Moderate Expected Utility

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Abstract

Individual choice data often violates strong stochastic transitivity (SST) while conforming to moderate stochastic transitivity (MST). We propose a slightly stronger version of the MST postulate, which we call MST+, and we show that MST and MST+ retain significantly more predictive power than weak stochastic transitivity (WST). Our first theorem shows that a binary choice rule satisfies MST+ if and only if it can be represented by a *moderate utility model*: a utility function describes the value of each option, and a distance metric determines their degree of comparability. Our second theorem introduces the *moderate expected utility model* and shows how our parameters can be identified from choice data over lotteries.

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1 Introduction

Consider a decision maker who is most likely to choose option x in a binary comparison against y, and, in turn, most likely to choose option y in a binary comparison against z. Denoting by $\rho(x, y)$ the probability of choosing x over y and by $\rho(y, z)$ the probability of choosing y over z, we have

$$\rho(x, y) \geq 1/2 \text{ and } \rho(y, z) \geq 1/2. \tag{1}$$

A simple test of the transitivity of the decision maker's choices may require the decision maker to choose x most often in a binary comparison against z,

$$\text{If (1) holds, then } \rho(x, z) \geq 1/2. \tag{WST}$$

This basic postulate is known as *weak stochastic transitivity.* WST is the most permissive condition under which an analyst may obtain a coherent ranking over the choice options from binary choice data.

A more stringent transitivity criterion which is well-studied in the literature is *strong stochastic transitivity*:

$$\text{If (1) holds, then } \rho(x, z) \geq \max\left\{\rho(x, y), \rho(y, z)\right\}. \tag{SST}$$

Choice models that satisfy SST (such as the classic Logit model) are typically simple to analyze but fail to accommodate many empirically relevant phenomena.

In this paper, we consider a less studied, intermediate condition called *moderate stochastic transitivity*:

$$\text{If (1) holds, then } \rho(x, z) \geq \min\left\{\rho(x, y), \rho(y, z)\right\}. \tag{MST}$$

In Section 2, we show that MST allows for many empirically relevant choice patterns ruled out by SST, and yet has significantly more empirical bite than WST.

Our main contribution is to characterize a family of parametric models of individual choice that generate the entire range of observable choice behavior that satisfies MST. This family can prove useful in applications where SST is violated, while at the same time offering more predictive power than WST models.

Our main results are two representation theorems for choice behavior that exhibits a moderate degree of transitivity. First, we introduce a slight strengthening of MST,

$$\text{If (1), then } \begin{cases} \rho(x,z) > \min\{\rho(x,y), \rho(y,z)\} \\ \text{or} \\ \rho(x,z) = \rho(x,y) = \rho(y,z) \end{cases} \tag{MST+}$$

which we call *moderate stochastic transitivity plus*, or MST+.

Theorem 1 shows that binary choice behavior over a finite set of alternatives satisfies MST+ if and only if it is a *moderate utility model* (MUM). A binary choice rule ρ is a MUM if there exists a utility function u and a distance metric d such that

$$\rho(x,y) = F\left(\frac{u(x) - u(y)}{d(x,y)}\right) \tag{MUM}$$

where F is a strictly increasing transformation with $F(t) = 1 - F(-t)$ for all $t \in \mathbb{R}$. In a MUM, the decision maker's ability to discriminate among a pair of options x and y depends both on the difference in value $u(x) - u(y)$ and on the difficulty of comparing the options given by the distance $d(x,y)$.

Note the role of the distance metric d: for a given difference in value $u(x) - u(y)$, larger values of the distance $d(x,y)$ drive choice probabilities closer to 1/2. In other words, a larger distance means options are more difficult to compare. The abstract metric d does not have to be the standard metric of Euclidean space: in applications, d takes the form of a statistical distance between random variables, angular distance between vectors in multi-attribute settings, and so on. In Section 5, we show that specific functional forms of u and d yield several familiar models from the discrete choice estimation literature as particular instances of MUM.

Theorem 2 enriches the domain of choice options to include lotteries over the alternatives and obtains the identification of the MUM parameters. By imposing the assumptions of continuity, linearity, and convexity, in addition to MST+, our *moderate expected utility model* (MEM) characterization identifies (i) a unique von Neumann-Morgenstern expected utility function over lotteries; (ii) a norm, induced by an inner product on the relevant linear space, that is unique up to two scaling factors; and (iii) a monotonic transformation F that is unique up to the same two scaling factors.

Section 5 relates our MUM and MEM representations to the existing literature. We show that some familiar models used to address failures of SST are particular instances of MUM. A natural example of a MUM used in the discrete choice estimation literature is the classic binary probit model with correlated variables, which is also an example of a random utility model (RUM). Despite having a non-empty intersection, we show that MUMs neither nest nor are nested in the set of binary RUMs. We conclude with a brief discussion of the possible extensions of our model to choice over more than two options.

2 Moderate stochastic transitivity

Let Z be a finite set of choice options. A (binary, stochastic) *choice rule* on Z is a function $\rho : Z^2 \to [0,1]$ such that $\rho(x,y) + \rho(y,x) = 1$ for every $x, y \in Z$. The number $\rho(x,y)$ denotes the probability that the decision maker selects option x in a binary comparison against y.

Let $\wedge$ and $\vee$ denote the min and max operators, respectively, so that $a \wedge b = \min\{a, b\}$

and $a \vee b = \max\{a, b\}$. The two most commonly studied notions of transitivity for binary choice data are weak stochastic transitivity (WST) and strong stochastic transitivity (SST):

(WST) $\rho(x, y) \wedge \rho(y, z) \geq 1/2 \Longrightarrow \rho(x, z) \geq 1/2$

(SST) $\rho(x, y) \wedge \rho(y, z) \geq 1/2 \Longrightarrow \rho(x, z) \geq \rho(x, y) \vee \rho(y, z)$

In this paper we focus on a less studied, intermediate form of transitivity called moderate stochastic transitivity (MST):

(MST) $\rho(x, y) \wedge \rho(y, z) \geq 1/2 \Longrightarrow \rho(x, z) \geq \rho(x, y) \wedge \rho(y, z)$

The definitions clearly imply that SST $\Rightarrow$ MST $\Rightarrow$ WST. Our main results characterize the set of choice rules that satisfy a slightly stronger version of MST, namely

(MST+) $\rho(x, y) \wedge \rho(y, z) \geq 1/2 \Longrightarrow \rho(x, z) > \rho(x, y) \wedge \rho(y, z)$ or $\rho(x, z) = \rho(x, y) = \rho(y, z)$.

The strengthening is necessary to obtain our representation results. Note, however, that the only difference between MST and MST+ is that the knife-edge case

$$\rho(x, y) \vee \rho(y, z) > \rho(x, z) = \rho(x, y) \wedge \rho(y, z)$$

is allowed by MST but ruled out by MST+. Hence, MST and MST+ are empirically indistinguishable, that is, no finite amount of data allows an analyst to tell them apart.

