Monotone Comparative Statics under Monopolistic Competition

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Abstract

We let heterogeneous firms face decisions on an arbitrary number of complementary activities in a monopolistically-competitive industry. One key insight is that firm-level complementarities may assert themselves much more clearly at the industry level than at the firm level of analysis. The response of an individual firm to exogenous changes in the parameters of its profit maximisation problem is ambiguous due to indirect effects through changes in industry competition. Only in special cases are firm-level comparative statics monotone. Turning to the industry level, we provide sufficient conditions for first-order stochastic dominance shifts in the equilibrium distributions of the activities regardless of the ambiguities prevailing at the firm level. As all these results apply to many recent and well-known models of international trade, they are more than just a theoretical curiosity.

Keywords: Complementary Activities; Firm Heterogeneity; Supermodularity; Monotone Comparative Statics

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1 Introduction

More than two decades ago, Milgrom and Roberts (1990a) argued that strategy and structure in modern manufacturing firms reflect widespread complementarities among many diverse activities undertaken by firms. Drawing on Topkis (1978), they emphasised how such complementarities—and the corresponding mathematical property of supermodularity—make firms’ decisions exhibit monotone comparative statics. That is, the optimal levels of the activities are monotonic in parameters of the profit maximisation problem that influence the set of available activities or the attractiveness of these activities all else equal. Since this seminal contribution, the monotonicity theorems developed by Topkis (1978), and later by Milgrom and Shannon (1994) and Athey (2002), have played important roles in the comparative statics of firms. One reason is their virtue of focusing on the properties of the optimisation problem that are essential for obtaining monotone comparative statics and doing away with superfluous assumptions. For example, these monotonicity theorems allow activities to be discrete choice variables and the profit function to be nonconcave, nondifferentiable, and discontinuous at some points. When applying these monotonicity theorems, many studies on firm-level complementarities in organisational economics assume that the competitive environment of the firm is exogenous.\(^1\) While being convenient and serving as a natural baseline, such partial analysis is not sufficient when studying exogenous shocks that affect all firms in an industry. In this case, firms are not only directly affected by the exogenous changes but also indirectly affected through changes in the competitive environment. In studies of games with strategic complementarities or substitutes, where monotonicity theorems have also been applied, such an indirect effect is central.\(^2\)

The present paper shows that the presence of an indirect effect has a number of interesting implications for the comparative statics in a setting of monopolistic competition among heterogeneous firms. One main result is that firm-level complementarities can manifest themselves much more clearly at the industry level than at the firm level. A technical insight is that monotonicity theorems can be indeed very useful in the analysis of firms in monopolistic competition; a workhorse market structure in many diverse fields of

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economics. Specifically, we put forward a model of monopolistic competition where heterogeneous firms each make a combined decision on an arbitrary number of activities. Firms endogenously enter and exit the industry. The demand level of the industry reflects the intensity of competition and adjusts to ensure a zero expected value of entry. Importantly, the activities faced by firms are complementary with each other and with the demand level of the industry.\footnote{Since our focus is on the implications of complementarities, we simply assume their existence rather than investigate how they may arise. Topkis (1995) and Mrazova and Neary (2012) consider conditions for complementarities to arise and provide examples of specific models.}

At the firm level, we investigate how the decisions of individual firms respond to exogenous increases in industry-wide parameters of the profit maximisation problem. These parameter changes do not only have a non-negative direct effect (given the demand level) on the equilibrium decisions of individual firms but also an indirect effect operating through changes in the demand level. Only when the indirect effect is nonnegative and thus aligned with the direct effect can we be sure that the firm-level comparative statics are monotone. We provide sufficient conditions for this to be the case by restricting the nature of the exogenous changes such that the demand level increases. The indirect effect is however generally ambiguous and, as a consequence, so are the comparative statics at the firm level.

The centerpiece of our contribution is the provision of sufficient conditions for monotone comparative statics at the industry level. By this we mean that exogenous increases in industry-wide parameters of the profit maximisation problem lead to first-order stochastic dominance shifts in the equilibrium distribution of any activity. This means that the share of active firms undertaking at least a given level of any activity increases and so does the average level of any activity. The equilibrium distribution of any activity does not only depend on the levels chosen by firms conditional on being active (the level effect) but also on the endogenous selection of which firms are active (the selection effect). The presence of the selection effect implies that monotone comparative statics at the firm level are neither necessary nor sufficient for monotone comparative statics at the industry level. We are therefore able to identify sufficient conditions for monotone comparative statics at the industry level regardless of the ambiguities prevailing at the firm level. These conditions ensure that the level and the selection effects induced by changes in the demand level cancel out, thus making the indirect effect neutral at the
industry level. This is in some sense a knife-edge result. We also show how weaker conditions ensure monotone comparative statics at the industry level when the demand level can be predicted to either rise or fall. An implication of our results is that one may observe that many, or even all, firms reduce their levels of a given activity while the equilibrium distribution of this activity shifts towards higher levels thereby increasing the average level of this activity.

Since the seminal work by Melitz (2003), models with monopolistic competition among heterogeneous firms have become mainstream in the literature on international trade. Many of these recent trade models feature firm-level complementarities and this paper provides a unifying framework for these. The framework is sufficiently general to encompass, at least symmetric-market versions of, well-known contributions such as Melitz (2003), Antràs and Helpman (2004), Helpman et al. (2004), Melitz and Ottaviano (2008), Helpman and Itskhoki (2010), Helpman et al. (2010), Arkolakis (2010), Amiti and Davis (2011), Bernard et al. (2011), Bustos (2011), Arkolakis and Muendler (2011), Caliendo and Rossi-Hansberg (2011), Mayer et al. (2011), and Kasahara and Lapham (forthcoming). By relying on monotonicity theorems for comparative statics, our mathematical techniques differ substantially from the standard approach in these studies. Applying our results to these models reveals that the common assumption of Pareto-distributed firm-specific productivity parameters leads to monotone comparative statics at the industry level. The Pareto assumption represents the knife-edge case where the level and the selection effects induced by changes in the demand level cancel out. Because our results provide strong and testable implications for many recent and well-known models from international trade, our point that firm-level complementarities may assert themselves more clearly at the industry level than at the firm level not only represents a theoretical curiosity.

Bernard et al. (2003) and Arkolakis et al. (2012) have noted the discrepancy between firm- and industry-level effects of trade liberalisation on firm markups using models with heterogeneous firms and endogenous selection. But, to the best of our knowledge, we are the first to provide a general and thorough analysis of how firm-level complementarities can imply monotone comparative statics at the industry level despite ambiguities in firm-level responses. In illustrating the merits of the mathematical tools behind monotonicity theorems for analysing the above-mentioned trade models, this paper is also related to Mrazova and Neary (2012). These two authors emphasise
the role of supermodularity (complementarity) in shaping the sorting pattern of firms in a given equilibrium. Our approach differs by not only focusing on a given equilibrium but rather conducting comparative statics across equilibria. Finally, vaguely related is Costinot (2009) who examines the role of log-supermodularity in generating comparative advantage. We emphasise that, although the present paper has close ties to the international-trade literature, nothing in the formulation of our framework limits the relevance or application of our results to trade-related issues.

The remainder of the paper is organised as follows. Section 2 briefly reviews the central mathematical results from Topkis (1978, 1995) and Milgrom and Shannon (1994) that we draw upon in our analysis as these may be unfamiliar to some readers. Section 3 develops our framework and presents our central assumption of complementarities. Section 4 emphasises the ambiguities in firm-level responses and provides special cases with monotone comparative statics. Next, Section 5 characterises the equilibrium distributions of activities and derives sufficient conditions for monotone comparative statics at the industry level. Section 6 presents concrete applications of both our framework and results. Finally, Section 7 offers some concluding remarks.

2 Mathematical Preface

Let $X \subseteq \mathbb{R}^n$ and $T \subseteq \mathbb{R}^m$ be partially ordered sets with the component-wise order. For two vectors $x', x'' \in \mathbb{R}^n$, we let $x' \vee x''$ denote the component-wise maximum and $x' \wedge x''$ denote the component-wise minimum. The set $X$ is a lattice if for all $x', x'' \in X$, $x' \vee x'' \in X$ and $x' \wedge x'' \in X$. The set $S \subseteq X$ is a sublattice of $X$ if $S$ is a lattice itself. For two sets $S', S'' \subseteq \mathbb{R}^n$, we say that $S''$ is higher than $S'$ and write $S' \leq_s S''$ if for all $x' \in S'$ and all $x'' \in S''$, $x' \vee x'' \in S''$ and $x' \wedge x'' \in S'$. If a set becomes higher, then we say that the set is increasing.

