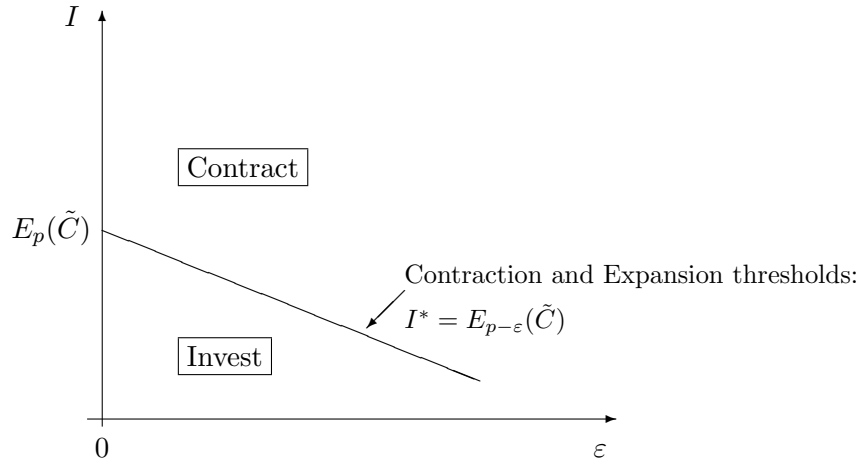
$$I \succ \tilde{C} \iff \min_{\pi \in \Pi} E_{\pi}(I) > \min_{\pi \in \Pi} E_{\pi}(\tilde{C}), \quad \text{i.e.,} \quad I \succ \tilde{C} \iff I > E_{p-\varepsilon}(\tilde{C}), \quad \text{or} \quad I > 2. \quad (12)$$

As in the SEU case, the largest value of I at which the *DM* leaves the status quo for the alternative is unique and given by $I^* = 2$, independent of the status quo. The willingness to

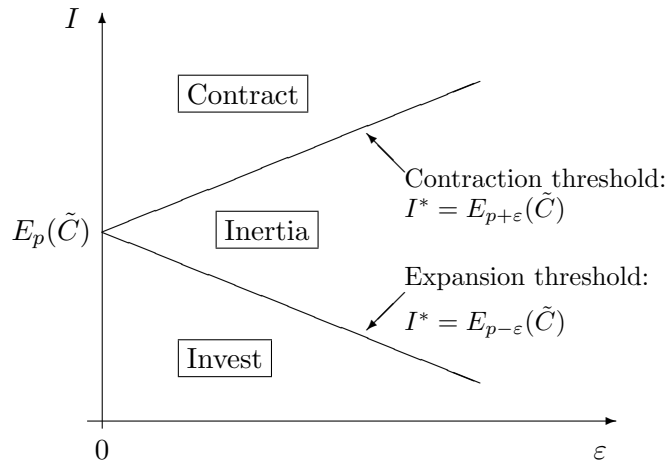
Figure 1: Investment and contraction decision with MEU and IBEU preferences

The figure displays the value of contraction and expansion as a function of the degree of ambiguity ε . Panel A consider the case of MEU preferences and Panel B consider the case of IBEU preference in which the status quo is the sure thing I in expansion decisions and the gamble \tilde{C} in contraction decisions.

Panel A: MEU



Panel B: IBEU



invest, represented by I^* , is lower under MEU than under SEU. It is worth noting, however, that I^* under MEU would be identical to I^* under SEU but with the more pessimistic but unique

prior distribution $p' = 0.2$. Moreover, this critical “price” is the same whether buying or selling the gamble. This suggests that based on revealed preferences, it is not possible to distinguish a MEU DM who sees a gamble as ambiguous from one who has a unique but pessimistic view of the gamble.

Empirically, MEU and IBEU predict different behavior regarding to the investment and contraction decisions. While under MEU, DM are reluctant to invest and eager to contract, under IBEU, they are reluctant to invest *and* reluctant to contract.

3 A simple dynamic model of Investment

In this section, we construct a dynamic real option model of a self-financed entrepreneur. The model illustrates how the difference between MEU and IBEU in the contraction decision illustrated in the one-period example of the previous section impacts the investment decision before option exercise date. The model also illustrates how the change of status quo over time may create a time inconsistency problem for the IBEU preferences. This is a novel source of time inconsistency which complements the time inconsistency issues in MEU preferences caused by learning and updating (see, e.g., Al-Najjar and Weinstein (2009)). Finally, we will be able to position our dynamic model in the context of the existing literature on real option with ambiguity.

3.1 Entrepreneurs

The focus of our study is on a special class of decision makers that we refer to as “Entrepreneurs.” There seems to be two broad dimensions along which entrepreneurs are assumed to be different from other DMs .

1. *Technology*: Entrepreneurs are seen as individuals who create new productive opportunities in the economy. We will implement this view by assuming that the Entrepreneur has monopoly access to ambiguous investment opportunities.
2. *Ambiguity attitude*: Entrepreneurs are also seen as individuals who see a world that is different from others. Some researchers describe them as being more tolerant of ambiguity or more decisive (e.g., Rigotti (2004), Amarante, Ozgür, and Phelps (2011), and Amarante, Ghossoub, and Phelps (2011)). While this may be the case, we feel it is important to

consider situations where Entrepreneurs can see either more ambiguity or less ambiguity than others in the economy.

3.2 Setup

We consider a simple model of a risk neutral Entrepreneur (E) with an investment opportunity that has payoffs over three dates, 0, 1, and 2, defining two periods. E has the ability to turn investment into future cash flows at time $t = 2$ through an ambiguous project. Details of the technology are presented below. E has monopoly access to the project and has the ability to affect output through expansion and contraction decisions.

We assume initially that the E has sufficient funds to self finance and chooses to do so. This setting gives us an opportunity to examine the real consequences of ambiguity without the added effects of financing. In Section 4 we relax this assumption and introduce the need for E to arrange external financing from a risk neutral financier F .

3.2.1 Technology

The project's productivity is described by the random variable $\tilde{\theta}_2$ representing cash flows at time $t = 2$. E 's subjective view of the cash flow $\tilde{\theta}_2$ is that it is the endpoint of a binomial stochastic process $(\tilde{\theta}_t)_{t=0,1,2}$. Starting at the unambiguous value θ_0 , the process evolves to $\tilde{\theta}_1$ at time $t = 1$ where the random variable $\tilde{\theta}_1$ takes the value θ_u ('up state') or θ_d ('down state'). Conditionally on being in the 'up state' (resp. 'down state') at time $t = 1$, the projects cash flow are given by $\tilde{\theta}_2 = \tilde{\theta}_2^u$ (resp. $\tilde{\theta}_2 = \tilde{\theta}_2^d$) where the $\tilde{\theta}_2^u$ (resp. $\tilde{\theta}_2^d$) takes the value θ_{uu} (resp. θ_{du}) in the 'up-up state' (resp. 'down-up state') or θ_{ud} (resp. θ_{dd}) in the 'up-down state'(resp. 'down-down state'). We assume that $\theta_{uu} \geq \theta_{ud} \geq \theta_{du} \geq \theta_{dd}$.

We capture the ambiguity by assuming that E is not completely confident about the transition probabilities along the tree and therefore he thinks in terms of a *set* of transition probability measures,

$$\Pi = \{\pi : p - \varepsilon \leq \pi \leq p + \varepsilon\} \quad (13)$$

in which π represents the one-step-ahead beliefs about an "up" shock in productivity. To simplify exposition, we assume that the same set Π describes conditional beliefs at every realized $\tilde{\theta}_1$, reflecting the view that an up shock in productivity are unaffected by the current value

of $\tilde{\theta}_1$. This assumption essentially imposes independence between successive realizations of productivity shocks.⁹ This in turns implies that, upon observing the realization of $\tilde{\theta}_1$, E can only refine the set of future paths (binomial subtrees) but cannot learn about the law of motion governing the process $(\tilde{\theta}_t)_{t=0,1,2}$.¹⁰

3.2.2 Decisions

The project requires that one unit of capital be installed at $t = 0$ at a cost of I_0 . This unit of capital will produce cash flows $\tilde{\theta}_2$ at $t = 2$ as well as the ability to expand or contract capacity at $t = 1$. For simplicity, we assume that the project delivers no cash flow at time 1.

If $\tilde{\theta}_1 = \theta_u$ at time 1 ('up state'), the entrepreneur can keep the same project scale and get the random cash flows $\tilde{\theta}_2^u$. Alternatively, he can expand the firm by paying the amount I_1 at $t = 1$ and get the cash flows $\lambda\tilde{\theta}_2^u$ at $t = 2$ which $\lambda > 1$.