Choice models that satisfy MST+ are convenient for two reasons. First, the MST+ condition holds in many applications in which the more restrictive SST condition is violated. Hence, a choice model that satisfies MST+ but allows for violations of SST may provide the flexibility that is needed to accommodate empirically relevant choice phenomena. Second, MST+ is significantly more restrictive than WST, and restricting the analysis to models that conform to MST+ results in greater out-of-sample predictive power.

Reviewing some of the evidence, Mellers, Chang, Birnbaum and Ordonez (1992, p. 348) note that "weak and moderate stochastic transitivity are often satisfied, although a few exceptions have been noted", while "[s]trong stochastic transitivity is frequently violated". We provide three examples to illustrate how the flexibility provided by MST+ is useful compared to the more stringent SST. The classic Example 1 suggests that violations of SST must be expected when some pairs of alternatives are easier to compare than others. Examples 2 and 3 illustrate how ease of comparison may drive violations of SST in individual choice experiments with human and non-human subjects alike.

Example 1 (attributed to L. J. Savage, adapted from Tversky (1972)). *An individual has a difficult time comparing a trip to Paris, denoted P and a trip to Rome, denoted R, so that she is equally likely to pick either option $\rho(P, R) = 1/2$. The individual still has trouble deciding if the trip to Paris is enhanced by a €5 bonus, denoted by P^+. In other words, $\rho(P^+, R)$ is still approximately $1/2$. But when pressed to decide between the two Paris trip options, the individual clearly prefers the bonus, so that $\rho(P^+, P)$ is close to 1. SST requires that $\rho(P^+, R) \geq \rho(P^+, P)$ which is intuitively violated in this case, while MST+ only requires the more plausible inequality $\rho(P^+, R) > \rho(P, R)$.*

The lesson from Savage's Example 1 is that utility values cannot be the only factor determining the difficulty of comparing two options. A small monetary bonus makes the choice comparison very easy among the two Paris trips. The same monetary bonus has negligible impact, however, on the difficulty of comparing a trip to Paris and a trip to Rome. Hence, all models in which choice probabilities depend solely on utility (such as the classic Logit model) fail to capture this intuitively plausible behavior.

Example 2 (Choice data over lotteries). *Soltani, De Martino and Camerer (2012) recorded thousands of choices by 21 male Caltech undergraduates using simple lotteries (p, m) that pay m dollars with probability p in the lab. A high risk lottery h and a low risk lottery ℓ*

were fine-tuned to each individual to be approximately indifferent, (i.e., equally likely to be chosen in a binary comparison). Slightly perturbed versions of h *and* ℓ *were then offered for comparison against several types of 'decoy' lotteries. Figure 1 depicts the relative location of two decoy lotteries* 1 *and* 2 *with respect to* h *and* ℓ*. Decoy lottery* 1 *dominates* ℓ *and was chosen 95% of the time against* ℓ *but only 78% of the time against* h*. Thus, choice frequencies violate SST in the direction* $1 \to \ell \to h$*. Decoy lottery* 2*, on the other hand, is dominated by* ℓ *and was chosen 4% of the time against* ℓ *and 33% of the time against* h*. Hence, choice frequencies also violate SST in the direction* $h \to \ell \to 2$*. It is easy to verify that MST+ holds in both cases.*

Example 3 (Animal studies)**.** *Lea and Ryan (2015) recorded hundreds of mating decisions by female túngara frogs. Female túngara frogs choose mates based on the sound of their call. Figure 2 shows how the calls of the three male options* A*,* B *and* C *were differentiated along two attributes. In the binary choice data, option* B *is chosen in 63% of the trials against* A*; option* A *is chosen in 84% of the trials against* C*; and option* B *is chosen in 69% of the trials against* C*. Choices therefore satisfy MST+ but violate SST.*

Relaxing SST to MST+ allows the analyst to address the range of empirical phenomena illustrated by the examples above. At the same time, MST+ retains significantly more empirical bite than WST. To see this, suppose the choice rule ρ on Z satisfies WST. Enumerate the n options in $Z = \{x^1, x^2, \ldots, x^n\}$ in such a way that $\rho(x^i, x^j) \geq 1/2$ whenever $i \leq j$. For the sake of simplicity, let us assume that choice probabilities differ whenever possible, so that the set $\{\rho(x, y) \in [0, 1] : x \neq y\}$ has maximum cardinality with $n(n-1)$ elements.

When $Z = \{x^1, x^2, x^3\}$ has three alternatives, WST allows ρ to have six strict orderings:

$$\text{WST}\left\{\begin{array}{l} \left.\begin{array}{l} \rho(x^1,x^3) > \rho(x^1,x^2) > \rho(x^2,x^3) \\ \rho(x^1,x^3) > \rho(x^2,x^3) > \rho(x^1,x^3) \\ \rho(x^1,x^2) > \rho(x^1,x^3) > \rho(x^2,x^3) \\ \rho(x^2,x^3) > \rho(x^1,x^3) > \rho(x^1,x^2) \end{array}\right\}\text{MST} \\ \rho(x^1,x^2) > \rho(x^2,x^3) > \rho(x^1,x^3) \\ \rho(x^2,x^3) > \rho(x^1,x^2) > \rho(x^1,x^3) \end{array}\right.$$

MST+, which is equivalent to MST in this case, rules out the last two of the six strict orderings, where $\rho(x^1, x^3) < \rho(x^2, x^3) \wedge \rho(x^1, x^2)$. Let $\#WST(n) = [n(n-1)/2]!$ denote the number of strict orderings allowed by WST when Z has n options, and likewise, let $\#MST(n)$ denote the number of strict orderings allowed by MST+. The ratio $\#MST(n)/\#WST(n)$ can be interpreted as a measure of the restriction imposed on observable choice data by MST+ compared to WST. In the case $n = 3$ we just showed the ratio $\#MST(3)/\#WST(3)$ is equal to 2/3. This ratio decreases to less than 1/4 when $n = 4$ and less than 1/17 when $n = 5$. In fact, the ratio is arbitrarily small when n is large:

Proposition 1. $\lim_{n\to\infty} \#MST(n)/\#WST(n) = 0$.

We prove Proposition 1 in the Appendix. There, we also show that the ratio between $\#SST(n)$ and $\#MST(n)$ goes to zero when n is large. To summarize, a model that 'spans' the entire range of choice behavior allowed by the MST+ postulate can be useful in two ways: on the one hand, it allows the analyst the flexibility that is needed to deal with empirical violations of SST. On the other hand, it imposes significant restrictions out of sample —allowing the analyst to make sharper predictions— than the more lenient WST postulate. We introduce such a model in the next section.