Let $X$ be a lattice. The function $h : X \times T \to \mathbb{R}$ is supermodular in $x$ on
for each \( t \in T \) if for all \( x', x'' \in X \) and \( t \in T \),

\[
h(x', t) + h(x'', t) \leq h(x' \land x'', t) + h(x' \lor x'', t). \tag{1}
\]

Supermodularity of \( h \) in \( x \) implies that the return from increasing several elements of \( x \) together is larger than the combined return from increasing the elements separately.\(^7\) This follows from the fact that a higher value of one subset of the elements in \( x \) increases the value of increasing other subsets of elements. Supermodularity thus implies that the elements of the vector \( x \) are (Edgeworth) complements. If \( h \) is smooth, supermodularity is equivalent with \( \partial^2 h / \partial x_i \partial x_j \geq 0 \) for all \( i, j \) where \( i \neq j \). By (1), it follows that any function \( h \) is trivially supermodular in \( x \) when \( x \) is a single real variable. The function \( h(x, t) \) has increasing differences in \( (x, t) \) if for \( x' \leq x'' \), \( h(x'', t) - h(x', t) \) is monotone nondecreasing in \( t \). Increasing differences mean that increasing \( t \) raises the return from increasing \( x \). If \( h \) is smooth, increasing differences are equivalent with \( \partial^2 h / \partial x_i \partial t_j \geq 0 \) for all \( i, j \). The following monotonicity theorem is due to Topkis (1978).

**Theorem 1.** Let \( X \subseteq \mathbb{R}^n \) be a lattice, \( T \subseteq \mathbb{R}^m \) be a partially ordered set, \( S \) be a sublattice of \( X \), and \( h : X \times T \rightarrow \mathbb{R} \). If \( h(x, t) \) is supermodular in \( x \) on \( X \) for each \( t \in T \) and has increasing differences in \( (x, t) \) on \( X \times T \), then \( \arg \max_{x \in S} h(x, t) \) is monotone nondecreasing in \( (t, S) \).

If the set of maximisers only contains one element, this unique maximiser is nondecreasing in \( (t, S) \). In the remainder of this paper, we restrict attention to cases where the set of maximisers is a nonempty and complete sublattice.\(^8\) This implies that the set of maximisers has greatest and least elements and Theorem 1 implies that these greatest and least elements are nondecreasing functions of \( (t, S) \). We also follow the convention of focusing on the greatest element in the set of maximisers, effectively treating the maximiser as unique.\(^9\) The monotone comparative statics result of Theorem 1 could also be obtained under the weaker assumption that \( h \) is quasisupermodular in \( x \) and exhibits the single crossing property in \( (x, t) \).\(^10\) However,

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\(^7\)To see this, rewrite (1) into \[ [h(x', t) - h(x' \land x'', t)] + [h(x'', t) - h(x' \land x'', t)] \leq h(x' \lor x'', t) - h(x' \land x'', t). \]

\(^8\)General sufficient conditions for this are found in Milgrom and Shannon (1994).

\(^9\)We share this approach with Bagwell and Ramey (1994) and Holmstrom and Milgrom (1994). All results also hold when one focuses on e.g. the least element.

\(^10\)See Appendix A for formal definitions and a theorem.
the properties supermodularity and increasing differences more accurately represent the standard notion of complementarity, are well-known, and are easy to characterise for smooth functions. Therefore we use these assumptions throughout the main part of the paper while keeping in mind that all our results will also hold under quasisupermodularity and the single crossing property.

3 The Model

After paying a sunk entry cost of $f_e$ units of the numéraire, atomistic firms enter an industry characterised by monopolistic competition. Upon entry into this industry, a firm receives a productivity level, $\theta \in \mathbb{R}_+$, and a vector of characteristics, $\gamma \in \mathbb{R}^p$. Individual firms are fully characterised by the pair $(\theta, \gamma)$ which is a realisation of the random variables $(\Theta, \Gamma)$. We let $\Theta$ and $\Gamma$ be independently distributed with c.d.f. $F$ and $G$, respectively. Let productivity be a continuous variable and $F$ be $C^1$ on the interior of its support. In line with labelling $\theta$ as productivity, assume that firms with higher $\theta$ are able to earn weakly higher profits.

After observing its realisation $(\theta, \gamma)$, a firm has to choose whether to start producing or to exit the industry. If a firm chooses to produce, it has to make a decision, $x = (x_1, \ldots, x_n)$, where $x_i$ denotes the chosen level of activity $i$. An activity refers to any variable at the discretion of the firm. The level of an activity can be either discrete or continuous. We let $x \in X$ where $X \subseteq \mathbb{R}^n$ is the set of all conceivable, but not necessarily available, decisions. The set $X$ is assumed to be a lattice which, loosely speaking, means that undertaking a higher level of any activity may require, but importantly, cannot prevent undertaking a higher level of another activity. Restricting attention to lattices allows complementarities to take effect. The profitability of the decisions in $X$ is influenced by a vector of exogenous industry-wide parameters, $\beta \in B$ with $B \subseteq \mathbb{R}^m$. Further, the actual choice set of all firms is restricted to a set of available decisions, $S \subseteq X$, with $S$ being a sublattice of $X$. Our comparative statics are going to focus on changes in $(\beta, S)$ which

\footnote{The distinction between what we call productivity, $\theta$, and other firm characteristics, $\gamma$, is made since we impose some assumptions on $\theta$ that we will not impose on $\gamma$. $\theta$ could in principle represent any firm characteristic that conforms to the assumptions we impose on $\theta$. If one wishes to analyse a situation with a single source of firm heterogeneity, one can simply ignore $\gamma$ in the following.}
determines the attractiveness (all else equal) and availability of activities. Firm profits also depend on a common endogenous aggregate statistic. We will think of and refer to this variable, $A \in \mathbb{R}^+$, as the demand level and let firm profits be nondecreasing in $A$. In line with monopolistic competition among atomistic firms, individual firms perceive $A$ as exogenous.\textsuperscript{12} To get a sense of what $A$ could be, consider a model where identical consumers’ preferences are additively separable across varieties of a differentiated good. In this case, the inverse marginal utility of income enters into the profit function of firms through the demand function and would constitute the demand level.\textsuperscript{13} Note that, $A$ could also comprise endogenous variables such as factor prices as long as all endogenous variables outside the control of the firm can be combined into the single variable $A$.

### 3.1 Profits, Complementarities, and the Optimal Decision

Profits, $\pi$, of a firm with characteristics $(\theta, \gamma)$ depend on the decision, $x$, the demand level, $A$, and the industry parameters, $\beta$. Formally,

$$\pi = \pi(x, \theta, \gamma; A, \beta),$$

where the semicolon separates firm-specific from industry-wide variables. The following assumption summarises the complementarities that are central to our analysis.

**Assumption 1.** For all $(\theta, \gamma, A, \beta)$, the profit function, $\pi(x, \theta, \gamma; A, \beta)$, is supermodular in $x$ on $X$ and has increasing differences in $(x, \theta)$, $(x, A)$, and $(x, \beta)$ on $X \times \mathbb{R}^+$, $X \times \mathbb{R}^+$, and $X \times B$, respectively.

Supermodularity in $x$ implies that the activities are complementary with each other. Further, the assumption of increasing differences implies that productivity, the demand level, and the elements in $\beta$ are all complementary to the $n$ activities.\textsuperscript{14} While $\theta$, $A$, and $\beta$ thus influence firms’ decisions in

\textsuperscript{12}This setup also encompasses the case of perfect competition. To see this, let all firms share the same characteristics $(\theta, \gamma)$, $f_e = 0$, and $A$ could simply be the endogenous price level. For our industry-level analysis to be interesting firm heterogeneity is however central.

\textsuperscript{13}See e.g. Section 6.2.

\textsuperscript{14}Note that $\beta$ only contains those parameters that comply with Assumption 1. As other parameters are kept constant throughout our analysis, we simply abstract from these.
similar ways, they play very different roles in our model. Productivity, \( \theta \), is a firm characteristic that helps us characterise the equilibrium sorting of firms, \( A \) is an endogenous demand level to be determined in equilibrium, and \( \beta \) are exogenous parameters with respect to which we conduct comparative statics.

Proper ordering of activity levels and parameters is crucial for profits to satisfy Assumption 1.\(^{15}\) Even after proper ordering, Assumption 1 may not apply to all conceivable activities that firms can face. However, if one can redefine or reduce the decision of firms to a form where the corresponding profit function satisfies our assumptions, then our results can be applied to the activities that constitute that decision.\(^{16}\) Consequently, we do not necessarily require that all possible activities faced by firms are complementary. Finally, note that (2) does not have to represent a certain payoff to firms. In case of uncertainty after a firm has realised its characteristics and made its decision, the objective function, \( \pi \), could be interpreted as expected profits. For an illustrative example, see Athey and Schmutzler (1995).