If $\tilde{\theta}_1 = \theta_d$ at time 1 ('down state'), the entrepreneur can keep the same project scale and get the cash flows $\tilde{C}_2 = \tilde{\theta}_2^d = \{\theta_{du}, \theta_{dd}\}$. Alternatively, he can shut dow the firm and get immediately the cash flow S and nothing at time $t = 2$. In order to simplify the analysis we assume that the entrepreneur can only expand in the up state and can only contract in the down state.

3.3 Optimal expansion and contraction decisions

We analyze the expansion and contraction decision in the presence of ambiguity under both IBEU and MEU an compare them with the benchmark SEU case of no ambiguity. In all cases we consider a risk neutral entrepreneur E .

To solve the intertemporal investment problem we follow a *recursive* approach to the analysis, determining first the expansion/contraction decisions at time 1 and then the investment decision at time 0. As in Strotz (1955), the initial investment decisions are based on the anticipation of the contraction/expansion policies that E will later choose.¹¹ Our optimal investment decision is therefore time consistent. In Section 3.4 we compare the recursive solution of this section to

⁹A concrete way to understand our assumptions on ambiguity is to assume that nature flips a first coin at time $t = 1$ to determine the current shock (up or down). At time $t = 2$ a second independent coin is used to determine if the second shock is an up or a down movement. Both coins are ambiguous with a set of priors given in (13).

¹⁰Because we take the one-step-ahead priors as primitive in our description of ambiguity, the resulting unconditional set of priors over the the state space satisfies by construction the rectangularity condition of Epstein and Schneider (2003) which guarantee time consistency of MEU preferences.

¹¹See Siniscalchi (2011) for an analysis of the time inconsistency issue in dynamic choice under ambiguity.

the solution in which E precommits at time 0 on his investment strategy and study potential sources of time-inconsistency in the presence of ambiguity.

3.3.1 The Expansion Decision

At $t = 1$ in the “up” state ($\tilde{\theta}_1 = \theta_u$) the entrepreneur is able to expand capacity by exchanging the current unit of capital for $\lambda > 1$ units of productive capital by paying I_1 . Hence, in state u the utility derived from retaining the status quo ($\tilde{\theta}_2^u$) for E under a particular prior $\pi \in \Pi$ is

$$E_\pi(\tilde{\theta}_2^u)$$

while the alternative is to expand capacity providing the utility

$$\lambda E_\pi(\tilde{\theta}_2^u) - I_1.$$

We analyze the expansion decisions under SEU, IBEU and MEU.

1. **SEU.** If E is ambiguity neutral with the subjective prior $\pi = p$, (i.e. $\varepsilon = 0$), he will make decisions based on SEU implying an expansion if and only if

$$(\lambda - 1)E_p(\tilde{\theta}_2^u) > I_1 \tag{14}$$

In this case (14) is the familiar NPV rule.

2. **IBEU.** If E thinks the project’s productivity is ambiguous, $\varepsilon > 0$, and has IBEU preferences, then he will expand if and only if

$$(\lambda - 1)E_\pi(\tilde{\theta}_2^u) > I_1, \quad \text{for all } \pi \in \Pi. \tag{15}$$

The left hand side of the inequality (15) is increasing in π because $\theta_{uu} > \theta_{ud}$ and therefore it binds at the most “pessimistic” prior $\pi = p - \varepsilon$. E will thus expand if and only if

$$(\lambda - 1)E_{p-\varepsilon}(\tilde{\theta}_2^u) > I_1. \tag{16}$$

3. **MEU.** If E thinks the project's productivity is ambiguous, $\varepsilon > 0$, and has IBEU preferences, then he will expand if and only if

$$\min_{\pi \in \Pi} (\lambda E_{\pi}(\tilde{\theta}_2^u) - I_1) > \min_{\pi \in \Pi} E_{\pi}(\tilde{\theta}_2^u). \quad (17)$$

Because, $\theta_{uu} > \theta_{ud}$, the minimum in both sides of (17) is attained at $\pi = p - \varepsilon$ and thus E will expand if and only if

$$(\lambda - 1)E_{p-\varepsilon}(\tilde{\theta}_2^u) > I_1 \quad (18)$$

From the above analysis we see that under both IBEU and MEU, the condition under which a firm will expand, as given by (16) and (18) are identical and indistinguishable from the expansion decision of a pessimistic SEU entrepreneur.

3.3.2 The Contraction Opportunity

At $t = 1$ in the “down” state ($\tilde{\theta}_1 = \theta_d$), E can convert capacity into an alternative, unambiguous use that has an immediate risk free salvage value of S . Hence, in this state E 's subjective value of retaining the status quo ($\tilde{\theta}_2^d$) under a particular prior $\pi \in \Pi$ is

$$E_{\pi}(\tilde{\theta}_2^d)$$

while the alternative is to contract capacity providing S . We analyze the contraction decisions under SEU, IBEU and MEU.

1. **SEU.** If E is ambiguity neutral with the subjective prior $\pi = p$ (i.e. $\varepsilon = 0$), he will make decisions based on SEU implying a contraction decision if and only if

$$S > E_p(\tilde{\theta}_2^d) \quad (19)$$

As with the expansion decision, in this case the decision criteria (19) is the familiar NPV rule.

2. **IBEU**. If in the presence of ambiguity ($\varepsilon > 0$) E makes decisions based on IBEU, then he will contract if and only if

$$S > E_{\pi}(\tilde{\theta}_2^d), \quad \text{for all } \pi \in \Pi. \quad (20)$$

Because the right hand side of the last inequality is increasing in π , it binds at the most optimistic prior $\pi = p + \varepsilon$ and the contraction condition can be characterized with the inequality

$$S > E_{p+\varepsilon}(\tilde{\theta}_2^d) \quad (21)$$

3. **MEU**. If in the presence of ambiguity ($\varepsilon > 0$), E makes decisions based on MEU, then he contract if and only if

$$S > \min_{\pi \in \Pi} E_{\pi}(\tilde{\theta}_2^d) \iff S > E_{p-\varepsilon}(\tilde{\theta}_2^d) \quad (22)$$

While it is difficult to distinguish SEU from MEU and IBEU for expansion decisions beyond relative pessimism, a contraction decision delivers very different results. While the MEU decision (22) is indistinguishable from SEU decision (19) with a pessimistic belief, the IBEU decision is clearly different from both SEU and MEU. A comparison of (22) with (21) shows that when facing a contraction an MEU decision maker is pessimistic while an IBEU decision maker is optimistic. Hence, even when E can sell a firm for S and may even consider S to be larger than her expected payoff from continuing under some priors, he will not contract because there are some priors under which the asset value would be higher than S .

It has been observed that managers are reluctant to divest or shut down projects that have not done well, a result that has been explained by agency problems, reputation concerns and asymmetric information (see, for example, Boot (1992) and Weisbach (1995)). Our explanation is in terms of ambiguity aversion under symmetric information. If DMs are ambiguity averse and base decisions based on IBEU they will be reluctant to terminate a project because of the importance that potential regret plays in their decision making.

3.3.3 Initial investment

At $t = 0$ the status quo is simply to not invest and keep I_0 and the alternative is an investment that will lead to the expansion and contraction decisions given above. As we explained above, we take the expansion and contraction decisions as given and evaluate them as of $t = 0$. While it is possible to analyze the investment decision in generality, we will focus on the special case where $E_p(\tilde{\theta}_2^d) \leq S \leq E_{p+\varepsilon}(\tilde{\theta}_2^d)$ to illustrate how the differences in the exercise strategies of the contraction option for the SEU, the MEU and the IBEU entrepreneur impact the initial investment policy at time $t = 0$. Moreover, given that the investment policy for the expansions are identical under IBEU and MEU, we also assume that $I_1 \geq (\lambda - 1)E_p(\tilde{\theta}_2^u)$ in order to simplify the calculations by making the expansion option unattractive. We analyze now the initial investment decision under SEU, IBEU and MEU.