3 Moderate utility model

A choice rule ρ on a finite set Z is a *moderate utility model (MUM)* if there is a utility function $u : Z \to \mathbb{R}$, a distance metric $d : Z^2 \to \mathbb{R}_+$ and a strictly increasing function F, such that for all $x \neq y$,

$$\rho(x, y) = F\left(\frac{u(x) - u(y)}{d(x, y)}\right) \tag{2}$$

where F satisfies $F(t) = 1 - F(1 - t)$ for all t. The utility u represents the value of each option. It is easy to see that $\rho(x, y) \geq 1/2$ if and only if $u(x) \geq u(y)$ for any $x, y \in Z$. The ratio $[u(x) - u(y)]/d(x, y)$ can be interpreted as the strength of preference for option x over option y, while the function F maps strength of preference to choice probabilities.

Taking the distance d in (2) to be the special case of the discrete metric $d(x, y) = 1$ if $x \neq y$ and $d(x, x) = 0$ for all x, we obtain the classic *Fechnerian model*

$$\rho(x, y) = F\left(u(w) - u(x)\right) \tag{3}$$

in which the ability to discriminate among x and y depends solely on the difference between the values of x and y (Fechner, 1859; Debreu, 1958; Davidson and Marschak, 1959; Fudenberg et al., 2015). The role of a non-trivial distance metric d in a MUM is to make the choice probabilities of options that are more difficult to compare closer to 1/2.

Example 4. *A concrete example of a MUM used in the discrete choice estimation literature is the* binary probit model*, first proposed by Thurstone (1927). In a probit model there is a Gaussian vector $X = (X_1, \ldots, X_n)$, each coordinate X_i corresponding to an option $x^i \in Z$, such that $\rho(x^i, x^j) = \mathbb{P}\{X_i > X_j\}$ for all $x^i, x^j \in Z$. Note that*

$$\rho(x^i, x^j) = \mathbb{P}\left\{\frac{X_i - X_j - \mathbb{E}[X_i - X_j]}{\sqrt{Var(X_i - X_j)}} > \frac{\mathbb{E}[X_i - X_j]}{\sqrt{Var(X_i - X_j)}}\right\} = \Phi\left(\frac{\mathbb{E}[X_i - X_j]}{\sqrt{Var(X_i - X_j)}}\right)$$

which is a special case of (2) when $u(i) = \mathbb{E}[X_i]$ *is the utility function,* $d(i,j) = \sqrt{Var(X_i - X_j)}$ *is the distance metric (once we rule out perfectly correlated variables), and* $F = \Phi$ *is cdf of the standard Gaussian distribution.*

A non-trivial distance metric d gives MUMs the flexibility that is needed to accommodate empirical violation of SST. For example, consider how the MUM model accommodates the choices over trips in Example 1. Let the trip to Paris and the trip to Rome have utilities $u(P) = U(R) = 0$, respectively, while the trip to Paris with the €5 bonus has utility $u(P^+) = 1$. Let the distance metric be given by $d(P, P^+) = \varepsilon > 0$ and $d(P, R) = d(P^+, R) = 1/\varepsilon > 0$. Finally, let $F = \Phi$ be the standard Gaussian cdf. Applying (2) we have $\rho(P, R) = 1/2$, $\rho(P^+, P) = \Phi(1/\varepsilon)$ and $\rho(P^+, R) = \Phi(\varepsilon)$. Taking $\varepsilon > 0$ small, we obtain $\rho(P^+, P)$ close to one and $\rho(P^+, R)$ close to 1/2 as desired. Here, the two Paris trip options are close according to the metric d, which makes them easy to compare. The trip to Rome option is far from the other options according to d and therefore difficult to compare.

A MUM can also address Examples 2 and 3 by explicitly relating the abstract utility u and distance d to the attribute space. It is important to note that d can differ from the standard Euclidean distance. For example, Hausman and Wise (1978) relate the difficulty of comparing two choice options to the angular distance between the vectors of their observable attributes. For instance, options 1 and ℓ in Figure 1 form a small angle with respect to the origin, so that $d(1, \ell)$ is small, while options 1 and h form a wider angle with respect to the origin, so that $d(1, h)$ is large. Hence, a decision maker may ascribe to h and ℓ the same utility values, and yet have an easier time comparing 1 to ℓ than to h. Similarly, options A and C are much closer in angular distance than options B and C in Figure 2. Options A and B may be close in value, but frogs find option C much easier to compare to A than to B. Hence, a MUM can address both situations in which the ease of comparison involves dominance (Figure 1) and non-dominance (Figure 2) in the attribute space.

Halff (1976) proposed the MUM definition (2) and showed that all MUMs satisfy MST.

In our first characterization theorem, below, we show that MUMs also satisfy the stronger MST+ condition. In fact, we show that MST+ is both necessary and sufficient for a choice rule to be a MUM.

Theorem 1. *A choice rule ρ on a finite Z is a MUM if and only if it satisfies MST+.*

We prove Theorem 1 in the Appendix. Necessity is shown in two steps: first, we show every MUM satisfies MST (this is the step already proved by Halff (1976)). Then, we show that a MUM must also satisfy MST+. For sufficiency, we explicitly construct the utility u and distance d; we show that d satisfies the properties of a metric (the key property being the triangle inequality); and we show that an ordinal representation with u and d holds:

$$\rho(w,x) \geq \rho(y,z) \text{ if and only if } \frac{u(w)-u(x)}{d(w,x)} \geq \frac{u(y)-u(z)}{d(y,z)} \tag{4}$$

Then, it is straightforward to find a transformation F such that the cardinal representation of equation (2) holds.

To obtain the identification of the MUM parameters, in the next section we enrich the choice domain to include all lotteries over the finite set Z. With a finite set of options, however, an analyst who observes ρ can still obtain ordinal information about u and d. Following Natenzon (2019), we say that a pair of options $\{w,x\}$ is *easier to compare* than $\{y,z\}$ when $|\rho(w,x)-1/2| > |\rho(y,z)-1/2|$. A pair is easier to compare when the choice probabilities are more extreme (closer to zero or one). When $\{x,y\}$ is easier to compare than $\{x,z\}$ we also say that x is easier to compare to y than to z. An option x is *revealed preferred* to y, denoted $x \succ y$, when $\rho(x,y) > 1/2$. We also write $x \succcurlyeq y$ when $\rho(x,y) \geq 1/2$. Finally, an option x is *revealed more similar* to y than to z when x is easier to compare to y than to z and the revealed preferred relation is either $x \succ y \succcurlyeq z$ or $z \succcurlyeq y \succ x$ and transitive.