Faced with the profit function in (2), a firm makes the optimal decision, \( x^* \), under the constraint that \( x \in S \) while taking \( \theta, \gamma, A, \) and \( \beta \) as given. Formally we have that\(^ {17}\)

\[
x^*(\theta, \gamma; A, \beta, S) = \arg \max_{x \in S} \pi(x, \theta, \gamma; A, \beta).
\]

**Lemma 1.** The optimal decision, \( x^*(\theta, \gamma; A, \beta, S) \), is monotone nondecreasing in \( (\theta, A, \beta, S) \).

Lemma 1 follows readily from Assumption 1 and Theorem 1 and is simply the manifestation of the complementarities discussed above. Importantly, these comparative statics are partial in nature since the endogeneity of \( A \) is ignored. In the analysis of equilibrium comparative statics, Lemma 1 is nevertheless used repeatedly. The profits obtained under the optimal decision

\(^{15}\)If a function is supermodular in \((x_1, x_2)\), then it is not supermodular in \((-x_1, x_2)\). Such a reordering trick is useful if one activity is a substitute for all others (Milgrom and Roberts, 1995). If a function has increasing differences in \((x_1, x_2, \beta)\), then it does not have increasing differences in \((x_1, x_2, -\beta)\).

\(^{16}\)For an example of how to obtain a supermodular profit function in a specific case with a core group of complementary activities and a group of additional activities, see Milgrom et al. (1991).

\(^{17}\)Our focus on the greatest element among maximisers in case of nonuniqueness allows us to treat \( x^* \) as unique. Therefore, we use "=" and not "\( \in \)" in (3).
are denoted by $\pi^*$ which is defined as

$$\pi^*(\theta, \gamma; A, \beta, S) \equiv \pi(x^*(\theta, \gamma; A, \beta, S), \theta, \gamma; A, \beta).$$

Remember that more productive firms earn higher profits and firms earn higher profits the higher is the demand level, i.e., $\pi^*$ is nondecreasing in $(\theta, A)$.

### 3.2 Entry and Equilibrium Distribution of Activities

Firm profits upon entry are bounded below by zero because when $\pi^*$ happens to be negative, the firm exits the industry and forfeits the sunk cost of entry. The expected profits upon entry, $\Pi$, are thus given by

$$\Pi(A, \beta, S) \equiv \int \int \max\{0, \pi^*(\theta, \gamma; A, \beta, S)\} \, dG(\gamma) \, dF(\theta),$$

and are assumed to be finite. We assume free entry and an unbounded pool of potential entrants. In equilibrium, we therefore have that the expected profits upon entry are equal to the cost of entry,

$$\Pi(A, \beta, S) = f_e. \quad (4)$$

This free-entry condition pins down the endogenous demand level, $A$, as a function of $\beta$ and $S$. We assume the existence and uniqueness of an equilibrium with an $A$ that satisfies (4).\(^{18}\) The optimal decisions of firms conditional on being active, the equilibrium demand level, and the distributions of productivities and characteristics together give rise to endogenous distributions of the activities among active firms. We denote the c.d.f. of the equilibrium distribution of activity $i$ by $H_i(x_i; \beta, S).$\(^{19}\) These distributions are the focus of our industry-level analysis and are treated in detail in Section 5. Before we consider the industry level, we proceed with an analysis of firm-level comparative statics.

\(^{18}\)We do neither specify the forces of adjustment, such as entry of new firms or competition over inputs, nor the process that leads to (4) being satisfied.

\(^{19}\)Note that this c.d.f. does not condition on $\gamma$. It is the distribution of the activity among all active firms in the industry.
4 Firm-Level Comparative Statics

We now investigate the equilibrium responses of individual firms to increases in the industry-wide parameters \((\beta, S)\).\(^{20}\) Both increases in \(\beta\) and increases in \(S\) provide firms with an incentive to increase their levels of all activities, all else equal. Increases in \(\beta\) do so by increasing the relative profitability of undertaking higher levels of the activities while increases in \(S\) do so by shifting upwards the set of available decisions. These incentives can be brought about in two distinct ways. By increasing \(\beta\) the relative attractiveness of higher levels of activities is increased both if profits associated with higher levels increase and if profits associated with lower levels decrease. Analogously, the set of available decisions is shifted upwards both if higher levels of activities become available or if lower levels become unavailable. While these two approaches to increasing \(\beta\) and \(S\) are not mutually exclusive, their distinct effects on firm profits will prove important. In the following, we shall refer to increases in \((\beta, S)\) as carried out with the carrot (stick) if expected profits upon entry, \(\Pi\), increases (decreases) for a given demand level. If the profits of all firms increase (decrease) for a given demand level, we say that \((\beta, S)\) has been increased purely by the carrot (stick).\(^{21}\)

Let us define the equilibrium decision of a firm conditional on being active as

\[
\tilde{x}^*(\theta, \gamma; \beta, S) \equiv x^*(\theta, \gamma; A(\beta, S), \beta, S).
\] (5)

From the right hand side of (5), it is clear that changes in \((\beta, S)\) have a direct effect on firm decisions but, since the whole industry is affected, such changes also have an indirect effect through changes in the demand level. This dichotomy is important, so let us be more formal. When we consider a change from \((\beta', S')\) to \((\beta'', S'')\), where either \(\beta\) or \(S\) could remain unchanged, we make the following decomposition of the total effect on \(\tilde{x}^*\).

\[
\Delta \tilde{x}^* = x^*(\theta, \gamma; A', \beta'', S'') - x^*(\theta, \gamma; A, \beta', S')
+ x^*(\theta, \gamma; A'', \beta'', S'') - x^*(\theta, \gamma; A', \beta'', S'').
\]

\(\Delta \tilde{x}^*\) Direct Effect

\(\Delta \tilde{x}^*\) Indirect Effect

\(^{20}\)In this analysis, the special properties of \(\theta\) relative to \(\gamma\) are irrelevant and one could abstract from \(\theta\) for the remainder of this section.

\(^{21}\)It is quite common in the literature to analyse introductions of new activities, meaning changes from \(S'\) to \(S''\) such that \(S' \subset S''\) and \(S' \leq S''\). Since all of the decisions available under \(S'\) are also available under \(S''\), such introductions are a pure carrot approach.
where $A' = A(\beta', S')$ and $A'' = A(\beta'', S'')$. The direct effect on $\tilde{x}^*$ is thus the change in the decision prompted by changes in $(\beta, S)$ when $A$ is held constant. The indirect effect on $\tilde{x}^*$ stems from $A$ not being constant but rather being a function of the conditions affecting firms’ decisions.

The sign of the direct effect follows readily from Lemma 1.

**Corollary 1.** An increase in $(\beta, S)$ has a nonnegative direct effect on the equilibrium decision, $\tilde{x}^*$.

By now, the conclusion of Corollary 1 should come as no surprise. Regardless of whether the carrot or the stick is employed in increasing $(\beta, S)$, firms get an incentive to increase their levels of all activities. The inherent complementarities among activities ensure that this is manifested in an increase in the optimal decisions, all else equal. This is the central result in the analyses of complementary activities in the organisational-economics papers cited in the introduction.

### 4.1 The Indirect Effect on $\tilde{x}^*$

Whereas the direct effect of an increase in $(\beta, S)$ on $\tilde{x}^*$ is unambiguously nonnegative, the sign of the indirect effect depends on the extent to which the carrot or the stick is employed.

**Lemma 2.** The indirect effect on the equilibrium decision, $\tilde{x}^*$, is nonpositive (nonnegative) if $(\beta, S)$ is increased by use of the carrot (stick).

In general, an increase in $(\beta, S)$ can have either a positive or a negative indirect effect on $\tilde{x}^*$. To see this, note two things. By Lemma 1, the sign of the indirect effect has the same sign as the change in the demand level. Second, the demand level responds to satisfy the free-entry condition. Whether this results in a lower or higher $A$ depends crucially on whether the increases in $(\beta, S)$ increase or decrease expected profits upon entry given the demand level. That is, whether the increase in $(\beta, S)$ is brought about by the carrot or the stick. This may not be obvious ex ante. In certain cases, where the change in $(\beta, S)$ tends to either increase or decrease the profits of all firms, one can however easily predict the sign of the indirect effect on $\tilde{x}^*$.

**Corollary 2.** The indirect effect on the equilibrium decision, $\tilde{x}^*$, is nonpositive (nonnegative) if $(\beta, S)$ is increased purely by the carrot (stick).
As mentioned in the introduction, studies of models with strategic interaction among firms also point out an indirect effect. While similar in nature to what we describe, the mechanism through which this effect arises is different. Further, the sign of the indirect effect in these models is typically exogenously determined by firms’ activities being either strategic complements or substitutes. In our model, the sign of the indirect effect is in general endogenously determined.

4.2 The Total Effect on $\tilde{x}^*$

By combining the insights of the two preceding sections, we obtain the following proposition.

Proposition 1. The total effect on the equilibrium decision, $\tilde{x}^*$, is non-negative (ambiguous) for all firms if $(\beta, S)$ is increased by use of the stick (carrot).