1. **SEU.** Because $E_p(\tilde{\theta}_2^d) \leq S$ we see that E will contract at $t = 1$ under SEU preferences. Internalizing this policy at the ex ante stage, E will invest at $t = 0$ if and only if

$$I_0 \leq pE_p(\tilde{\theta}_2^u) + (1 - p)S \equiv I_0^{*SEU}$$

2. **IBEU.** Because $S \leq E_{p+\varepsilon}(\tilde{\theta}_2^d)$, E will not contract at time $t = 1$ under IBEU preferences. Given this policy, E will abandon the status quo wealth of I_0 at $t = 0$ if and only if

$$I_0 < \pi_0 E_{\pi'}(\tilde{\theta}_2^u) + (1 - \pi_0) E_{\pi''}(\tilde{\theta}_2^d), \quad \text{for all node-specific priors } (\pi_0, \pi', \pi'') \in \Pi^3. \quad (23)$$

Given that the productivity $\tilde{\theta}_2^u$ and $\tilde{\theta}_2^d$ are increasing in the state and that $\tilde{\theta}_2^d \leq \tilde{\theta}_2^u$ almost in all states, the inequality (23) is binding at $\pi_0 = \pi' = \pi'' = p - \varepsilon$. Thus the investment decision can be characterized by the inequality

$$I_0 < (p - \varepsilon)E_{p-\varepsilon}(\tilde{\theta}_2^u) + (1 - p + \varepsilon)E_{p-\varepsilon}(\tilde{\theta}_2^d) \equiv I_0^{*IBEU}, \quad (24)$$

3. **MEU.** Since $E_{p-\varepsilon}(\tilde{\theta}_2^d) < S$, E will contract at time $t = 1$ under MEU preferences. Accounting for this policy E will undertake the project at $t = 0$ if and only if

$$I_0 \leq \min_{(\pi_0, \pi') \in \Pi^2} \pi_0 E_{\pi'}(\tilde{\theta}_2^u) + (1 - \pi_0)S. \quad (25)$$

The constraint in (25) is clearly binding at $\pi_0 = \pi' = p - \varepsilon$ and the investment decision is thus characterized by

$$I_0 < (p - \varepsilon)E_{p-\varepsilon}(\tilde{\theta}_2^u) + (1 - p + \varepsilon)S \equiv I_0^{*MEU}, \quad (26)$$

Notice that $I_0^{*IBEU} < I_0^{*MEU}$ and therefore the impact of future delays in contractions by the IBEU entrepreneur is to adopt a more conservative investment policy than that of a MEU entrepreneur. This result has important implications if we wish to rely on observed corporate decisions to infer perceived ambiguity and ambiguity attitude of entrepreneurs. Suppose, for example that one wants to use the cross section of firms' investment decisions to gauge the dispersion of beliefs about productivity in a particular industry. The above result suggests that if we mistakenly assume firms act according to an MEU decision rule when the IBEU rule is the correct one, we will severely overestimate the degree of ambiguity about productivity in the industry. As in structural estimation of corporate decision models, this example points to the importance of the specific theoretical framework in drawing inferences from observed behavior.

3.4 Commitment and real investment decisions

In this section, we illustrate how the inertia assumption combined with the multiple prior of the IBEU induce time inconsistency. If E is able to pre-commit at $t = 0$ to a particular option exercise policy, then his investment decision at time $t = 0$ can be different from the investment policy resulting from the recursive approach (Strotz (1955)) that we utilized when analyzing the investment problem in the previous section. To make the point in a transparent way, we adopt the following assumption

Assumption 1. *Suppose the technology parameters satisfy*

$$\theta_{uu} = \theta_{ud} = \bar{\theta} > \theta_{du} > S > \theta_{dd} \quad (27)$$

and

$$I_1 > (\lambda - 1)\bar{\theta}. \quad (28)$$

Condition (27) is a technological condition that eliminates the risk in the up node. Condition (28) says that it is never optimal to expand in the up state. Both condition (27) and (28) are done without loss of generality to simplify the proof of the proposition. The following proposition

shows that in presence of a contraction option, time inconsistency can arise in our investment problem.

Proposition 1. *Assume Assumption 1 holds and the technological parameters $(I_0, S) \in \mathcal{R}_0 \cap \mathcal{R}_1$ where $\mathcal{R}_0 = \left\{ (p - \varepsilon)\bar{\theta} + (1 - p + \varepsilon)E_{p-\varepsilon}(\tilde{\theta}_2^d) \leq I_0 < (p - \varepsilon)\bar{\theta} + (1 - p + \varepsilon)S \right\}$ and $\mathcal{R}_1 = \{E_{p-\varepsilon}(\tilde{\theta}^d) \leq S \leq E_{p+\varepsilon}(\tilde{\theta}^d)\}$. When E 's preferences are driven by IBEU, the commitment optimal policy requires to invest and contract whereas the recursive optimal policy is to not invest at all.*

As we show in the proof of the proposition, \mathcal{R}_0 is a region of investment cost I_0 and salvage values S where E faced with the status quo of keeping I_0 and not investing, will commit to invest and contract but will reject a commitment to invest and continue. Once the investment at time $t = 0$ is undertaken, at time $t = 1$ E cannot compare the contraction option with the continue policy in region \mathcal{R}_1 . In fact, because $S \leq E_{p+\varepsilon}(\tilde{\theta}^d)$, there are always some optimistic beliefs under which the continue option is better than the contraction outcome. Due to the inertia assumption, he would therefore rather continue than contract at time $t = 1$. Anticipating at $t = 0$ that the contraction option cannot be sustained in the future, E follows the status quo policy by rejecting the investment: in region \mathcal{R}_1 we have $E_{p-\varepsilon}(\tilde{C}_2) \leq I_0$ and thus there are always some pessimistic beliefs under which not investing (status quo) dominates investing.

Notice that when there is a single prior ($\varepsilon = 0$), the set \mathcal{R}_1 becomes a singleton and the time inconsistency disappears. This illustrates the well-known time consistency property of SEU preferences. Under MEU preferences, E 's decisions are also time consistent. But, in this case time consistency for MEU is a consequence of the absence of learning and updating in multi-prior context. In general, in the presence of learning and updating with MEU preferences, the change in the set of priors can also induce a disagreement between ex ante preference and interim preferences (see Al-Najjar and Weinstein (2009), Siniscalchi (2011), and Epstein and Schneider (2010)).¹² Finally, notice that the temporal switch in belief induced by the contraction option is required for constructing a time inconsistency problem i.e., without contraction option the problem will be time consistent also under IBEU preferences.

¹²Learning does not induce dynamic inconsistency in the context of IBEU preferences if one adopts a prior-by-prior Bayesian updating procedure. However, as we show in Section 5, the change of status quo over time may create a tension between time 0 and time 1 IBEU preferences.

4 External Financing

In this section we consider the case where the entrepreneur needs to finance the project and we start with a one period financing model. A risk neutral entrepreneur E has access to a project that delivers cash flows $\tilde{\theta}$ at $t = 1$ but has no funds and must obtain financing from a risk neutral financier F . The cash flow can take the value θ_u in the ‘up’ state and the value θ_d in the down state. We assume that, when F feels that the project is ambiguous, he thinks that the probability of the ‘up state’ (π_F) belongs to the set

$$\Pi_F = \{\pi_F : p - \varepsilon_F < \pi_F < p + \varepsilon_F\}. \quad (29)$$

Similarly, when E feels that the project is ambiguous he thinks that the probability of the ‘up state’ (π_E) belongs to the set

$$\Pi_E = \{\pi_E : p - \varepsilon_E < \pi_E < p + \varepsilon_E\}. \quad (30)$$

We consider two extreme cases. In the first case, F has multiple priors ($\varepsilon_F > 0$) and E has a single prior ($\varepsilon_E = 0$). In the second case the roles are reversed: E has multiple priors ($\varepsilon_E > 0$) and F has a single prior ($\varepsilon_F = 0$). As we mentioned, we consider the case of a single period financing arrangement in this section. The next section studies contracting in a dynamic setting.

4.1 Multi-prior F and single-prior E

Suppose that $\varepsilon_E = 0$ and $\varepsilon_F > 0$. We study the financing arrangement when F has either IBEU or MEU preferences.

4.1.1 IBEU preferences

When F has IBEU preferences, every sale of security from E to F is a negative NPV transaction for E . In fact, under these preferences of F and the fact that E 's prior belong to Π_F , the maximum amount that F is willing to pay for a security is always smaller than the NPV of this security according to E 's prior. As a result, if E will always prefer self financing, whenever possible, to outside financing. If E does not have capital to invest, he must finance the project by issuing a security \tilde{D} delivering the cash flows D_u in the ‘up state’ and D_d in the ‘down

state'.¹³ Because financing is costly for E , as we show in the next proposition, he will raise just the necessary I . The set of feasible securities \mathcal{D} is

$$\mathcal{D} = \left\{ \tilde{D} \mid 0 \leq \tilde{D} \leq \tilde{\theta}, E_{\pi_F}(\tilde{D}) \geq I \text{ for all } \pi_F \in \Pi_F \right\} \quad (31)$$

E solves the following problem

$$\sup_{\tilde{D} \in \mathcal{D}} V_E(\tilde{D}) = -I + \inf_{\pi_F \in \Pi_F} E_{\pi_F}(\tilde{D}) + E_p(\tilde{\theta} - \tilde{D}). \quad (32)$$

In the above expression, $\inf_{\pi_F \in \Pi_F} E_{\pi_F}(\tilde{D})$ is the amount raised from a multi-prior IBEU financier, and $E_p(\tilde{\theta} - \tilde{D})$ is E 's cash flow, net of the repayment promises from the security \tilde{D} . Because E 's belief p is an element of the set Π_F , $\inf_{\pi_F \in \Pi_F} E_{\pi_F}(\tilde{D}) < E_p(\tilde{D})$ and therefore the following inequality is always true

$$V_E(\tilde{D}) \leq -I + E_p(\tilde{\theta}), \quad (33)$$

i.e., the financing arrangement is always a negative NPV transaction. When inequality (33) binds then the financing arrangement is a zero NPV transaction. This happens in two important cases. The first case is when the set Π_F is the singleton $\Pi_F = \{p\}$ (i.e. $\varepsilon_F = 0$) and the second case is when the contract \tilde{D} has state independent payoffs, that is $D_u = D_d$. The following proposition characterizes the optimal financing arrangement.