Proposition 2. *Let ρ be a MUM with parameters (u,d,F). Then:*

(i) *An option x is revealed preferred to y if and only if $u(x) > u(y)$;*

(ii) *An option x is revealed more similar to y than to z only if $d(x,y) < d(x,z)$.*

Proof. Let ρ be a MUM as in (2). Then $\rho(x,y) > 1/2$ if and only if $[u(x)-u(y)]/d(x,y) > 0$ if and only if $u(x) > u(y)$ proving (i). Suppose $x \succ y \succcurlyeq z$, and that x is easier to compare to y than to z, that is, $\rho(x,y) > \rho(y,z) \geq 1/2$. Then (2) implies $[u(x)-u(y)]/d(x,y) > [u(x)-u(z)]/d(x,z)$ and (i) implies $u(x)-u(y) \leq u(x)-u(z)$, hence we must have $d(x,z) > d(x,y)$. The same reasoning applies when $z \succcurlyeq y \succ x$. □

To illustrate items (i) and (ii) in Proposition 2, consider again the choice data from Example 3. Option B is revealed preferred to A and C, and yet option A is revealed easier to compare to C than to B. Thus, option A is also revealed more similar to C than to B. By (i) and (ii) the analyst concludes that any MUM that generates this data must satisfy the inequalities $u(B) > u(A) > u(C)$ and $d(A,C) < d(B,C)$. Likewise, revealed preference and revealed similarity analysis imply that every MUM that generates the data in Example 2 must satisfy $u(1) > u(\ell) = u(h) > u(2)$, $d(2,\ell) < d(2,h)$ and $d(1,\ell) < d(1,h)$.

It is worth noting that the inequalities obtained for the abstract distance metric d from the revealed similarity analysis above agree with the inequalities implied by the standard Euclidean distance in the space of observable attributes in both Figure 1 and Figure 2. In empirical applications, however, non-Euclidean distance functions (such as angle distance) may provide a better fit than the standard Euclidean distance when taking a MUM model to the data (see, for example, Hausman and Wise, 1978).

4 Moderate expected utility

We continue to let Z be a finite set of objects and we extend the domain of choice alternatives to the set of all lotteries over Z, denoted by Δ. We identify Δ with the $n-1$ dimensional simplex $\{x \in [0,1]^n : x_1 + \cdots + x_n = 1\}$. The function $U : \Delta \to [0,1]$ is an *expected utility function* if it is linear and onto. A choice rule $\rho : \Delta^2 \to [0,1]$ is a *moderate expected utility model* (MEM) if there exist an expected utility function U, a norm $\|\cdot\|$ induced by an inner product, and a strictly increasing transformation F, such that, for any lotteries $x \neq y$ in Δ,

$$\rho(x,y) = F\left(\frac{U(x) - U(y)}{\|w - x\|}\right). \tag{5}$$

Example 5. *For a concrete example of a MEM, extend the binary probit model of Example 4 to the set of lotteries over the finite set Z by letting*

$$\rho(x,y) = \mathbb{P}\{X'x > X'y\} = \Phi\left(\frac{u'x - u'y}{\sqrt{(x-y)'\Lambda'\Lambda(x-y)}}\right)$$

where $u = \mathbb{E}[X]$ is the mean and $\Lambda'\Lambda = Var(X)$ is the covariance matrix of the Gaussian vector X. This decision maker is a (random) expected utility maximizer, and her Bernoulli index is given by the random vector X. This model is a special case of (5), where $U(x) = u'x$ is the linear transformation given by the mean vector u, $F = \Phi$ is the cdf of the standard Gaussian distribution, and the norm $\|\cdot\|$ is induced by the inner product $\langle x, y\rangle = (\Lambda x)'(\Lambda y)$ where the linear transformation Λ is the square root of the covariance matrix $\Lambda'\Lambda$.

Every MEM satisfies MST+. This can be shown by repeating the argument for a MUM in the proof of Theorem 1. Compared to the MUM, however, the MEM is defined in the richer domain of lotteries contained in a linear vector space; it imposes linearity on the utility function U; and it requires the distance metric to be a norm induced by an inner product. These assumptions carry additional testable implications beyond MST+.

First, the requirement that U is onto $[0,1]$ implies that a MEM cannot be constant, that is $\rho \neq 1/2$. Second, every MEM is *continuous* at every point in the domain except along the diagonal $\{(x,x) \in \Delta^2 : x \in \Delta\}$. Third, every MEM is *linear*, that is, for all $0 < \alpha < 1$

and any lotteries $x, y, z \in \Delta$ we have $\rho(x, y) = \rho(\alpha x + (1-\alpha)z, \alpha y + (1-\alpha)z)$. And finally, every MEM ρ is *convex*, that is, whenever $\rho(x, y) = 1/2$ and $\rho(x, z) = \rho(y, z) > 1/2$, we have $\rho\left(x/2 + y/2, z\right) > \rho(\alpha x + (1-\alpha)y, z)$ for all $\alpha \neq 1/2$.

Continuity and linearity are familiar postulates from the random choice literature (see, for example, Gul and Pesendorfer, 2006), while convexity deserves some discussion. As in Example 1, suppose an individual is equally likely to choose a trip to Paris (P) and a trip to Rome (R) so that $\rho(P, R) = 1/2$. Suppose that either trip is more likely to be chosen than staying home (H), that is, $\rho(P, H) > 1/2$ and $\rho(R, H) > 1/2$. Suppose, moreover, that each trip is chosen over staying at home with the exact same probability, that is, $\rho(P, H) = \rho(R, H)$. Under these assumptions, convexity requires that, among all the lotteries $\alpha P + (1-\alpha)R$ that give a trip to Paris with probability α and a trip to Rome with probability $1-\alpha$, the even coin-flip ($\alpha = 1/2$) be the lottery most likely to be chosen in a binary comparison against staying at home. In particular, the fifty-fifty lottery between Paris and Rome must be chosen more often against staying at home than either Paris or Rome for sure. Our second Theorem shows these properties, in addition to MST+, are both necessary and sufficient for a MEM representation.

Theorem 2. *ρ is a MEM iff it is non-constant, continuous, linear, convex, and MST+.*

We prove Theorem 2 in the Appendix. Necessity is straightforward. To prove sufficiency, we first show that ρ has a unique linear extension to the $n-1$ dimensional hyperplane that contains Δ. Transitivity, linearity and continuity allow us to invoke a standard result to obtain the expected utility function U. The indifference sets $I(y) := \{x \in \Delta : U(x) = U(y)\}$ are then parallel hyperplanes of dimension $n-2$. To construct the norm, we fix one indifference set $I(y)$, and one lottery x with $U(x) > U(y)$, as illustrated in Figure 3. Convexity and symmetry imply the contour sets $\{y \in I : \rho(x, y) \geq \alpha\}$ are concentric ellipsoids, centered at the point that maximizes $y \mapsto \rho(x, y)$ on I. These ellipsoids are dilations of one another. We take one such ellipsoid to be the unit ball that defines the norm in the $n-2$ dimensional subspace parallel to $I(y)$. Then, we the norm is extended include one more dimension, which turns out to be the direction of maximum choice probability. In the obtained MEM representation, the expected utility U turns out to be unique, while the norm $\|\cdot\|$ and the transformation F are unique up to two scaling factors, as we show next.