The direct effect on the equilibrium decisions of some firms may very well be zero.\footnote{This is e.g. the case for a firm that, given $A$, maintains its decision following an increase in $S$.} It is therefore necessary (and sufficient) to have a nonnegative indirect effect on $\tilde{x}^*$, in order for the total effect to be unambiguously non-negative for all firms. In cases where $(\beta, S)$ is increased by use of the carrot, additional or more attractive activities, that is new opportunities, may actually turn out to be a detrimental threat for some firms as these opportunities become available to all firms and thus depress the demand level. Since firms and activities differ in their characteristics, either of the direct or the indirect effects may dominate the other for a given activity in a given firm.\footnote{When the sign of the total effect is ambiguous, it may vary both across activities within a given firm and across different firms for a given activity. We provide an example of this in Section 6.1.} Despite the fewer or less attractive activities that may result from using the stick, firms are provided with an incentive to increase the levels of their activities that is only reinforced by the fact that these apparently worse conditions affect all firms and therefore ease competition ($A$ increases).

Proposition 1 highlights that analysing complementary activities in industry equilibrium refines the comparative statics obtained in some earlier studies. In their analyses of complementary activities, Milgrom and Roberts (1990a, 1995), Milgrom et al. (1991), Holmstrom and Milgrom (1994), Athey (1991), Holmstrom and Milgrom (1994), Athey
and Schmutzler (1995), and Topkis (1995) assume that all variables that affect profits, but are outside the control of the firm, are exogenous.\textsuperscript{24} As insightful as their analyses are, this means that they do not capture the feedback through the indirect effect. It is exactly this feedback from the endogenous demand level which implies that the decisions of individual firms do not generally exhibit monotone comparative statics in \((\beta, S)\) in spite of the complementarities imposed by Assumption 1.\textsuperscript{25} In a nutshell, one contribution of the present section is to show that even though increases in \((\beta, S)\) provide firms with an incentive to increase activities all else equal, the responses of firms also depend on the endogenous competitive effects on the industry. Only in special cases are the comparative statics monotone for all firms.

**Corollary 3.** The total effect on the equilibrium decision, \(\tilde{x}^*\), is nonnegative for all firms if \((\beta, S)\) is increased purely by use of the stick.

The main arguments above have not depended on firm heterogeneity being nondegenerate. Thus, the mechanisms described are still at play if the distributions \(F\) and \(G\) are degenerate such that all firms share the same characteristics and make the same decision \(\tilde{x}^*\). One qualification to our results must however be given. With homogeneous firms, an increase in \(S\) will unambiguously increase \(\tilde{x}^*\). To see this, note first that if the initial \(\tilde{x}^*\) becomes unavailable, then only higher decisions are possible. Second, if the initial \(\tilde{x}^*\) does not become unavailable, we know, by the definition of the strong set order \((\leq_s)\) in Section 2, that possible lower decisions available ex post were also available ex ante. By a revealed preference argument, possible lower decisions cannot be optimal after \(S\) has increased. The total effect of an increase in \(\beta\) remains ambiguous.\textsuperscript{26}

\textsuperscript{24}Our nonnegative direct effect on \(\tilde{x}^*\) is completely equivalent to the firm-level monotone comparative statics obtained in these earlier studies. These papers do not feature an indirect effect and therefore the total effect is always nonnegative.

\textsuperscript{25}The reason why we do not obtain monotone comparative statics in \(\beta\) is that our assumption of the profit function having increasing differences in \((x, \beta)\) is partial. The assumption holds for a given \(A\) which by no means ensures increasing differences in \((x, \beta)\) once we recognise the endogeneity of \(A\). Similarly, the conditions for monotone comparative statics in \(S\) only hold for a given \(A\).

\textsuperscript{26}A good example, which fits into our framework, is the Krugman (1979) model. As shown by Zhelobodko et al. (2012), the total output of firms (an activity) may either increase or decrease following an increase in market size \((\beta)\) depending on how the "relative love of variety" varies with consumption.
5 Industry-Level Comparative Statics

This section considers how the equilibrium distributions of activities respond to changes in \((\beta, S)\). First, let us formalise our notion of monotone comparative statics at the industry level.

**Definition 1.** We say that the industry level exhibits monotone comparative statics when increases in \((\beta, S)\) induce first-order stochastic dominance (FOSD) shifts in the equilibrium distributions of all activities. That is, 
\[ H_i(x_i; \beta, S) \text{ is nonincreasing for all levels, } x_i, \text{ of all activities, } i = 1, \ldots, n. \]

Consequently, industry-level monotone comparative statics imply that the share of active firms that undertake at least any given level of any activity increases. The equilibrium distributions of the activities thus shift towards higher values of all activities and the average level of any activity increases.

Without endogenous exit, such that all firms choose to be active upon entry, the industry-level comparative statics follow directly from the firm-level comparative statics.\(^{27}\) In this case, the industry level exhibits monotone comparative statics if the total effect on \(\tilde{x}^*\) is nonnegative for all firms. If the total effect on \(\tilde{x}^*\) is ambiguous, then so are the industry level comparative statics in general. In this section, we therefore analyse the more interesting case where some firms endogenously shut down upon entry. With endogenous exit the equilibrium distributions of the activities are both affected by the levels of activities undertaken by firms conditional on being active (level effect) and on the endogenous selection of which firms are active (selection effect). The selection effect introduces a discrepancy between the firm-level comparative statics (the level effect) and the industry-level comparative statics. On the one hand, monotone comparative statics at the firm level do not ensure monotone comparative statics at the industry level. On the other hand, it is possible that the industry level exhibits monotone comparative statics regardless of ambiguity at the firm level, as we show in the present section.

5.1 The Equilibrium Sorting of Firms

To characterise the equilibrium distributions of activities, first consider the sorting of firms based on productivity. Remember that firms with higher

\(^{27}\)This includes the case without firm heterogeneity.
productivity levels are able to earn higher profits. Thus, given $\gamma$, the self-selection of firms into being active or exiting obeys a threshold rule with respect to productivity; all firms with productivities above this threshold are active and all firms with productivities below exit. Denoting this threshold by $\theta_a$, we have that

$$\theta_a(\gamma; A, \beta, S) \equiv \inf\{\theta : \pi^*(\theta, \gamma; A, \beta, S) > 0\}.$$  

For the reasons discussed above, we focus on the case with endogenous exit and assume that no draw of firm characteristics, $\gamma$, is sufficiently favourable to ensure that a firm is able to produce profitably regardless of its productivity draw. That is, $\theta_a(\gamma; A, \beta, S) > \theta_0$ for all $\gamma$ where $\theta_0 \geq 0$ is the lowest possible productivity draw.

To characterise the sorting of active firms into the activities, consider the cross-section of firms in a given equilibrium. In this case, $(A, \beta, S)$ is given. By Lemma 1, we therefore know that higher productivity firms choose weakly higher levels of all activities, conditional on $\gamma$. \(^{28}\) Let $\theta_i$ be the lowest level of productivity at which a firm undertakes at least level $x_i$ of activity $i$, given $\gamma$. Bounding this threshold from below by $\theta_a$, it is given by

$$\theta_i(x_i, \gamma; A, \beta, S) \equiv \max\{\theta_a, \inf\{\theta : x_i^*(\theta, \gamma; A, \beta, S) \geq x_i\}\}. \quad (6)$$

Although the sorting pattern we just described is convenient for characterising an equilibrium, it has some undesirable features if productivity is the only source of firm heterogeneity. First, the strict relationship between productivity and the level of any activity seems unrealistically stark. Moreover, this very relationship introduces a gap between the range of available decisions in $S$ and the range of decisions observed in equilibrium. \(^{29}\) Given the wide variety of firm decisions seen in reality, such a limitation on the observable decisions is undesirable. Allowing for additional sources of firm heterogeneity through $\Gamma$ alleviates these issues. While the strict sorting pattern based on productivity holds for a given realisation of characteristics, $\gamma$,

\(^{28}\)This pattern of firm sorting into the various activities can be seen as a straightforward extension of an insight of Mrazova and Neary (2012) to our case of multidimensional firm decisions.

\(^{29}\)For example, if we have two binary activities, then we can have four possible decisions, $S = \{(0,0),(1,0),(0,1),(1,1)\}$. However, if $x' = (1,0)$ is the optimal decision for one firm, then $x'' = (0,1)$ cannot be the optimal decision for some other firm since this would contradict Lemma 1.
it does not necessarily hold across firms with different characteristics. If we consider all firms at once, this can break the strict relationship between the level of a given activity and productivity and thereby increase the number of observable decisions.\textsuperscript{30}

With $\Theta$ and $\Gamma$ being independently distributed, we can identify restrictions on the distribution of productivities, $F$, that give us monotone comparative statics at the industry level without restricting neither the distribution of characteristics, $G$, nor how $\gamma$ affects profits. These features are comforting to the extent that many potential sources of firm heterogeneity are not easily observable.