Proposition 2. *Suppose E has single prior beliefs (SEU) and F has multiple prior beliefs on the productivity of a project and that F 's preferences are driven by IBEU. If self-financing is not possible then it is optimal to finance the project with a security that $\tilde{D}^* = (D_u^*, D_d^*)$ that has the form*

$$\tilde{D}^* = (D_u^*, D_d^*) = \begin{cases} (I, I) & \text{if } 0 \leq I \leq \theta_d \\ \left(\theta_d + \frac{1}{p - \varepsilon_F}(I - \theta_d), \theta_d \right) & \text{if } \theta_d \leq I \leq E_{p - \varepsilon_F}(\tilde{\theta}) \end{cases} \quad (34)$$

If $I > E_{p - \varepsilon_F}(\tilde{\theta})$ financing is not feasible.

The intuition for the proposition is that each time E issues a security, he is going to lose money. When possible he wants to avoid issuing securities. If he must raise money, he will first

¹³We utilize the abuse of notation $\tilde{D} = (D_u, D_d)$ to define a security that pays the cash flow D_u in the 'up state' and D_d in the 'down state'. Furthermore, the notation \tilde{D} designates both the security and also the (random) cash flow generated by this security.

issue a security with constant payoff because their NPV is insensitive to beliefs. In fact both E and F agree on the valuation of a constant payoff security (riskless bond). If issuing riskless bonds does not allow to raise enough money to finance the investment, then E start issuing state contingent payoff securities (risky bond) up to a point where the NPV of the whole firm under the worst belief, $\pi_F = p - \varepsilon_F$, is larger than I in which case, the project is too costly and E abandons it. By issuing the security \tilde{D}^* E raises the exact amount I to invest and gets the utility

$$V^E(\tilde{D}^*) = E_p(\tilde{\theta}) - E_p(\tilde{D}^*) = \underbrace{\theta_d + p(\theta_u - \theta_d) - I}_{\text{NPV under SEU}} - \underbrace{\frac{\varepsilon_F}{p - \varepsilon_F} (I - D_d^*)}_{\text{Loss due to F's Ambiguity}}$$

The term $\frac{\varepsilon_F}{p - \varepsilon_F} (I - D_d^*)$ represents thus the additional loss that E incurs when issuing a security to an ambiguity sensitive financier. The additional loss is zero if $\varepsilon_F = 0$ or if $D_d^* = \theta_d = I$ (in which case $\tilde{D}^* = (I, I)$).

Notice that if E has some cash $W_0 < I$ then the above proposition applies by changing I to $I - W_0$. We therefore have a pecking order in the security issuance decision. The first best is to use cash to finance a project. If this is not possible, E would try to issue risk-less debt and if this also is not possible, financing will occur through risky bonds securities.

4.1.2 MEU preferences

If F has MEU preferences, then given the set of priors Π_F the maximum price that they are willing to pay for a security \tilde{D} is $\inf_{\pi_F \in \Pi_F} E_{\pi_F}(\tilde{D})$. Because

$$\inf_{\pi_F \in \Pi_F} E_{\pi}(\tilde{D}) \geq I \iff E_{\pi_F}(\tilde{D}) \geq I \text{ for all } \pi_F \in \Pi_F,$$

the feasible set of securities for the case of MEU preferences corresponds to the set (31) obtained for the case of IBEU preference. Therefore Proposition 2 will also hold when F has MEU preferences and the same order for securities will prevail. We conclude then that we will observe the same type of contracts when the financiers use MEU or IBEU to make decisions.

4.2 Multi-prior E and single-prior F

Suppose that $\varepsilon_E > 0$ and $\varepsilon_F = 0$. We study the financing arrangement when E has either IBEU or MEU preferences. In what follows we denote by $V_E^{\pi_E}(\tilde{D})$ E 's subjective valuation of the cash flow obtained from starting the project by issuing \tilde{D} under the belief π_E .

4.2.1 IBEU preferences

The set of feasible contracts is:

$$\mathcal{G} = \left\{ \tilde{D} \mid 0 \leq \tilde{D} \leq \tilde{\theta}, E_p(\tilde{D}) \geq I \right\} \quad (35)$$

Assuming E has a status quo of zero wealth, he will finance the project with the security \tilde{D} if and only if

$$V_E^{\pi_E}(\tilde{D}) = -I + E_p(\tilde{D}) + E_{\pi_E}(\tilde{\theta} - \tilde{D}) > 0 \quad \text{for all } \pi \in \Pi_F. \quad (36)$$

This constraint defines the set of implementable securities. By (35), for every feasible contract $\tilde{D} \in \mathcal{G}$, we have $\tilde{\theta} \geq \tilde{D}$, and therefore any feasible security $\tilde{D} \in \mathcal{G}$ satisfies the constraint (36) and can be implemented. The only restrictive constraint is the financing constraint $E_p(\tilde{D}) \geq I$. If this constraint is satisfied, then E is willing to start the project with any financing arrangement $\tilde{D} \in \mathcal{G}$. The following proposition summarizes the financing arrangement in this case.

Proposition 3. *Suppose F has single prior beliefs (SEU) and E has multiple prior beliefs and IBEU preferences. If $I \leq E_p(\tilde{\theta})$, the entrepreneur chooses to start the project by issuing any security $\tilde{D} \in \mathcal{G}$. Due to incomplete preferences, the entrepreneur is unable to rank the different financing options.*

The set \mathcal{G} of feasible contracts is described in Figure 2. Notice that this set always contains the point $\tilde{D} = (\theta_u, \theta_d)$ which corresponds to the equity contract that sells the entire project to F . Note also that for $\theta_d < I$ (Panel A), riskless debt $\tilde{D} = (I, I)$ is not feasible while for $\theta_d > I$ (Panel B), riskless debt is one of the possible financing arrangements.

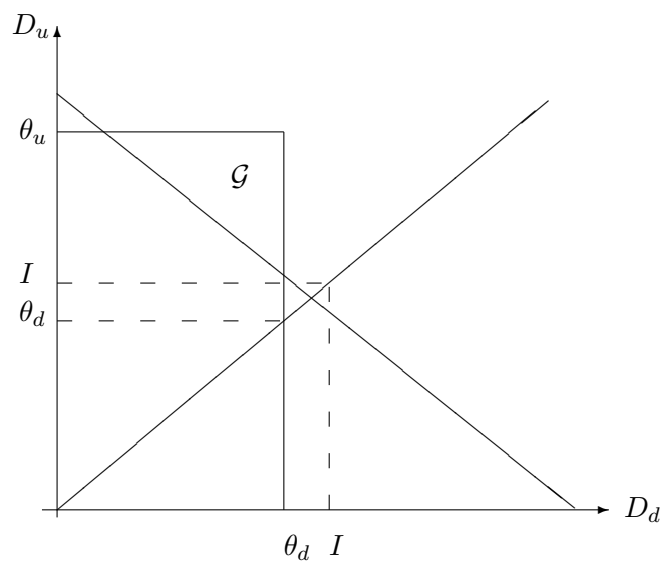
Collecting the result from Proposition 2 and Proposition 3, we see that it is the financier's set of priors which determine whether the project is started.¹⁴ Notice also that the optimal security

¹⁴This is because the entrepreneur's has no attractive outside options in our model and as a result, if financing is possible E will be better off starting the project. If we make the alternative assumption that E must give up some opportunities if he start the project, then his preference will have a more important impact on the decision to start the project.