Proposition 3. *Let $(U_1, \|\cdot\|_1, F_1)$ be a MEM representation of ρ. Then $(U_2, \|\cdot\|_2, F_2)$ is also a MEM representation of ρ if and only if:*

(i) $U_1 = U_2 = U$

(ii) $\ker(U)^{\perp_1} = \ker(U)^{\perp_2}$

(iii) *There exist $A, B > 0$ such that:*

  (a) $\|x\|_1 = A\|x\|_2$ *for all* $x \in \ker(U)$*,*

  (b) $\|x\|_1 = B\|x\|_2$ *for all* $x \in \ker(U)^\perp$*, and*

  (c) Letting $T := F_1^{-1}(\max_{x,y} \rho(x, y))$*, we have for each* $t \in [-T/B, T/B]$*,*

$$F_2(t) = F_1\left(\frac{tAT}{\sqrt{T^2 + t^2A^2 - t^2B^2}}\right).$$

We prove Proposition 3 in the Appendix. Item (i) says the normalized expected utility function U is unique. The uniqueness of U up to an affine transformation comes from the standard vNM expected utility representation result. Full uniqueness comes from our normalization requiring that U is a function onto $[0, 1]$.

Item (ii) in Proposition 3 says the inner products $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ that generate the norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively, must agree on which vectors are orthogonal to $\ker(U)$, the null space of U.

Item (iii) in Proposition 3 says the norm $\|\cdot\|$ and the transformation F in the MEM representation are unique up to two scaling factors $A, B > 0$. The norm $\|\cdot\|$ is defined on the

$n-1$ dimensional subspace of $\mathbb{R}^n$ that is parallel to Δ, namely, the set of vectors $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ such that $x_1 + \cdots + x_n = 0$. The constant $A > 0$ rescales the norm in the $n-2$ dimensional subspace $\ker(U)$, while the constant $B > 0$ rescales the norm along the single dimensional space $\ker(U)^{\perp}$. It is easy to see that since

$$F_1\left(\frac{U(x) - U(y)}{\|x-y\|_1}\right) = \rho(x, y) = F_2\left(\frac{U(x) - U(y)}{\|x-y\|_2}\right)$$

we must have, for each $x \neq y$,

$$F_2^{-1}(\rho(x, y)) = \frac{\|x-y\|_1}{\|x-y\|_2} \times F_1^{-1}(\rho(x, y))$$

that is, F_2 can be obtained from F_1 by rescaling each point in the domain of F_1 by the ratio of the two norms. Item (iii.c) shows how one explicitly obtains F_2 from F_1 and the values of $A, B > 0$. In particular, when $A = B > 0$ we have $F_2(t) = F_1(At)$ for all t.

To provide some intuition for the existence of the two scaling factors $A, B > 0$, consider two lotteries x, y on the same indifference plane I and a lottery z with lower utility, as shown in Figure 3. Suppose $\rho(x, z) > \rho(y, z)$. In the representation, we have the inequality

$$\frac{U(x) - U(z)}{\|x-z\|} > \frac{U(x) - U(y)}{\|x-y\|}$$

Let $\hat{x}$ be the projection of z onto the indifference plane I. By orthogonality of the projection, we have $\|x-z\|^2 = \|x-\hat{x}\|^2 + \|\hat{x}-z\|^2$ and $\|y-z\|^2 = \|y-\hat{x}\|^2 + \|\hat{x}-z\|^2$, and hence

$$\begin{aligned}
\frac{U(x) - U(z)}{\|x-z\|} > \frac{U(x) - U(y)}{\|x-y\|} &\Leftrightarrow \|x-z\| < \|x-y\| \\
&\Leftrightarrow \|x-\hat{x}\|^2 + \|\hat{x}-z\|^2 < \|y-\hat{x}\|^2 + \|\hat{x}-z\|^2 \\
&\Leftrightarrow A^2\|x-\hat{x}\|^2 + B^2\|\hat{x}-z\|^2 < A^2\|y-\hat{x}\|^2 + B^2\|\hat{x}-z\|^2
\end{aligned}$$

that is, the norm of the two orthogonal components can be separately rescaled while preserving the inequality. Hence, the rescaling preserves the ordinal representation for $\rho(\cdot, z)$ on the indifference plane I. By linearity, it also preserves the ordinal representation in the entire domain of ρ. To preserve the cardinal representation, we adjust F accordingly using the factors $A, B > 0$ as stated in item (iii.c) of Proposition 3.

To summarize, we showed the utility U in the MEM representation is unique, while the norm $\|\cdot\|$ and the mononotic transformation F are unique up to the same two scaling factors $A, B > 0$. In the next section, we explain how our results relate to the existing literature.

5 Related literature

Our Theorem 1 closes an open question posed by Halff (1976). Halff (1976) proposed the definition (2) of a MUM, proved that every MUM satisfies MST, and left open the question of sufficiency. Our Theorem 1 answers the question posed by Halff by showing that, while MST is not sufficient for a choice rule to be a MUM, the slightly stronger MST+ condition is both necessary and sufficient.

The MUM characterized in Theorem 1 generalizes several nested models of stochastic binary choice in the literature, as shown in Figure 4. The most restrictive model in Figure 4 is the binary *Logit model* in which choice probabilities are given by the formula

$$\rho(x, y) = \frac{e^{u(x)}}{e^{u(x)} + e^{u(y)}} = \frac{1}{1 + e^{-[u(x)-u(y)]}} \tag{6}$$

for some utility function $u : Z \to \mathbb{R}$. Luce (1959) showed formula (6) is equivalent to the *product rule*

(PR) $\quad \rho(x, y)\rho(y, z)\rho(z, x) = \rho(x, z)\rho(z, y)\rho(y, x)$

which can be interpreted as saying that the probability of observing a choice cycle in the direction $x \succ y \succ z \succ x$ is always equal to the probability of observing a choice cycle in the opposite direction.

A generalization of formula (6) is the *Fechnerian utility model* from psychophysics where

$$\rho(x, y) = F(u(x) - u(y)) \tag{7}$$

for some utility function $u : Z \to \mathbb{R}$ and some strictly increasing $F : \mathbb{R} \to (0, 1)$. The testable implications of formula (7) are well studied (see Debreu (1958) and references therein). A result in Fudenberg et al. (2015) shows the Fechnerian formula is equivalent to two postulates when the set of options is finite. The first postulate is the mild assumption of *positivity*, which requires $\rho(x, y) > 0$ for every x, y. The second, more substantive postulate is *acyclicity*, which, when specialized to a binary choice setting, rules out cycles of the form

$$\rho(w^i, x^i) \geq \rho(y^i, z^i) \tag{8}$$

for all $i = 1, \dots, n$ with at least one strict inequality, whenever $\{w^i, x^i\} = \{y^{f(i)}, z^{f(i)}\}$ and $w^i = y^{g(i)}$ for some permutations $f, g : \{1, \dots, n\} \to \{1, \dots, n\}$.