5.2 The Level and Selection Effects

On the basis of the sorting pattern described above, we now characterise the equilibrium distributions of the activities and highlight the effects at play. In the following, we focus on a particular level $x_i$ of activity $i$ which could be any level of the $n$ activities. Applying the law of large numbers, let $s_a \equiv 1 - F(\theta_a)$ be the share of firms with characteristic $\gamma$ that are active and $\bar{s}_a \equiv \int s_a \, dG(\gamma)$ be the overall share of active firms. Similarly, let $s_i \equiv 1 - F(\theta_i)$ denote the share of firms with characteristic $\gamma$ undertaking at least level $x_i$ of activity $i$ and let $\bar{s}_i \equiv \int s_i \, dG(\gamma)$ be the corresponding share across characteristics.\textsuperscript{31}

Using these shares, the c.d.f. of the equilibrium distribution of activity $i$ can be expressed as

$$H_i(x_i; \beta, S) = 1 - \frac{\bar{s}_i(x_i; A(\beta, S), \beta, S)}{\bar{s}_a(A(\beta, S), \beta, S)}.$$  \hfill (7)

Denote by $\Delta H_i$ the change in $H_i$ induced by an increase from $(\beta', S')$ to $(\beta'', S'')$. Using $t' = (\beta', S')$ and $t'' = (\beta'', S'')$ for shorthand notation, this change can be decomposed into the level and selection effects mentioned above,

\textsuperscript{30}For examples where multidimensional firm heterogeneity has this purpose, see Eaton et al. (2011), Amiti and Davis (2011), and Kasahara and Lapham (forthcoming).

\textsuperscript{31}Note that $s_a \geq s_i$ and therefore $\bar{s}_a \geq \bar{s}_i$. 
\[ \Delta H_i = \frac{\bar{s}_i(x_i; A', t')}{\bar{s}_a(A', t')} - \frac{\bar{s}_i(x_i; A', t'')}{\bar{s}_a(A', t'')} + \frac{\bar{s}_i(x_i; A', t'')}{\bar{s}_a(A', t'')} - \frac{\bar{s}_i(x_i; A', t'')}{\bar{s}_a(A', t'')} \]

Direct Level Effect

\[ \frac{\bar{s}_i(x_i; A'; t'')}{\bar{s}_a(A', t'')} - \frac{\bar{s}_i(x_i; A'; t'')}{\bar{s}_a(A', t'')} + \frac{\bar{s}_i(x_i; A'; t'')}{\bar{s}_a(A', t'')} - \frac{\bar{s}_i(x_i; A'; t'')}{\bar{s}_a(A', t'')} \]

Indirect Level Effect

Direct Selection Effect

Indirect Selection Effect

where again, \( A' = A(\beta', S') \) and \( A'' = A(\beta'', S'') \). Remember that the (total) level effect is due to changes in the levels of activity \( i \) undertaken by firms conditional on being active. In (8), this is represented by changes in the share of firms undertaking at least a given level of activity \( i \), \( \bar{s}_i \). The (total) selection effect is due to changes in the range of active firms which is represented by changes in the share of active firms, \( \bar{s}_a \). Each of these two effects has a direct component induced by changes in \( (\beta, S) \) for a given \( A \) and an indirect component induced by changes in \( A \). Together, the direct and indirect level effects correspond to the firm-level responses analysed in Section 4. The additional effect of an increase in \( (\beta, S) \) on \( H_i \) through the direct and indirect selection effects is clearly a source of discrepancy between firm- and industry-level comparative statics.

5.3 Industry-Level Monotone Comparative Statics: the General Case

The present section provides conditions that imply monotone comparative statics \( (\Delta H_i \leq 0) \) in the general case where the demand level may either rise or fall. These conditions are relaxed in the next section by conditioning on use of either the carrot or the stick.

Let us start by considering the direct level effect on \( H_i \). We know from Section 4 that the direct effect of an increase in \( (\beta, S) \) on firms’ decisions is nonnegative. This tends to increase the share of firms that undertake at least a given level of activity \( i \), \( \bar{s}_i \).\(^{32}\) The direct level effect in (8) is therefore nonpositive which work in favor of a FOSD shift in \( H_i \). Since this effect may actually be zero for some levels of activity \( i \), we must ensure that the sum of

\(^{32}\)To see this formally, note that by Lemma 1, \( x^* \) is nondecreasing in \( (\theta, \beta, S) \). Thus it follows from (6) that \( \theta_i \) is nonincreasing in \( (\beta, S) \) given \( (\gamma, A) \). Therefore, \( s_i \) is increasing for all \( \gamma \) which means that \( \bar{s}_i \) increases.
the remaining effects in (8) is also nonpositive. First off, because the demand level could be constant such that both the indirect level and selection effects are zero, this requires that the direct selection effect is itself nonpositive.

For the direct selection effect to be nonpositive, we need that the direct effect on the share of active firms is nonpositive. Intuitively, the marginal active firms have low productivity and therefore undertake low levels of the activities conditional on being active. An increase in the share of active firms would therefore work against a FOSD shift in $H_i$. To ensure that the direct effect on $\bar{s}_a$ is nonpositive for all $G$, we need that the cutoffs $\theta_a$ are nondecreasing for all $\gamma$ given $A$. To ensure that this is the case, we impose the following restriction which applies in the remainder of the paper.

**Restriction 1.** Let the following inequality hold for all $\gamma$,

$$\pi^*(\theta'_a, \gamma; A', \beta'', S'') \leq 0,$$

where $\theta'_a = \theta_a(\gamma; A', \beta', S')$, $\beta' \leq \beta''$, and $S' \leq S''$.

Under Restriction 1, the direct effect on the optimised profits of the ex-ante marginal active firms is nonpositive. This implies that the direct selection effect does not tend to make previously unprofitable firms want to become active.\(^{33}\) Note that under Restriction 1, $(\beta, S)$ can still be increased by both the carrot or the stick.\(^{34}\) Hence, Restriction 1 does not limit the firm-level responses to be either ambiguous or unambiguous. Further, Restriction 1 only implies that the direct effect on $\theta_a$ is nonnegative. The total effect is still ambiguous due to $\theta_a$ being affected by the demand level as well.

Finally, we need to ensure that the sum of the indirect effects on $H_i$ is nonpositive. An increase in $A$ tends to make all firms weakly increase their levels of activity $i$. This makes the indirect level effect nonpositive. At the same time, an increase in $A$ will allow some previously inactive low-productivity firms to produce profitably. The indirect selection effect is therefore nonnegative, c.f. the discussion above. These two effects are reversed when $A$ decreases but are obviously still opposing. Therefore, in order to ensure that

\(^{33}\)Formally, the direct effect on $\theta_a$ is nonnegative, i.e. $\theta_a(\gamma; A', \beta', S') \leq \theta_a(\gamma; A', \beta'', S'')$, since $\pi^*$ is nondecreasing in $\theta$ and (9) holds with equality for $\theta_a(\gamma; A', \beta'', S'')$.

\(^{34}\)One the one hand, Restriction 1 is clearly satisfied when using purely the stick, since the direct effect on profits of all firms is negative. On the other hand, one can design a carrot approach where (9) holds with equality and the direct effect on the profits of all other firms is nonnegative while being strictly positive for some firms.
the sum of these indirect effects on \( H_i \) is nonpositive in both cases, we must have that the indirect level and selection effects are exactly offsetting. In other words, the percentage changes in \( \bar{s}_a \) and \( \bar{s}_i \) induced by the change in \( A \) have to be equal.

To formulate the conditions that ensure this, we utilise that productivity is only identified up to a positive monotone transformation in our setup. Given a "primitive" productivity variable, \( \phi \) with c.d.f. \( F_\Phi \), one can instead work with any positive monotone transformation, \( \theta = V(\phi) \). Without loss of generality we can therefore let the distribution of (transformed) productivities, \( F \), have constant hazard rate. This merely amounts to choosing the transformation, \( V \), such that \( F(\theta) = F_\Phi(V^{-1}(\theta)) \) has constant hazard rate, \( \lambda(\theta) = \lambda \).

**Proposition 2.** W.l.o.g. let \( F \) have constant hazard rate. For all \( G \), increases in \( (\beta, S) \) induce FOSD shifts in the equilibrium distributions of all activities if and only if the optimal decision, \( x^* \), and the choice to exit depend on \( \theta \) and \( A \) only through \( \theta + Z(A, \beta, S) \).

The constant hazard rate implies by construction that the density at any productivity level is constant relative to the probability mass above it. This means that the percentage changes (induced by a change in \( A \)) in the share of active firms and the share of firms undertaking at least a given level of activity \( i \) are equal for a given \( \gamma \) if the changes in the thresholds \( \theta_a \) and \( \theta_i \) are equal. The conditions in Proposition 2 ensure exactly this. Further, they ensure that the effect does not depend on \( \gamma \), such that the conclusion holds when aggregating over \( \gamma \). For details of the proof, see Appendix B.