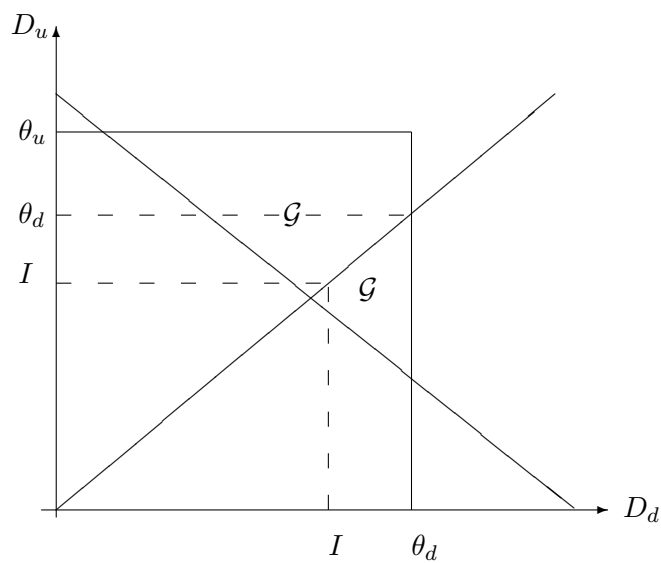
Figure 2: Feasible contracts, IBEU case.

The figure displays the set \mathcal{G} of feasible contracts for the case in which E exhibit IBEU preferences and F is ambiguity neutral.

Panel A: $\theta_d < I$



Panel B: $\theta_d > I$



from Proposition 2 is contained the set \mathcal{G} is all subcases. Thus financing with this contract is an acceptable option for E for all configuration of preferences for E and F.

4.2.2 MEU Preferences

When E makes MEU choices and F makes SEU choices, the set of feasible contracts is:

$$\mathcal{G} = \left\{ \tilde{D} \mid 0 \leq \tilde{D} \leq \tilde{\theta}, E_p(\tilde{D}) \geq I \right\}$$

E chooses the optimal contract by solving the following optimization problem

$$\sup_{\tilde{D} \in \mathcal{G}} V_E(\tilde{D}) = -I + E_p(\tilde{D}) + \inf_{\pi_E \in \Pi_E} E_{\pi_e}(\tilde{\theta} - \tilde{D}),$$

E will start the project whenever this quantity is positive.

Proposition 4. *When E has multiple prior beliefs and MEU preferences and F has single prior beliefs, E will start the project if and only if*

$$I \leq E_p(\tilde{\theta}).$$

If this condition holds, then it is optimal to sell the whole firm, i.e., $\tilde{D} = (\theta_u, \theta_d)$.

There are multiple optima because Figure 2 shows that the set \mathcal{G} intersects the set of contracts leaving E with flat payoff ($\theta_u - D_u = \theta_d - D_d$) then any element D of this intersection gives E the value

$$V_E(\tilde{D}) = -I + E_p(\tilde{D}) + \inf_{\pi} E_{\pi}(\tilde{\theta} - \tilde{D}) = -I + E_p(\tilde{D}) + E_p(\tilde{\theta} - \tilde{D}) = -I + E_p(\tilde{\theta})$$

and therefore the security \tilde{D} is also an optimal choice.

4.3 Multi-prior F and E

In this section we assume that E has a multiple prior set Π_E and F has a multiple prior set Π_F and we make the assumption that

$$\Pi_E \subseteq \Pi_F \iff \varepsilon_E \leq \varepsilon_F.$$

4.3.1 IBEU preferences

We assume that both F and E have IBEU preferences. The set of contracts satisfying the financing constraints is

$$\mathcal{H} = \left\{ \tilde{D} \mid 0 \leq \tilde{D} \leq \tilde{\theta} \text{ and } \inf_{\pi_F \in \Pi_F} E_{\pi}(\tilde{D}) \geq I \right\}$$

If E finances the project with $\tilde{D} \in \mathcal{H}$, his subjective valuation under prior $\pi_E \in \Pi_E$ is

$$V_E^{\pi_E}(\tilde{D}) = -I + \inf_{\pi_F \in \Pi_F} E_{\pi_F} \tilde{D} + E_{\pi_E}(\tilde{\theta} - \tilde{D}).$$

Because $\tilde{D} \leq \tilde{\theta}$ any contract from \mathcal{H} is implementable and E is willing to start the firm and finance it with any $\tilde{D} \in \mathcal{H}$. In the next proposition we show that the structure of the set Π_E is irrelevant for this result provided that $\Pi_E \subseteq \Pi_F$. Now unlike the case where F has a single prior beliefs, it is possible to rank the financing contracts as the following proposition shows.

Proposition 5. *The optimal financing arrangement when both E and F have the multiple priors $\Pi_E \subset \Pi_F$ and have IBEU preferences is as follow.*

1. *If $I > E_{p-\varepsilon_F}(\tilde{\theta})$, then E cannot finance the project.*
2. *If $\theta_d \leq I \leq E_{p-\varepsilon_F}(\tilde{\theta})$ then E will accept to finance the project with any \tilde{D} in the set \mathcal{H} .¹⁵ Moreover, all the contracts in the set \mathcal{H} are dominated (from E 's perspective) by the contract*

$$\tilde{D}^* = \left(\theta_d + \frac{1}{p-\varepsilon}(I - \theta_d), \theta_d \right).$$

3. *If $0 \leq I \leq \theta_d$, then E will accept to finance the project with any \tilde{D} in the set \mathcal{H} . Moreover, all the contracts in the set \mathcal{H} are dominated (from E 's perspective) by the risk free contracts of the form*

$$\tilde{D} = (\gamma, \gamma) \quad \text{with } I \leq \gamma \leq \theta_d.$$

¹⁵Notice that when $\theta_d \leq I \leq E_{p-\varepsilon}$ the set \mathcal{H} can also be defined as

$$\mathcal{H} = \{D \mid D \leq \theta \text{ and } E_{p-\varepsilon}(D) \geq I\}$$

Notice that in all subcases, the only dominating contracts are the one which are optimal when E uses SEU to make decisions and F uses IBEU to make decisions. The above proposition says that if we only select the dominating contracts, the set of priors of E is irrelevant in the context of our problem provided that it is included in the F 's set of priors.

Notice that the assumption $\Pi_E \subseteq \Pi_F$ seems crucial for the above proposition. Under this assumption the structure of the set of E 's prior is irrelevant and everything is as if E has a single prior and SEU preferences. It can be shown that if we assume instead that $\Pi_F \subseteq \Pi_E$ we will have multiple contract which are not comparable as in the case where F has a single prior (section 4.2).

4.3.2 MEU preferences

Here again, the assumptions on the sets Π_E and Π_F are going to be important. If the set Π_F is the largest then E will issue risk free securities when they satisfy the financing constraints. If this is not possible they will issue a state contingent security that gives up all the firm in the down state. If the set Π_E is the largest then, E will sell the whole firm. TBC

5 Dynamic contracting under ambiguity

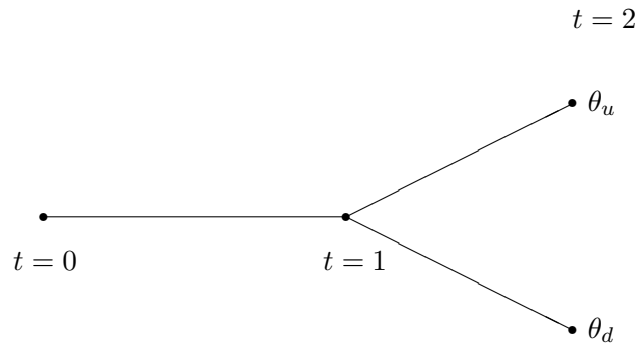
In Section 3.4 we showed how the change is status quo together with the multi-prior feature of IBEU can generate time inconsistency in E 's decisions. This time inconsistency emerges endogenously with IBEU in a dynamic context and creates a new economic force that may generate a demand for any security that will help E to commit to the policy that he currently prefers. The demand for this particular form of securities does not exist under SEU or MEU preferences. The purpose of this section is to illustrate how the demand for this type of securities can be addressed in an optimal contract setting.

E has access to an investment opportunity generating a cash flow θ_u in the “up” state and θ_d in the “down” state at time $t = 2$. The cash flow of the project is then $\theta = (\theta_u, \theta_d)$. A cost of I needs to be paid to begin the project and E does not have enough cash to pay for it. To simplify the model, we assume that the E has no personal wealth and if he wants to start the project, he must issue some securities to the financiers (F) in order to start the project. The securities are issued at $t = 0$ and they have an option feature that can be exercised at the intermediate

time $t = 1$. There is no new information being revealed between time $t = 0$ and $t = 1$, i.e. the information structure is summarized by Figure 5.

Figure 3: Information structure for dynamic contracting

The figure displays the information structure we consider in studying the dynamic contracting problem in the presence of ambiguity. Securities are issued at time $t = 0$ and they contain an option feature that can be exercised at time $t = 1$. Payoffs are received at time $t = 2$. No information is revealed between time $t = 0$ and $t = 1$.



We suppose that F has multiple priors in the set $\Pi = [p - \varepsilon, p + \varepsilon]$ and we will consider both MEU and IBEU preferences. At this stage, we do not commit to any preferences for E (we just discuss valuation and not contracting).