Formula (7) can be further generalized to *simple scalability* (Krantz, 1964) which requires

$$\rho(x, y) = F(u(x), u(y)) \tag{9}$$

for some utility function u and a real valued function F which is strictly increasing in the first argument and strictly decreasing in the second. Tversky and Russo (1969) showed that simple scalability is equivalent to positivity and a slightly stronger version of SST:

(SST+) $\rho(x, y) \wedge \rho(y, z) \geq 1/2 \Longrightarrow \rho(x, z) \geq \rho(x, y) \vee \rho(y, z)$, and
$\rho(x, y) \wedge \rho(y, z) > 1/2 \Longrightarrow \rho(x, z) > \rho(x, y) \vee \rho(y, z)$

which, compared to the original SST postulate, imposes the additional requirement that a strict inequality in the hypothesis entails a strict inequality in the conclusion.

It can be seen immediately by inspecting the formulas that (6) $\Rightarrow$ (7) $\Rightarrow$ (9). To see that the simple scalability model (9) is nested in MUM, note that SST+ immediately implies MST+. The failure of the reverse implications is also easily seen by examples.

An additional postulate, not shown in Figure 4, is the *quadruple condition* considered by Debreu (1958):

(QC) $\rho(w,x) \geq \rho(y,z)$ if and only if $\rho(w,y) \geq \rho(x,z)$

In a setting where Z is infinite, and under an additional stochastic continuity assumption, Debreu (1958) showed that QC implies the Fechnerian utility model (7). It is also immediate from the definitions that a Fechnerian utility model (7) satisfies QC. When Z is finite, however, our next example shows that QC, while necessary, is not sufficient for ρ to be a Fechnerian utility model.

Example 6. *Let* $Z = \{1,2,3,4,5\}$ *and let* ρ *be a choice rule on* Z *with*

$$1 > \rho(5,1) > \rho(5,2) > \rho(4,1) > \rho(4,2) > \rho(3,1) >$$
$$\rho(5,3) > \rho(4,3) > \rho(3,2) > \rho(5,4) > \rho(2,1) > 1/2$$

Verifying that ρ *satisfies QC is tedious but straightforward. This* ρ *does not admit a Fechnerian representation as in (7), since* $\rho(5,4) > \rho(2,1)$ *and* $\rho(3,1) > \rho(5,3)$ *would imply* $u(5)-u(4)+u(3)-u(1) > u(2)-u(1)+u(5)-u(3)$ *and the representation would require* $\rho(3,2) > \rho(4,3)$*, a contradiction.*

QC is easily seen to imply SST+. Suppose $\rho(x,y) \wedge \rho(y,z) \geq 1/2$. Then QC and $\rho(y,z) \geq 1/2 = \rho(x,x)$ imply $\rho(y,x) \geq \rho(z,x)$ and hence $\rho(x,z) \geq \rho(x,y)$. Also, QC and $\rho(x,y) \geq 1/2 = \rho(z,z)$ imply $\rho(x,z) \geq \rho(y,z)$ and hence $\rho(x,z) \geq \rho(x,y) \vee \rho(y,z)$. The same argument with strict inequalities in the hypothesis implies a strict inequality in the conclusion and SST+ obtains. The converse implication fails, as our next example shows.

Example 7. *Let* $Z = \{1, 2, 3, 4\}$ *and let* ρ *be a choice rule on* Z *with*

$$1 > \rho(4,1) > \rho(4,2) > \rho(3,1) > \rho(2,1) > \rho(4,3) > \rho(3,2) > 1/2$$

which is easily verified to satisfy SST+ but fails QC, since $\rho(4,2) > \rho(3,1)$ *but* $\rho(4,3) < \rho(2,1)$.

Several familiar discrete choice models used to address violations of SST in the literature are particular instances of MUM. Examples 4 shows the classic multinomial probit is a MUM. The Bayesian probit model (Natenzon, 2019) restricted to binary choice is equivalent to a probit model, and therefore a MUM. Another example, below, is the elimination-by-aspects model proposed by Tversky (1972).

Example 8 (Tversky's EBA)**.** *The choice rule* ρ *on a finite* Z *is an* elimination-by-aspects (EBA) *rule if there exist a mapping* A *that takes each option* $x \in Z$ *to a set of aspects* $A(x)$ *that* x *possesses, and a measure* m *over the set of all aspects such that*

$$\rho(x,y) = \frac{m\,[A(x)] - m\,[A(y)]}{m\,[A(x) \setminus A(y)] + m\,[A(y) \setminus A(x)]}.$$

Every EBA is a MUM with $u(x) = m[A(x)]$ *for all* $x \in Z$*,* $d(x,y) = m\,[A(x) \setminus A(y)] + m\,[A(y) \setminus A(x)]$ *and* F *given by the strictly increasing function* $F(t) = 1/2 + t/2$*.*

Probit and EBA are also instances of the *random utility model* (RUM). A choice rule ρ on a finite Z is a RUM if there exists a probability measure μ over the strict orderings on Z such that $\rho(x,y)$ equals the probability under μ of the event in which x beats y. Block and Marschak (1959) and Falmagne (1978) characterize the set of RUMs in an abstract setting of choice options when choice data for all finite menus is available. Gul and Pesendorfer (2006) impose linearity and provide a characterization of RUM in the richer setting of lotteries. A review of the literature that tackles the characterization of binary choice RUMs is provided by Fishburn (1992). Example 4 shows the MUM and RUM families have a non-empty intersection. Next, we show that neither MUM nor RUM nest each other.

Example 9. *We slightly modify an example given in de Souza (1983) to obtain a choice rule that satisfies MST+ but is not a RUM. Let* $Z = \{1, 2, 3, 4, 5, 6\}$ *and let the choice rule* ρ *on* Z *be given by*

$$
\begin{aligned}
\rho(4,5) = \rho(4,6) = \rho(2,5) = \rho(2,3) = \rho(1,6) = \rho(1,3) &= 1 \\
\rho(2,6) = \rho(1,5) &= \frac{1}{2} + \varepsilon \\
\rho(2,4) = \rho(1,4) = \rho(3,5) = \rho(3,6) &= \frac{1}{2} + \frac{\varepsilon}{2} \\
\rho(3,4) = \rho(1,2) = \rho(5,6) &= \frac{1}{2} + \frac{\varepsilon}{3}
\end{aligned}
$$

where $0 < \varepsilon < 3/10$*. It is straightforward to verify that* ρ *satisfies MST+. Now suppose* ρ *is a RUM generated by the probability* μ *on the set of strict orderings over* Z*. Since* $\rho(2,3) = \rho(4,6) = 1$*, for any strict ordering in the support of* μ *in which* $3 \succ 4$ *we also have* $2 \succ 3 \succ 4 \succ 6$ *and therefore* $2 \succ 6$*. This shows* μ *must assign zero probability to the intersection of events* $3 \succ 4$ *and* $6 \succ 2$*. By the same reasoning,* μ *must assign zero probability to the intersection of events* $3 \succ 4$ *and* $5 \succ 1$*; and* μ *must also assign zero probability to the intersection of events* $6 \succ 2$ *and* $5 \succ 1$*. Since* μ *is a probability measure, this implies* $\rho(3,4) + \rho(5,1) + \rho(6,2) \leq 1$*. But instead we have* $\rho(3,4) + \rho(5,1) + \rho(6,2) = 3/2 - 5\varepsilon/3 > 1$ *and therefore* ρ *cannot be a RUM.*

A converse, well-known example shows that RUM models can violate MST+. Let μ assign equal probability to three strict orderings $x \succ y \succ z$, $y \succ z \succ x$ and $z \succ x \succ y$ over the options x, y and z. Then the binary choice rule ρ generated by μ has $\rho(x, y) = \rho(y, z) = \rho(z, x) = 2/3$ which violates WST, and therefore also violates MST+.