Applying Proposition 2 in a given context can be done in the following way. First check that firms’ decisions and choices to exit can be expressed such that they depend on the demand level and a transformation, \( \theta \), of primitive productivity only through \( \theta + Z(A, \beta, S) \). A sufficient condition is that profits, \( \pi \), do so.\(^\text{35}\) Next, monotone comparative statics at the industry level demands that this transformation of productivity has constant hazard rate.

As mentioned above, Restriction 1 does not restrict \( A \) to either rise or fall. Therefore, the industry-level monotone comparative statics in Proposition 2 are independent of whether the firm-level response is ambiguous or not. One reason is that the indirect effect through the demand level, which is the source

\(^{35}\)However, it is not necessary; see Section 6.3. Note that Lemma 1 implies that \( Z(A, \beta, S) \) is increasing in \( A \) since \( \theta \) and \( A \) both tend to increase the decision of firms.
of the possible ambiguity at the firm level, is made neutral at the industry level by the conditions in Proposition 2. Propositions 1 and 2 enable us to draw the somewhat surprising conclusion that firm-level complementarities can assert themselves more clearly at the industry level than at the firm level. Further, the FOSD shifts at the industry level could be magnified when including more complementary activities into the model. These points are not only theoretically interesting but also important to take into account when testing models that conform to our assumptions.

5.4 Industry-Level Monotone Comparative Statics: Carrot or Stick

While the effect on the demand level is ambiguous in general, the direction of change can often be predicted under certain increases in \((\beta, S)\). For example, this is the case when the increase in \((\beta, S)\) represents either a pure carrot or a pure stick approach. Once we consider special cases where \(A\) either rises or falls, we no longer need to be exactly on the knife edge where the indirect level and selection effects on \(H_i\) exactly balance. This leads us to the following proposition which is proved in Appendix C.

**Proposition 3.** Let \(\theta_a\) be invariant across \(\gamma\)'s and let the optimal decision, \(x^*\), and the choice to exit depend on \(\theta\) and \(A\) only through \(\theta + Z(A, \beta, S)\). For all \(G\), increases in \((\beta, S)\) by the carrot (stick) induce FOSD shifts in the equilibrium distributions of all activities if the distribution of productivities has nonincreasing (nondecreasing) hazard rate, \(\lambda(\theta)\).

To understand the intuition behind Proposition 3, remember that \(A\) induces the indirect level and selection effects through its effects on \(\theta_i\) and \(\theta_a\), respectively. Relative to the case with constant hazard rate, where the two effects balance, a nonincreasing hazard rate puts relatively more probability density at \(\theta_a\) (since \(\theta_a \leq \theta_i\)) which means that the indirect selection effect dominates the indirect level effect. This works in favour of a FOSD shift in the equilibrium distribution of activity \(i\) when \(A\) falls. Conversely, a non-

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36 For simplicity, let Restriction 1 hold with equality such that \(H_i\) is only affected by the direct level effect. By the insights of Milgrom and Roberts (1996), this direct effect would be larger in the presence of additional complementary activities all else equal.

37 A specific example is a trade liberalisation in the Melitz (2003) model which represents a pure carrot approach; see Section 6.1.
decreasing hazard rate means that the indirect level effect dominates which works in favour of FOSD shifts when $A$ increases.

5.5 A Noteworthy Special Case

In many models of international trade, the decision of firms can be formulated in such a way that profits only depend on the primitive productivity variable, $\phi$, and a demand-level variable, $A$, through their product $A\phi$.\(^{38}\) This arises naturally when consumers have CES preferences, in which case $A$ is a function of total expenditure and a price index, but can also arise in other cases; see Section 6. Since many of these models otherwise conform to our basic setup and feature complementarities as described in Assumption 1, it is interesting to see when the conditions of Propositions 2 and 3 are satisfied.

That profits only depend on $A$ and $\phi$ through $A\phi$ is equivalent to profits only depending on them through $\theta + \log A$ with $\theta = \log \phi$. Further, if profits only depend on the log of primitive productivity, $\theta$, and the demand level through $\theta + \log A$, then so do firms’ decisions and the choice to exit. Finally, the distribution of $\log \phi$ having constant hazard rate is equivalent with $\phi$ being Pareto distributed.\(^{39}\) Corollary 4 follows.

**Corollary 4.** Let profits only depend on primitive productivity, $\phi$, and a demand-level variable, $A$, through their product, $A\phi$. The conditions of Proposition 2 are satisfied if and only if $\log \phi$ has constant hazard rate, i.e., if and only if $\phi$ is Pareto distributed.

While the Pareto distribution is often used in these models of international trade due to other attractive features, our result points out a novel knife-edge property with strong implications for industry-level comparative statics in the presence of complementarities. Further, even though this literature consider many diverse activities, many of these models can be formulated such that labour input is one of these. Therefore, the comparative statics considered, such as trade liberalisations, very often lead to the prediction of FOSD shifts in the size distribution of firms (as measured by labor input).\(^{40}\)

\(^{38}\) This is e.g. the case in the list of nested trade models provided in the introduction. Section 6 provides two examples.

\(^{39}\) $\theta = \log \phi$ being distributed with constant hazard rate, \(\lambda\), implies $F(\theta) = 1 - e^{-\lambda(\theta - \theta_0)}$. Substituting for $\theta$ gives $F_\phi(\phi) = 1 - (\phi_0/\phi)^{-\lambda}$ where $\phi_0 = e^{\theta_0}$ is the lower bound on $\phi$. Thus $F_\phi$ is given by the Pareto.

\(^{40}\) To see how this works in the basic Melitz (2003) model, see Section 6.1.
Next, it is quite easy to come up with productivity distributions that satisfy the weaker distributional requirement in Proposition 3. If one wants monotone comparative statics at the industry level when employing the carrot (stick), one just has to pick a distribution of log-productivities, $F$, with a nonincreasing (nondecreasing) hazard rate. The corresponding distribution for primitive productivities is then obtained as $F_\Phi(\phi) = F(\log \phi)$.

5.6 Comparative Statics without Direct Effects

Some parameter changes have no direct effects on either firms’ decisions or their choice to exit. While this may be true for some elements of $\beta$, it is obviously the case for the sunk entry cost, $f_e$, which we will use to illustrate the implications. First off, the firm-level comparative statics are determined solely by the indirect effect on $\bar{x}^*$. From (4) it is clear that an increase in $f_e$ must increase $A$ and thereby the equilibrium decisions of all firms. At the industry level however, things may be different. Since $f_e$ does not have direct level or selection effects, only the indirect effect matters for the industry level. We can therefore conclude that the equilibrium distributions of the activities are completely unaffected if the conditions in Proposition 2 hold. Further, on the one hand, if $\lambda(\theta)$ is nondecreasing such that the indirect level effect dominates the indirect selection effect, then the equilibrium distributions of activities shift toward higher values when $f_e$ increases. On the other hand, if $\lambda(\theta)$ is nonincreasing such that the indirect selection effect dominates, then all of these distributions shift towards lower values, regardless of the unambiguous increase in the equilibrium decisions of all firms.

6 Applications

The present section shows how our firm- and industry-level results can be applied to some existing models and to a case with more generally-formulated preferences. The first example is the Melitz (2003) model. The second considers how our industry-level results can be applied with general additively separable preferences if one reinterprets $\theta$ as quality. The third example is the case of quadratic preferences and endogenous markups.\footnote{Our results can also be applied to a number of models from the trade literature covering a broad range of topics. Among these are Antràs and Helpman (2004), Helpman}
6.1 Exporting and Labour Input

Consider a two-country Melitz (2003) model. If we let the activities of the firms be export status, given by the indicator $1_{ex}$ for exporting, and total labour input for variable production, $l$, while requiring that the resulting output is optimally distributed across markets in case of exporting, then we obtain the profit function

$$\pi(x, \theta; A, \beta) = (1 + 1_{ex}\tau^{1-\sigma})^{\frac{1}{\sigma}} l^{\sigma-1} e^{\frac{\sigma-1}{\sigma}(\theta+\log A)} - l - f - 1_{ex}f_{ex}, \quad (10)$$

where $A$ is a demand shifter, $\sigma > 1$ is the elasticity of substitution from the CES preferences, $\tau > 1$ is an iceberg trade cost, $f$ is a fixed cost of production, and $f_{ex}$ is a fixed cost of exporting. Here, we define $\theta = \log \phi$, with $\phi$ being the primitive productivity variable of Melitz (2003).