We will consider two type of primitive contracts. Risky debt has the form

$$B^\beta = (B_u^\beta = \theta_d + \beta(\theta_u - \theta_d), B_d^\beta = \theta_d) \text{ with } 0 \leq \beta \leq 1.$$

When $\beta = 0$, the debt is safe and when $\beta > 0$ the debt is risky and has a face value B_u^β .

Equity has the form

$$Q^\alpha = (\alpha\theta_u, \alpha\theta_d) \text{ with } 0 \leq \alpha \leq 1.$$

Because the payoff in the up state is always larger for the class of securities that we consider, we know that if F makes decisions based on IBEU or MEU, E will be able to raise the amount

$$E_{p-\varepsilon}(D), \text{ for } D = B^\beta \text{ or } D = Q^\alpha.$$

Notice that the amount raised is equal under IBEU or MEU.

Now we turn to the cases where E issues a security with an option feature. Consider the security $BQ^{\alpha,\beta}$ which gives to the savers the possibility to convert the bond B^α to the equity Q^α . The conversion decision must be taken at time $t = 1$. Similarly, we consider the security $QB^{\alpha,\beta}$ which gives the savers to convert the equity to a bond at time $t = 1$. With both securities, F must decide if they exercise the option at time $t = 1$ by comparing the value under a particular prior π of the bond

$$\pi(\theta_d + \beta(\theta_u - \theta_d)) + (1 - \pi)(\theta_d) = \theta_d + \pi\beta(\theta_u - \theta_d)$$

with the value of the equity

$$\alpha\pi\theta_u + \alpha(1 - \pi)\theta_d.$$

With MEU, the decision is based on the worst prior $p - \varepsilon$ and the indifference frontier in the plan (α, β) is given by

$$\alpha = \frac{\theta_d + (p - \varepsilon)\beta(\theta_u - \theta_d)}{E_{p-\varepsilon}(\theta)}$$

The frontier splits the plan into a region Γ_1 where debt is preferred and a region Γ_2 where equity is preferred.

Therefore, both securities $BQ^{\alpha,\beta}$ and $QB^{\alpha,\beta}$ will have the same price which is given by

$$\text{Proceeds}(BQ^{\alpha,\beta}) = \text{Proceeds}(QB^{\alpha,\beta}) = \theta_d + (p - \varepsilon)\beta(\theta_u - \theta_d) \text{ if } (\alpha, \beta) \in \Gamma_1$$

and

$$\text{Proceeds}(BQ^{\alpha,\beta}) = \text{Proceeds}(QB^{\alpha,\beta}) = \alpha E_{p-\varepsilon}(\theta) \text{ if } (\alpha, \beta) \in \Gamma_2$$

With IBEU, the plan is split into three regions Γ'_1 , Γ''_1 and Γ_2 .

In the region Γ''_1 debt is preferred and in region Γ_2 equity is preferred. Region Γ'_1 is a region where equity and debt are not comparable (under some prior debt is preferred whereas under some other priors equity is preferred). The region Γ'_1 is described by the equation

$$\frac{\theta_d + (p + \varepsilon)\beta(\theta_u - \theta_d)}{E_{p+\varepsilon}(\theta)} \leq \alpha \leq \frac{\theta_d + (p - \varepsilon)\beta(\theta_u - \theta_d)}{E_{p-\varepsilon}(\theta)}$$

In the regions Γ''_1 and Γ_2 the proceeds are given as before

Figure 4: Conversion regions, MEU case.

The figure displays the conversion regions in the case of MEU preferences. In region Γ_1 debt is preferred and in region Γ_2 equity is preferred.

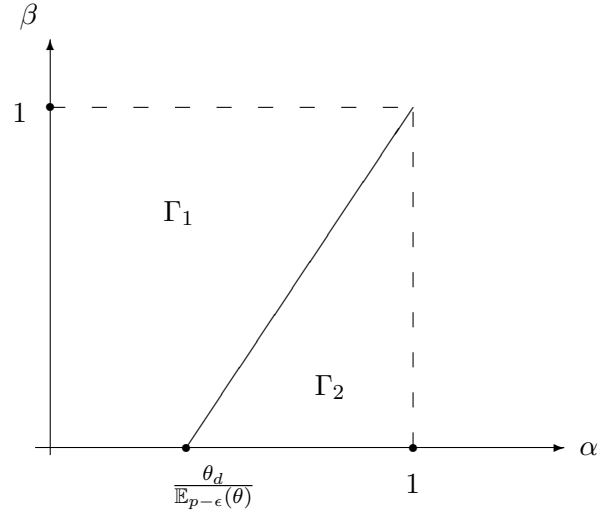
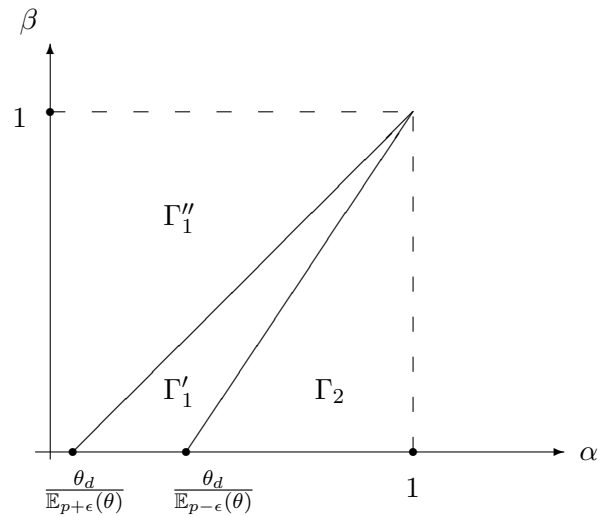


Figure 5: Conversion regions, IBEU case.

The figure displays the conversion regions in the case of IBEU preferences. In region Γ''_1 debt is preferred, in region Γ_2 equity is preferred and in region Γ'_1 equity and debt are not comparable.



$$\text{Proceeds}(BQ^{\alpha,\beta}) = \text{Proceeds}(QB^{\alpha,\beta}) = \begin{cases} \theta_d + (p - \varepsilon)\beta(\theta_u - \theta_d) & \text{if } (\alpha, \beta) \in \Gamma''_1 \\ \alpha E_{p-\varepsilon}(\theta) & \text{if } (\alpha, \beta) \in \Gamma_2 \end{cases}$$

But on the region Γ'_1 , the amount raised by E may change with the security that is issued. When pricing Security $BQ^{\alpha,\beta}$ with $(\alpha, \beta) \in \Gamma'_1$ at time $t = 0$, F knows that he will not exercise the conversion option because he will have a bond as a status quo. As a result, he will price this security as a straight bond. More formally, for any $(\alpha, \beta) \in \Gamma'_1$

$$\text{Proceeds}(BQ^{\alpha,\beta}) = \theta_d + (p - \varepsilon)\beta(\theta_u - \theta_d).$$

Similarly, F will price the security $QB^{\alpha,\beta}$ as an equity and for any $(\alpha, \beta) \in \Gamma'_1$,

$$\text{Proceeds}(QB^{\alpha,\beta}) = \alpha E_{p-\varepsilon}(\theta).$$

This result suggests that when issuing convertible equities (an equity with the option to convert it to a bond), it is possible E will be able to raise more money when F make decisions based on MEU than when they make decision based on IBEU.

Proposition 6. *For any $(\alpha, \beta) \in \Gamma'_1$, the proceeds from selling $QB^{\alpha,\beta}$ are larger when savers are MEU than when they are IBEU. More formally,*

$$\text{Proceeds}^{IBEU}(QB^{\alpha,\beta}) = \alpha E_{p-\varepsilon}(\theta) \leq \text{Proceeds}^{MEU}(QB^{\alpha,\beta}) = \theta_d + (p - \varepsilon)\beta(\theta_u - \theta_d)$$

The result in this proposition is a manifestation of the asymmetry that we already observed in the expansion/contraction options example of section 3. The proposition says that some securities with option features will be overvalued by MEU savers relative to IBEU savers.

Proposition 6 illustrates an important difference between IBEU and MEU. Consider a hybrid security A offering the ownership of security X with the option to convert it one period to security Y . Assume securities X and Y pay the cash flows after the first period. Consider the alternative hybrid security B offering the ownership of security Y with the option to convert it one period to security X . Assume that there is no information revelation about cash flows between the issuance date and the option exercise date.

When F uses SEU with the prior $\pi = p$, the *ex ante* valuation of the two securities is clearly identical and is given by

$$\text{Proceeds}(A) = \text{Proceeds}(B) = \text{Max} \{E_p(X), E_p(Y)\}$$

Notice that the security valuation at $t = 0$ is also identical if it was possible to commit to a particular policy exercise. From the perspective of time $t = 0$, the SEU saver will also pick the security delivering the highest expected payoff under the probability p even when commitment is possible.