Transitivity postulates are, by definition, restrictions imposed on binary choice behavior, and therefore our analysis is orthogonal to issues involving choice from non-binary menus. We conclude by showing how the model can be extended to choice over menus with more than two options.

Let $Z = \{x^1, \dots, x^n\}$ be set of options and let $X = (X_1, \dots, X_n)$ be a Gaussian vector where all the X_i's have the same variance $t > 0$. Consider the MUM from Example 4:

$$\rho(x^i, x^j) = \mathbb{P}\{X_i > X_j\} \tag{10}$$

Formula (10) can be easily extended to choice over more than two alternatives. Below, we give two examples with very different testable implications.

First, we may interpret the Gaussian variable X_i as the distribution of the utility i in a population of consumers. Under the population interpretation, a natural extension of (10) to a menu with k options is:

$$\rho(x_i, \{x_1, \dots, x_k\}) = \mathbb{P}\{X_i > X_j \text{ for all } j \neq i\} \tag{11}$$

which is the formula for the classic multinomial probit model.

Second, we may interpret the Gaussian variable $X_i = u(i) + \varepsilon_i$ as a noisy signal which represents the imperfect information obtained by an individual about the value $u(i)$ of each option i before making a choice. Suppose the individual's prior beliefs are that each value $u(i)$ is drawn iid from the same Gaussian distribution. If this individual maximizes the expected value of the chosen option given the signals, then:

$$\rho(x_i, \{x_1, \dots, x_k\}) = \mathbb{P}\{\mathbb{E}[u(i) - u(j) | X_1, \dots, X_k] > 0 \text{ for all } j \neq i\} \tag{12}$$

which is the formula for the Bayesian probit model (Natenzon, 2019).

While the binary formula (10) is a special case of both (11) and (12) with $k = 2$, these two extensions have very different behavioral implications. For example, the formula (11) based on the population interpretation satisfies the postulate of *regularity*:

(R) $\qquad \rho(x, A) \geq \rho(x, A \cup \{y\})$

which says that the choice probability of an option x in menu A cannot increase when a new option y is introduced. On the other hand, formula (12) based on the imperfectly informed individual interpretation can be used to accommodate well-documented empirical violations of this postulate, such as the attraction effect and the compromise effect (Natenzon, 2019).

The takeaway is that two stochastic choice models with very different testable implications may fall under the MUM category when restricted to binary choice. Conversely, since our analysis is entirely based on the strength of transitivity in binary choice, our results have no bearing on whether postulates that involve choice over more than two options, such as (R) above, may hold.

A Appendix: proofs

Proof of Proposition 1

Let Z be a finite set with n alternatives enumerated $x^1, x^2, \ldots, x^n$. Consider the set of choice rules ρ on Z which satisfy WST with $\rho(x^i, x^j) \geq 1/2$ whenever $i \leq j$ and for which the set $\{\rho(x, y) \in [0, 1] : x \neq y\}$ has maximum cardinality with $n(n-1)$ elements. Each such ρ induces a strict ordering $\succ_\rho$ of the $n(n+1)/2$ pairs $P_n := \{(x^i, x^j) : n \geq i > j \geq 1\}$ given by $(x^i, x^j) \succ_\rho (x^k, x^\ell)$ if and only if $\rho(x^i, x^j) > \rho(x^k, x^\ell)$. This set of choice rules ρ induces $\#WST(n) = [n(n-1)/2]!$ different strict orderings $\succ_\rho$ on P_n.

MST and MST+ allow the same number of different strict orderings over P_n which we denote $\#MST(n)$. Now consider the addition of alternative x^{n+1} to the set Z.

Lemma A.1. $\#MST(n+1) \leq [n(n-1)/2+1]^n \, \#MST(n)$

Proof. Take a single strict ordering over P_n compatible with MST. There are multiple ways to extend this strict ordering to incorporate the new pairs $(x^1, x^{n+1}), (x^2, x^{n+1}), \ldots, (x^n, x^{n+1})$ and obtain a strict ordering over P_{n+1} that is still compatible with MST. Since the original ordering has $n(n-1)/2$ pairs, there are $n(n-1)/2+1$ different positions to include (x^n, x^{n+1}). In this way we obtain $n(n-1)/2+1$ different strict orderings, all of which respect MST. The total number of strict orderings over $P_n \cup \{(x^n, x^{n+1})\}$ that satisfy MST is therefore $[n(n-1)/2+1] \, \#MST(n)$. Now we take one such strict ordering and extend it to incorporate a second pair (x^{n-1}, x^{n+1}). This pair can in principle be added into $n(n-1)/2+2$ different positions, but placing it in the very last position would violate MST, since MST requires $\rho(x^{n-1}, x^{n+1}) > \min\{\rho(x^{n-1}, x^n), \rho(x^n, x^{n+1})\}$. The total number of strict orderings over $P_n \cup \{(x^n, x^{n+1}), (x^{n-1}, x^{n+1})\}$ which satisfy MST must therefore be smaller or equal to $[n(n-1)/2+1]^2 \, \#MST(n)$. A simple inductive argument completes the proof. □

Lemma A.2. $\lim_{n\to\infty} \left[\prod_{k=1}^{n} \frac{n(n-1)/2+k}{n(n-1)/2+1}\right] = e$

Proof. The result can be shown by verifying that, for each n,

$$\left(1+\frac{1}{n}\right)^{n-1} \leq \left[\prod_{k=1}^{n} \frac{n(n-1)/2+k}{n(n-1)/2+1}\right] \leq \left(1+\frac{1}{n}\right)^{n}$$

and taking the limit as $n \to \infty$. We leave the details to the reader. □

Lemma A.1 implies that

$$\frac{\#MST(n+1)}{\#WST(n+1)} \leq \frac{\#MST(n)}{\#WST(n)} \frac{[n(n-1)/2]!}{[n(n+1)/2]!}[n(n-1)/2+1]^n$$
$$= \frac{\#MST(n)}{\#WST(n)} \left[\prod_{k=1}^{n} \frac{n(n-1)/2+1}{n(n-1)/2+i}\right]$$

and by Lemma A.2 the last expression in brackets goes to $1/e$ when n goes to infinity, where $e \approx 2.718$ is the base of the natural logarithm. Hence for all n sufficiently large the ratio $\#MST(n+1)/\#WST(n+1)$ is less than half of the ratio $\#MST(n)/\#WST(n)$, which completes the proof.