Now, let $x = (l, 1_{ex}), X = \mathbb{R}_+ \times \{0, 1\}$, and $\beta = (\tau^{-1}, -f_{ex})$. Then it is easy to verify that the profit function, (10), is supermodular in $x$ and has increasing differences in $(x, \theta)$, $(x, A)$, and $(x, \beta)$. Hence, our Assumption 1 regarding complementarities is satisfied. Since the model otherwise conform to our setup, we can apply our propositions to analyse the effects of different types of trade liberalisation. Starting with the firm level, we consider an opening to trade and incremental liberalisations of trade through decreases in $\tau$ and $f_{ex}$. Opening to trade implies introducing a new activity (exporting) by moving from $S' = \mathbb{R}_+ \times \{0\}$ to $S'' = \mathbb{R}_+ \times \{0, 1\}$. Both incremental liberalisations involve an increase in the attractiveness of exporting (an increase in $\beta$). All of these increases in $(\beta, S)$ are purely by means of the carrot, i.e., they have a nonnegative direct effect on the profits of any given firm. Therefore we know that upon either trade liberalisation, the demand level must fall. The indirect effects on the equilibrium decisions are thus negative. This implies that the total effect on the amount of labour employed in variable production is negative for some firms and positive for others. For example, when the iceberg trade cost is reduced, all exporters increase their use of labour while (ex post) nonexporters reduce their use of labour. While the positive direct effect dominates the negative indirect effect for exporters,
nonexporters are only affected by the negative indirect effect.\footnote{Referring to footnote 23, this shows that the direction of firm-level responses may vary across firms for a given activity. To see that responses can vary across activities within a given firm, one could split total labour for variable production into that used for the production to the domestic market and the export market. Doing so, the model still conforms to our assumptions and it can easily be verified that upon a decrease in $\tau$, exporters increase their use of labour for production to the export market while decreasing the use for labour for production to the domestic market.}

Moving on to the industry level, we can invoke Proposition 2 since the trade liberalisations described above have no direct effects on the profits of the least productive firms (which do not export) wherefore Restriction 1 holds. Thus, we know that if log-productivities, $\theta$, have a distribution with constant hazard rate, then trade liberalisations give rise to monotone comparative statics at the industry level. As mentioned in Section 5.5, this corresponds to primitive productivity, $\phi$, being Pareto distributed. In this case, either of the three types of trade liberalisation induces industry level shifts towards exporting and more labour in variable production.\footnote{Relating to the discussion in Section 5.3, this conclusion would not be affected by extending the model to include additional complementary activities.} The effects on the equilibrium distribution of labour input are illustrated in Figure 1.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1}
\caption{Equilibrium distribution of labour input before (dotted) and after (solid) three trade liberalisations.}
\end{figure}

The flat parts, which occur at the share of active firms that do not export, are due to the discrete increase in optimal labour input when exporting. These flat parts shift downwards since the share of active firms which do not export decreases. It is also possible to obtain monotone compara-
tive statics at the industry level without the Pareto distribution. Since the demand level is decreasing in all cases, Proposition 3 tells us that whenever log-productivities, $\theta$, are distributed with nonincreasing hazard rate, the equilibrium distributions shift towards exporting and more labour input in variable production.

The fact that improved export opportunities are beneficial to some firms and detrimental to others is a well-known result of the Melitz (2003) model. However, our industry-level results illustrate conditions under which the industry level exhibits monotone comparative statics regardless of firm-level responses being ambiguous. Even though a range of firms may downsize, the equilibrium distribution of labour input unambiguously shifts towards firms becoming larger under certain distributional assumptions.

6.2 Additively Separable Utility and Quality

Quite general preference structures can give rise to a profit function that satisfies the conditions in Propositions 2 and 3. With a slight reinterpretation of $\theta$ as (a transformation of) quality, we show that our industry-level results are more widely applicable than in just the case of CES preferences.

Let consumers have preferences $U = \int \phi_j u(c_j) \, d\bar{j}$ with $j$ indexing varieties of a differentiated good and $c_j$ being consumption of variety $j$. In this case, $\phi_j$ should be interpreted as quality of variety $j$ instead of the productivity of the firm producing it. With identical consumers maximising utility subject to the budget constraint, $\int p_j c_j \, d\bar{j} = I$, where $p_j$ is the price of variety $j$ and $I$ is income, the inverse demand function reads $p_j = A\phi_j u'(c_j)$, with $A$ being the inverse marginal utility of income. In this case, the revenue of a firm reads $r(q_j) = A\phi_j u'(q_j/L)q_j$, where $q_j$ is total output and $L$ is market size (the number of consumers). Now, we could go on to make assumptions on the cost structure and then write up a profit function which depends on both the choice of $q$, a number of additional activities, and some parameters such that our assumptions of complementarities are met. However, we confine the discussion to noting that with this revenue function, the profit function is well on its way to depend on $A$ and $\phi$ only through their product, $A\phi$. Drawing upon Section 5.5, one can then easily identify distributions of quality that can result in monotone comparative statics at the industry level in case the

\footnote{Note that in this case, $\phi_j$ represents quality in the sense that it scales up marginal utility, $u'(c_j)$, or marginal willingness to pay, for a given level of consumption.}
profit function exhibits complementarities.

6.3 Variable Markups

This example is based on the Melitz and Ottaviano (2008) model but abstracts from trade. Consumers have quadratic preferences leading to the linear demand function, \( q = L(p_{\text{max}} - p) \), where \( q \) is quantity, \( p \) is the price, \( p_{\text{max}} \) is the choke price, and \( L \) is market size. Firms have constant marginal costs given by the inverse of primitive productivity, \( \phi^{-1} \), and no fixed costs. In this closed-economy version of the model, the profit function can be written as

\[
\pi(x, \phi; A, \beta) = L\phi^{-2}(A\phi - x)(x - 1),
\]

where \( x = p/\phi^{-1} \) is the relative markup of the firm and \( A = p_{\text{max}} \). Let \( X = [1, \infty) \).\(^{48}\) Note that \( \phi^{-1} > A \) implies that a firm exits.

The profit function is nondecreasing in \( A \) and has increasing differences in \((x, A)\). Further, optimised profits are monotone increasing in \( \phi \) even though profits are not.\(^{49}\) While (11) does not have increasing differences in \((x, L)\), (11) does exhibit the single crossing property. Thus, we can still apply our firm-level results with \( \beta = L \).\(^{50}\) Note that \( L \) does not affect the optimal decision, \( x^* \), directly. Therefore, an increase in market size only has an indirect effect. As all firms can earn weakly higher profits the higher is \( L \) given \( A \), an increase in \( L \) has a negative indirect effect on the equilibrium decision. In total, a larger market makes all firms strictly reduce their markups.

What about the industry level? First, note that (11) exhibits the single crossing property in \((x, \phi)\) and that this is actually sufficient for Lemma 1.\(^{51}\) We can therefore proceed with our analysis. The model conforms to our setup in all other respects as well. Further, Restriction 1 is satisfied for an increase in \( L \) since this increase does not directly affect the optimal profits of the least productive firms. Finally, since \( A \) and \( \phi \) only matter for a firm’s decision on markups and exit through \( A\phi \) and Melitz and Ottaviano (2008) assume productivities, \( \phi \), to be distributed Pareto, the conditions of

\(^{48}\)No firm would want to choose a price lower than its marginal cost wherefore \( x \geq 1 \) is an uncontroversial restriction.

\(^{49}\)Solving for \( x^* = \frac{1}{4}(A\phi + 1) \) and plugging back into (11) yields \( \pi^* = \frac{L}{4}(A - \phi^{-1})^2 \).

\(^{50}\)As (11) is trivially supermodular in \( x \), increasing differences in \((x, \beta)\) is a sufficient condition for the partial comparative statics in Lemma 1. The single crossing property is both necessary and sufficient.

\(^{51}\)See Appendix A for details.
Corollary 4 are satisfied. We can therefore conclude that an increase in $L$ implies that the equilibrium distribution of markups, shifts towards higher values. However, this only holds in the weakest possible sense. Let $\theta = \log \phi$. As the increase in $L$ has zero direct effects on $\theta_0$ and $\theta_1$, the equilibrium distribution of markups is constant when primitive productivities, $\phi$, are distributed Pareto. Even though the model features pro-competitive effects of a larger market size, in that all firms strictly reduce their markups, the equilibrium distribution of markups is constant in $L$ due to selection effects (low-productivity firms with low markups exit).

Since $L$ only affects the equilibrium distribution of markups through the indirect effects and $A$ is decreasing, the equilibrium distribution of markups shifts towards lower values—in line with the intuition of the pro-competitive effect at the firm level—if log-productivities are distributed with increasing hazard rate. However, if the distribution of log-productivities has a decreasing hazard rate, then the equilibrium distribution of markups shifts towards higher values following an increase in $L$. In this case, sufficiently many low-productivity firms are driven out of business to ensure that the consumers are going to face higher markups on average when the market size has increased. The pro-competitive effects at the firm level therefore in no way ensure that the equilibrium distribution of markups shifts toward lower values.$^{52}$

### 7 Concluding Remarks

If one considers a given firm in isolation, then complementarities like the ones we analyse imply monotone comparative statics in the parameters of the firm’s profit maximisation problem. This has been known at least since Milgrom and Roberts (1990a). However, when placing firms in a context of monopolistic competition, this may no longer be the case due to indirect effects through changes in the demand level. New or better opportunities that become available to all firms may make a given firm scale down existing activities even when these existing activities are complementary to the activities affected. Thus, for some firms, such opportunities may turn out to be a threat detrimental to many dimensions of the firms’ operations.