When F is MEU, the *ex ante* valuation of the two securities is also identical and is given by

$$\text{Proceeds}(A) = \text{Proceeds}(B) = \text{Max} \left\{ \inf_{\pi} E_{\pi}(X), \inf_{\pi} E_{\pi}(Y) \right\}$$

The security valuation at time $t = 0$ does not change if it was possible for saver to commit to a particular policy decision. In this case, the MEU saver will still pick the security that offers the highest expected payoff according to the worst probability measure.

With IBEU the valuation of security A can be different from the valuation of Security B . Specifically, when the primitive security X is not comparable with security Y , the inertia assumption shows that

$$\text{Proceeds}(A) = \inf_{\pi} E_{\pi}(X)$$

whereas

$$\text{Proceeds}(B) = \inf_{\pi} E_{\pi}(Y)$$

This is a first important difference between MEU and IBEU: it seems that IBEU is sensitive to the sequencing of the options of hybrid securities whereas sequencing is irrelevant for SEU and MEU. The sequence of option is relevant for IBEU because it induces a particular path of status quo which in turn break down the indifference in the exercise choice.

The second difference is that when commitment on the exercise policy is possible, the valuation of the security A can be different from the valuation of the security A in the absence of commitment. An interesting situation occurs when X is not comparable to 0 whereas X dominates 0.¹⁶ In this case, the commitment exercise policy for the hybrid security A is to convert it to Y because the commitment solution uses 0 as a status quo. As a result, the valuation of security A under commitment is given by

$$\text{Proceeds}(A) = \inf_{\pi} E_{\pi}(Y)$$

¹⁶This can occur when $X = (-1, 3)$ and $Y = (1, 2)$. With a large enough set of priors, we see that X is not comparable with 0. The security Y always dominates 0. Again with a large enough set of priors X is not comparable with Y .

and it is different from its valuation when commitment is not possible.

6 Conclusion

We have examined the way in which ambiguity aversion, modeled as multiple priors, affects real investment and financing decisions. We considered two approaches to decision making under ambiguity which derive from two different relaxation of Savage's Subjective expected utility paradigm: the MEU approach, based on the relaxation of the independence axiom, and the IBEU, based on the relaxation of the completeness axiom. The two approaches can deliver considerably different, and sometime contradictory investment and financing decisions. Such heterogeneity in predicted choices can be useful for future empirical work that would attempt to determine which approach better describe observed corporate behavior over time.

A Appendix: Proofs

Proof of Proposition 1

To describe the recursive solution, we just need to specialize the analyzes in sections is readily available from 3.3. At time $t = 1$, the contraction decision is characterized by the inequality (21). Taking the investment policy at time $t = 1$ as a constraint, the recursive investment policy at time $t = 0$ is to invest if and only if

$$I_0 < (p - \varepsilon)\bar{\theta} + (1 - p + \varepsilon)S$$

when $S > E_{p+\varepsilon}(\tilde{\theta}^d)$ and to invest if and only if

$$I_0 < (p - \varepsilon)\bar{\theta} + (1 - p + \varepsilon)E_{p-\varepsilon}(\tilde{\theta}^d).$$

when $S \leq E_{p+\varepsilon}(\tilde{\theta}^d)$.

Under the commitment investment policy E must make one decision at time $t = 0$. E chooses between three policies: not investing (P0), Investing and continuing P1 and, investing and contracting P2. E takes P1 vs P0 if and only if

$$I_0 < (p - \varepsilon)\bar{\theta} + (1 - p + \varepsilon)E_{p-\varepsilon}(\tilde{\theta}^d) \equiv E_{p-\varepsilon}(\tilde{C}_2).$$

E takes P2 versus P0 if and only if

$$I_0 < (p - \varepsilon)\bar{\theta} + (1 - p + \varepsilon)S$$

It is easy to see with a figure that in Region \mathcal{R}_0 E wants to invest and contract (P2) and rejects the investment-continuation policy P1. In the sub-region $\mathcal{R}_0 \cap \mathcal{R}_1$, we see that the commitment investment policy requires to invest at time $t = 0$ and contract at time $t = 1$.

However, in the region \mathcal{R}_1 the recursive investment policy requires to not invest at all at $t = 0$. In fact, with the recursive approach E knows that if he invests at time $t = 0$, he will not contract at time $t = 1$: Under the condition $S < E_{p+\varepsilon}(\tilde{\theta}^d)$ there are always some optimistic beliefs that make the continue option more attractive than the contraction option. As a result, he only considers the path where he does not contract and this path is dominated by not investing in presence of the pessimistic beliefs. On the other hand, under commitment, E has no concern

for the comparison between S and θ^d because he can commit to a contraction. In the region \mathcal{R}_1 , the salvage value S is large enough (resp. I_0 is large enough) to make sure that the contraction option is acceptable (resp the continue option is rejected). ■

Proof of Proposition 2

We start with the observation that an optimal security must bind the constraint

$$\inf_{\pi \in \Pi_F} E_\pi(\tilde{D}) = I$$

If $\inf_{\pi \in \Pi_F} E_\pi(\tilde{D}) > I$, then we can decrease the payoff of \tilde{D} by a small amount in one of the two states and we will then improve E 's utility.¹⁷

Now, if $I \leq \theta_d$, then $\tilde{D} = (I, I) \in \mathcal{D}$ and E 's utility is maximized because Inequality (33) binds. On the other hand, if $I > E_{p-\varepsilon_F}(\tilde{\theta})$, then E is not going to be able to finance the project even if he sells the entire firm.

If $\theta_d \leq I \leq E_{p-\varepsilon_F}(\tilde{\theta})$ then it possible to finance the project. Assume that E finance with the project with a security that has $D_d < \theta_d$. It is then necessary to have $D_u > \theta_d$ to be able to raise I (because $I \geq \theta_d$). Because $D_u > D_d$, we have $I = \inf_{\pi \in \Pi_F} E_\pi(\tilde{D}) = E_{p-\varepsilon_F}(\tilde{D})$. We will now show that a small modification of \tilde{D} allows E to increase the project valuation. Consider the security \tilde{D}' defined by

$$D'_d = D_d + \eta, \quad D'_u = D_u - \eta \frac{p - \varepsilon_F}{1 - p + \varepsilon_F}$$

where η is a very small number. By construction, issuing \tilde{D}' allows to raise exactly I because $\inf_{\pi \in \Pi_F} E_\pi(\tilde{D}') = E_{p-\varepsilon_F}(\tilde{D}) = I$. It is also easy to verify that,

$$V_E(\tilde{D}') = V_E(\tilde{D}) + \eta \frac{\varepsilon_F}{1 - p + \varepsilon_F}$$

¹⁷To clarify this point, suppose that $D_u > D_d$, then the inf of $E_\pi(\tilde{D})$ is attained at $\pi = p - \varepsilon$. Consider the security $\tilde{D}' = (D_u - \eta, D_d)$ with η being a small number so that $E_{p-\varepsilon}(\tilde{D}') \geq I$. It can be checked that E 's utility derived from issuing \tilde{D}' is given by

$$V_E(\tilde{D}') = V_E(\tilde{D}) + \varepsilon \eta > V_E(\tilde{D}).$$

and therefore the entrepreneur prefers to issue \tilde{D}' . A similar reasoning can be used when $D_u > D_d$ by considering the security $\tilde{D}'' = (D_u, D_d - \eta)$.

and thus $V_E(\tilde{D}') > V_E(\tilde{D})$. We conclude then that the optimal security satisfies $D_d = \theta_d$ and the financing constraint requires that

$$D_u = \theta_d + \frac{1}{p - \varepsilon_F}(I - \theta_d).$$

■

Proof of Proposition 3

We have already observed that any security D in the set \mathcal{G} is going to generate a positive $V_E^\pi(\tilde{D})$ under any π in the set of E 's priors Π_E and thus the entrepreneur is happy to start the project by issuing \tilde{D} .