$^{52}$This relates to a remark made by Arkolakis et al. (2012). However, as our criterion is first-order stochastic dominance instead of monotone likelihood ratio dominance, our condition of a decreasing/nondecreasing hazard rate of the distribution of log-productivities is different from theirs (log-concavity/convexity of this distribution).
One main finding is that firm-level complementarities may manifest themselves much more clearly at the industry level than in the behaviour of individual firms. Despite the ambiguities at the firm level, we show that one may observe that the equilibrium distributions of the activities unambiguously shift towards higher values. Key to this result is the observation that these distributions depend not only on the activity levels undertaken by firms conditional on being active but also on the selection of active firms.

We believe that these results are worthwhile for several reasons. First, they provide general insights on the implications of firms-level complementarities in models of monopolistic competition; a workhorse market structure in many strands of the economics literature. Our industry-level results may prove particularly useful for the new macro literature featuring heterogeneous firms; see e.g. Ghironi and Melitz (2005). Second, our results provide strong and testable predictions—especially at the industry level—for a large number of recent trade models. We believe that it will be both useful and interesting to confront these predictions with industry-level data. On the one hand, such empirical investigations can shed light on the appropriateness of commonly-used functional form assumptions. On the other hand, this approach is likely to complement firm-level and structural estimations. We leave this task for future research. Third, we provide a flexible tool for modelling explanations of shifts at the industry level based on complementarities at the firm level. Our analysis clearly shows how incentives to undertake one activity at the firm level may induce unambiguous shifts in the equilibrium distributions of other activities at the industry level. Fourth and finally, we have illustrated another context in which monotonicity theorems are a powerful mathematical tool for conducting economic analysis.

A Quasisupermodularity and Single Crossing

Let $X$ be a lattice and $T$ be a partially ordered set. The real-valued function $h(x, t)$ is quasisupermodular in $x$ on $X$ if for all $x', x'' \in X$, $h(x', t) \geq h(x' \wedge x'', t)$ implies $h(x' \vee x'', t) \geq h(x'', t)$ and $h(x', t) > h(x' \wedge x'', t)$ implies $h(x' \vee x'', t) > h(x'', t)$. Hence, if an increase in a subset of the elements of $x$ raises $h$ at a given level of the remaining elements, exactly the same increase in the same subset of the elements of $x$ will increase $h$ when the remaining elements also increase. In the language of Milgrom and Shannon (1994), quasisupermodularity expresses a weak kind of complementarity between the
and their choices to exit only depend on the same, then the condition (12) surely holds. Further, when firms’ decisions $\theta$ and $\lambda$ which both integrate to one. The first step is to note that if (12) is to hold for any choice of $\theta$, then the constant hazard rate means that (12) is satisfied. Hence, if an increase in $x$ raises $h$ when $t$ is low, exactly the same increase in $x$ will raise $h$ when $t$ is high. One can verify by the relevant definitions that any supermodular function is also quasisupermodular and any function with increasing differences in $(x, t)$ also satisfies the single crossing property in $(x, t)$. Let $S \subseteq X$. The following monotonicity theorem is due to Milgrom and Shannon (1994).

**Theorem 2.** $\arg\max_{x \in S} h(x, t)$ is monotone nondecreasing in $(t, S)$ if and only if $h$ is quasisupermodular in $x$ on $X$ for each $t \in T$ and satisfies the single crossing property in $(x, t)$ on $X \times T$.

## B Proof of Proposition 2

To prove that the conditions in Proposition 2 are sufficient, first note that a zero (total) indirect effect in (8) is equivalent with

$$
\int \omega_a e^{-\int_{\lambda(x, \beta', S')} \lambda(\theta') d\theta'} dG(\gamma) = \int \omega_i e^{-\int_{\lambda(x, \beta', S')} \lambda(\theta') d\theta'} dG(\gamma), \quad (12)
$$

where we have used the definitions of $\bar{s}_a$ and $\bar{s}_i$. Further, we have defined the weights $\omega_a = s_a(\gamma; A', \beta', S') / \bar{s}_a(A', \beta', S')$ and $\omega_i = s_i(x_i; \gamma; A', \beta', S') / \bar{s}_a(x_i; A', \beta', S')$ which both integrate to one.

If $\lambda(\theta) = \lambda$ and the changes in $\theta_a$ and $\theta_i$ induced by the change in $A$ are the same, then the condition (12) surely holds. Further, when firms’ decisions and their choices to exit only depend on $A$ and $\theta$ through $\theta + Z(A, \beta, S)$, then $\theta_a + Z(A, \beta, S)$ and $\theta_i + Z(A, \beta, S)$ must be constant in $A$. This means that

$$
\theta_a(\gamma; A'', \beta, S) - \theta_a(\gamma; A', \beta, S) = Z(A', \beta, S) - Z(A'', \beta, S) \quad (13)
$$

and

$$
\theta_i(x_i; \gamma; A'', \beta, S) - \theta_i(x_i; \gamma; A', \beta, S) = Z(A', \beta, S) - Z(A'', \beta, S), \quad (14)
$$

which together with the constant hazard rate means that (12) is satisfied.

Next, we must prove that the conditions in Proposition 2 are also necessary. The first step is to note that if (12) is to hold for any choice of $G$, then
it must hold if $G$ is degenerate at any $\gamma$. With a constant hazard rate $\lambda$, this gives us

$$
\theta_a(\gamma; A'', \beta, S) - \theta_a(\gamma; A', \beta, S) = \theta_i(x_i, \gamma; A'', \beta, S) - \theta_i(x_i, \gamma; A', \beta, S). \quad (15)
$$

While we must have that the changes in $\theta_a$ and $\theta_i$ induced by a change in $A$ are equal, they cannot depend on $\gamma$. To see this, note that, in the general case with a nondegenerate $G$, the weights $\omega_a$ and $\omega_i$ in (12) are not equal and we cannot say how they depend on $\gamma$ (since we have not made any assumptions on how $\gamma$ affects profits). Thus, if (15) depends on $\gamma$, there exists some weights that imply that (12) does not hold. This means that both the right and left hand side of (15) can only depend on $A''$, $A'$, $\beta$, and $S$. This is equivalent with the firms’ decisions and their choice to exit depending on $\theta$ and $A$ only through $\theta + Z(A, \beta, S)$ for some function $Z$ which has to be nondecreasing in $A$ under Assumption 1.

### C Proof of Proposition 3

To prove this proposition, note that the (total) indirect effect on $H_i$ is non-positive when

$$
\int \omega_a e^{-\int_{\theta_a(\gamma; A''; \beta''; S'')}^{\theta_a(\gamma; A'; \beta''; S'')} \lambda(\theta) d\theta} dG(\gamma) \leq \int \omega_i e^{-\int_{\theta_i(x_i, \gamma; A''; \beta''; S'')}^{\theta_i(x_i, \gamma; A'; \beta''; S'')} \lambda(\theta) d\theta} dG(\gamma).
$$

When $\theta_a$ is independent of $\gamma$, this reduces to

$$
e^{-\int_{\theta_a(A''; \beta''; S'')}^{\theta_a(A'; \beta''; S'')} \lambda(\theta) d\theta} \leq \int \omega_i e^{-\int_{\theta_i(x_i, \gamma; A''; \beta''; S'')}^{\theta_i(x_i, \gamma; A'; \beta''; S'')} \lambda(\theta) d\theta} dG(\gamma). \quad (16)
$$

Because (13) and (14) hold, the length of the interval over which $\lambda(\theta)$ is integrated is the same on both sides of the inequality in (16). Remember that $\theta_i(x_i, \gamma; A', \beta'', S'') \geq \theta_a(A', \beta'', S'')$. Since $\theta_a(A'', \beta, S) \geq \theta_a(A', \beta, S)$ and $\theta_i(x_i, \gamma; A'', \beta, S) \geq \theta_i(x_i, \gamma; A', \beta, S)$ hold when $A$ falls, (16) holds for all $A' \geq A''$ when $\lambda(\theta)$ is nonincreasing. Conversely, $\theta_a(A'', \beta, S) \leq \theta_a(A', \beta, S)$ and $\theta_i(x_i, \gamma; A'', \beta, S) \leq \theta_i(x_i, \gamma; A', \beta, S)$ hold when $A$ rises. Hence, (16) holds for all $A' \leq A''$ when $\lambda(\theta)$ is nondecreasing. Since using the carrot (stick) corresponds to the former (latter) case, the proposition follows.
References