To show that these contracts are not comparable, let us consider a security \tilde{D} in \mathcal{G} . We have $V_E^\pi(\tilde{D}) = -I + E_p(\tilde{D}) + E_\pi(\tilde{\theta} - \tilde{D})$ for any $\pi \in \Pi_E$. Define the security $\tilde{D}' = (D_u + \eta, D_d)$ with η small enough so that $\tilde{D}' \in \mathcal{G}$. Direct calculations show $V_E^\pi(\tilde{D}') = -I + E_p(\tilde{D}) + E_\pi(\tilde{\theta} - \tilde{D}) + \eta(p - \pi)$ and thus

$$V_E^\pi(\tilde{D}') - V_E^\pi(\tilde{D}) = \eta(p - \pi)$$

which can be positive for some π and negative for some other π provided that p is the interior of Π_E . We thus conclude that \tilde{D} and \tilde{D}' are not comparable for E . ■

Proof of Proposition 4

If E finance the firm by issuing $\tilde{D} \in \mathcal{G}$, then he derives the utility

$$V_E(\tilde{D}) = -I + E_p(\tilde{D}) + \inf_{\pi \in \Pi_E} E_\pi(\tilde{\theta} - \tilde{D}).$$

If he issues $\tilde{D} = \tilde{\theta}$, then the derived utility is

$$V_E(\tilde{\theta}) = -I + E_p(\tilde{\theta}).$$

We can see that for any $\tilde{D} \in \mathcal{D}$ we have

$$V_E(\tilde{\theta}) - V_E(\tilde{D}) = E_p(\tilde{\theta} - \tilde{D}) - \inf_{\pi \in \Pi_E} E_\pi(\tilde{\theta} - \tilde{D}) \geq 0$$

and therefore $V_E(\tilde{\theta}) = \sup_{\tilde{D} \in \mathcal{G}} V_E(\tilde{D})$ and it optimal to sell the whole firm. ■

Proof of Proposition 5

Let us start with the case $\theta_d \leq I \leq E_{p-\varepsilon_F}(\tilde{\theta})$. Notice first that in this case, the set \mathcal{H} is above the 45 degree line and thus any $\tilde{D} \in \mathcal{H}$ satisfies $D_d \leq D_u$. Thus each time E 's finances the project with $\tilde{D} \in \mathcal{H}$, he will get the proceeds $E_{p-\varepsilon_F}(\tilde{D})$. Now, it is convenient to decompose \mathcal{H} as

$$\mathcal{H} = \bigcup_{I \leq \gamma \leq E_{p-\varepsilon_F}(\tilde{\theta})} \mathcal{H}_\gamma, \text{ where } \mathcal{H}_\gamma = \left\{ \tilde{D} \mid 0 \leq \tilde{D} \leq \tilde{\theta} \text{ and } E_{p-\varepsilon_F}(\tilde{D}) = \gamma \right\}.$$

Let us first prove that any $\tilde{D} \in \mathcal{H}_\gamma$ is dominated by $\tilde{D}^\#$ which is the unique element of \mathcal{H}_γ satisfying $D_d^\# = \theta_d$. Using the fact that both \tilde{D} and $\tilde{D}^\#$ belong to \mathcal{H}_γ , straightforward calculations show that $\tilde{D}^\# = (D_u - \alpha, \theta_d)$ where $\alpha > 0$ solves the equation

$$\alpha(p - \varepsilon_F) - (\theta_d - D_d)(1 - p + \varepsilon_F) = 0.$$

For any prior $\pi \in \Pi_E$,

$$V_E^\pi(\tilde{D}^\#) - V_E^\pi(\tilde{D}) = E_\pi(\tilde{D}) - E_\pi(\tilde{D}^\#) = \pi\alpha - (1 - \pi)(\theta_d - D_d)$$

Because $\Pi_E \subseteq \Pi_F$, we have $\pi \geq p - \varepsilon_F$ for any $\pi \in \Pi_E$ and comparing the last two equalities we see that $V_E^\pi(\tilde{D}^\#) - V_E^\pi(\tilde{D}) \geq 0$. We conclude that $\tilde{D}^\#$ is preferred to \tilde{D} . The second step is to show that any $\tilde{D} \in \mathcal{H}_\gamma \cap \{\tilde{D} \mid D_d = \theta_d\}$ for some $\gamma \in [I, E_{p-\varepsilon_F}(\tilde{\theta})]$ is dominated by \tilde{D}^* . First observe that financing the firm with a security $\tilde{D} \in \mathcal{H}_\gamma \cap \{\tilde{D} \mid D_d = \theta_d\}$ yield the utility

$$V_E^\pi(\tilde{D}) = -I + E_{p-\varepsilon_F}(\tilde{D}) + E_\pi(\tilde{\theta} - \tilde{D}) = -I + \gamma + E_\pi(\tilde{\theta} - \tilde{D})$$

for any $\pi \in \Pi_E$. On the other hand if E finances the project with \tilde{D}^* , he gets the utility

$$V_E^\pi(\tilde{D}^*) = -I + E_{p-\varepsilon_F}(\tilde{D}^*) + E_\pi(\tilde{\theta} - \tilde{D}^*) = E_\pi(\tilde{\theta} - \tilde{D}^*)$$

for any any $\pi \in \Pi_E$. Therefore

$$V_E^\pi(\tilde{D}^*) - V_E^\pi(\tilde{D}) = I - \gamma + E_\pi(\tilde{D}^* - \tilde{D}) = I - \gamma + \pi(D_u - D_u^*)$$

Using the fact that $(\tilde{D}^*, \tilde{D}) \in \mathcal{H}_I \times \mathcal{H}_\gamma$ and the fact that $D_d^* = D_d = \theta_d$ we get

$$V_E^\pi(\tilde{D}^*) - V_E^\pi(\tilde{D}) = (\gamma - I) \left[\frac{\pi}{p - \varepsilon_F} - 1 \right]$$

which is positive for any $\pi \in \Pi_E$, again because $\pi \geq p - \varepsilon_F$ (recall that $\Pi_E \subseteq \Pi_F$). To summarize, using the transitivity of IBEU, we have shown that \tilde{D}^* is preferred to any other implementable contract in (\mathcal{H}) and thus we can consider that the only “stable” contract is \tilde{D}^* .

Let us now turn to the case $I \leq \theta_d$. This case is different because the 45 degrees crosses the set \mathcal{H} and splits it into two subsets

$$\mathcal{H} = \mathcal{H}^+ \cup \mathcal{H}^-$$

where \mathcal{H}^+ (resp. \mathcal{H}^-) contains all the elements $\tilde{D} \in \mathcal{H}$ satisfying $D_u \geq D_d$ (resp. $D_u < D_d$). We will show here that while E is happy to finance the project with any security in the set \mathcal{H} he still prefers to finance the firm with a risk free security.

First, let us mention that if E finances the project with a security of the form $\tilde{D} = (\gamma, \gamma)$ with $\gamma \in [I, \theta_d]$ then he will get the utility

$$V_E^\pi(\tilde{D}) = -I + E_\pi(\tilde{\theta})$$

under the prior π . As a result, E is indifferent (prior by prior) between any two risk free contracts in \mathcal{H} . We will now focus on showing the dominance of the contract $\tilde{D}^f = (\theta_d, \theta_d)$ over all other contracts.

If E finances the project with contract $\tilde{D} \in \mathcal{H}^+$, the financiers will pay $\inf_{\pi \in \Pi_F} E_\pi(\tilde{D}) = E_{p-\varepsilon_F}(\tilde{D})$ and E gets the utility

$$V_E^\pi(\tilde{D}) = -I + E_{p-\varepsilon_F}(\tilde{D}) + E_\pi(\tilde{\theta} - \tilde{D})$$

for any prior $\pi \in \Pi_E$.

On the other hand, if E instead finances the project with the risk free security \tilde{D}^f he will get the utility

$$V_E^\pi(\tilde{D}^f) = -I + E_\pi(\tilde{\theta})$$

for any prior $\pi \in \Pi_E$ and thus

$$V_E^\pi(\tilde{D}^f) - V_E^\pi(\tilde{D}) = E_\pi(\tilde{D}) - E_{p-\varepsilon_F}(\tilde{D}).$$

Noticing that $\pi \geq p - \varepsilon_F$ and $D_u \geq D_d$ yields

$$V_E^\pi(\tilde{D}^f) \geq V_E^\pi(\tilde{D}) \text{ for all } \pi \in \Pi_E$$

meaning that E prefers \tilde{D}^f to any contract in \mathcal{H}^+ .

Now, if E finances the project with $\tilde{D} \in \mathcal{H}^-$, the financier F pays $\inf_{\pi \in \Pi_F} E_\pi(\tilde{D}) = E_{p+\varepsilon_F}(\tilde{D})$ he gets the utility

$$V_E^\pi(\tilde{D}) = -I + E_{p+\varepsilon_F}(\tilde{D}) + E_\pi(\tilde{\theta} - \tilde{D})$$

for any prior $\pi \in \Pi_E$. Thus

$$V_E^\pi(\tilde{D}^f) - V_E^\pi(\tilde{D}) = E_\pi(\tilde{D}) - E_{p+\varepsilon_F}(\tilde{D})$$

and recalling that $\pi \leq p + \varepsilon$ and $D_u \leq D_d$ gives

$$V_E^\pi(\tilde{D}^f) \geq V_E^\pi(\tilde{D}) \text{ for all } \pi \in \Pi_E.$$

■

Proof of Proposition 6

To be added.

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