Finally, we prove the additional claim, stated after Proposition 1, that

$$\lim_{n\to\infty} \#SST(n)/\#MST(n) = 0.$$

The choice probability $\rho(x^1, x^n)$ must be the highest choice probability in every ρ that satisfies SST. For each strict ordering of choice probabilities satisfying SST, there exist at least $n-2$ strict orderings which violate SST but satisfy MST: for each $k = 2, 3, \ldots, n-1$ change the value of $\rho(x^1, x^n)$ to be equal to $\max\{\rho(x^1, x^k), \rho(x^k, x^n)\} - \varepsilon$ for $\varepsilon > 0$ sufficiently small. It is immediate to see that each resulting ranking violates SST. To see that MST still holds, note that every inequality required by SST holds except those involving $\rho(x^1, x^n)$. In addition, SST implies that for each $k, j = 2, \ldots, n-1$, $\max\{\rho(x^1, x^k), \rho(x^k, x^n\}) > \min\{\rho(x^1, x^j), \rho(x^j, x^n\})$ hence for ε small we have $\rho(x^1, x^n) > \min\{\rho(x^1, x^j), \rho(x^j, x^n)\}$. Thus, $\#SST(n)/\#MST(n) \leq 1/(n-1) \to 0$ when $n \to \infty$. □

Proof of Theorem 1

For necessity, assume there exist u and d satisfying (2), and assume $\rho(x, y) \geq 1/2$ and $\rho(y, z) \geq 1/2$. If it were the case that $\rho(x, z) < \min\{\rho(x, y), \rho(y, z)\}$, then by (2) and the triangle inequality property of d it would follow that

$$\begin{aligned}
u(x) - u(z) &< d(x,z) \min\left\{\frac{u(x)-u(y)}{d(x,y)}, \frac{u(y)-u(z)}{d(y,z)}\right\} \\
&\leq [d(x,y) + d(y,z)] \min\left\{\frac{u(x)-u(y)}{d(x,y)}, \frac{u(y)-u(z)}{d(y,z)}\right\} \\
&\leq d(x,y)\frac{u(x)-u(y)}{d(x,y)} + d(y,z)\frac{u(y)-u(z)}{d(y,z)} \\
&= u(x) - u(z)
\end{aligned}$$

which is a contradiction. Hence, it must be the case that $\rho(x,z) \geq \min\{\rho(x,y), \rho(y,z)\}$. This first step of the necessity was also shown by Halff (1976).

Now suppose we have equality $\rho(x,z) = \min\{\rho(x,y), \rho(y,z)\}$. We consider the case $\min\{\rho(x,y), \rho(y,z)\} = \rho(x,y)$, while the remaining case is analogous and left to the reader. Representation (2) and the triangle inequality imply

$$\begin{aligned}
u(x) - u(y) + u(y) - u(z) &= u(x) - u(z) \\
&= d(x,z)\left[\frac{u(x)-u(y)}{d(x,y)}\right] \\
&\leq [d(x,y) + d(y,z)]\left[\frac{u(x)-u(y)}{d(x,y)}\right] \\
&= u(x) - u(y) + d(y,z)\left[\frac{u(x)-u(y)}{d(x,y)}\right].
\end{aligned}$$

Subtracting $u(x) - u(y)$ from both sides we obtain

$$\frac{u(y)-u(z)}{d(y,z)} \leq \frac{u(x)-u(y)}{d(x,y)}$$

and therefore (2) yields $\rho(x,y) = \rho(y,z) = \rho(x,z)$ as desired.

For sufficiency, suppose ρ satisfies MST+. In particular, ρ satisfies WST, and hence, by letting $x \succcurlyeq y$ if and only if $\rho(x,y) \geq 1/2$, we obtain a complete and transitive relation $\succcurlyeq$ over the finite set of options Z. The relation $\succcurlyeq$ induced by ρ divides the n alternatives in Z into $k \leq n$ indifference classes. Therefore, there exists a utility function $u : Z \to \{1, \ldots, k\}$ that is onto and represents $\succcurlyeq$, that is, $u(x) \geq u(x)$ if and only if $x \succcurlyeq y$ if and only if $\rho(x, y) \geq 1/2$.

Let $Y := \{\{x, y\} \subset Z : \rho(x, y) \neq 1/2\}$, and let m be the cardinality of the set $\{|\rho(x, y) - 1/2| : \{x, y\} \in Y\}$. Partition the set Y into m disjoint sets $Y_1 \cup Y_2 \cup \cdots \cup Y_m = Y$ such that for any two pairs $\{w, x\}$ and $\{y, z\}$ in Y we have $\{w, x\} \in Y_i$ and $\{y, z\} \in Y_j$ with $i \geq j$ if and only if $|\rho(w, x) - 1/2| \leq |\rho(y, z) - 1/2|$. Thus, the pairs in Y_1 have the highest value of $|\rho(x, y) - 1/2|$, while the pairs in Y_m have the lowest value of $|\rho(x, y) - 1/2|$ among the pairs in Y.

The result is trivial when Z has $n \leq 2$ alternatives so suppose $n \geq 3$. Define a constant $C = (n-1)^{[n(n-1)/2+1]} > 0$ and define the sequence $D_1, D_2, \ldots, D_m$ by:

$$D_1 = 0; D_j = (n-1)^{j-2} \text{ for } j = 2, \ldots, m.$$

Let $d : Z \times Z \to [0, \infty)$ be defined as follows:

$$d(x, y) = \begin{cases} 0, & \text{if } x = y \\ C, & \text{if } x \neq y \text{ and } \rho(x, y) = 1/2 \\ (C/2 + D_j)\,|u(x) - u(y)|\,, & \text{if } \{x, y\} \in Y_j \end{cases} \tag{13}$$

From the definition (13) it is immediate that d satisfies (i) $d(x, y) \geq 0$; (ii) $d(x, y) = 0$ if and only if $x = y$; and (iii) $d(x, y) = d(y, x)$ for all $x, y \in Z$. To show that d is a metric, it remains to verify the triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$. The inequality trivially holds when any two options among x, y, z are equal. Consider three distinct options $x, y, z \in Z$.

Case 1: $u(x) = u(y) = u(z)$. By the definition of u we have $\rho(x, y) = \rho(y, z) = \rho(x, z) = 1/2$. By the definition of d we have $d(x, z) = C < 2C = d(x, y) + d(y, z)$.

Case 2: $u(x) \neq u(y) = u(z)$. The definitions of u and d imply

$$\begin{aligned} d(x,y) + d(y,z) - d(x,z) &= (C/2 + D_i)\,|u(x) - u(y)| + C - (C/2 + D_j)\,|u(x) - u(z)| \\ &= (D_i - D_j)\,|u(x) - u(z)| + C \\ &\geq -(n-1)^{m-2}(n-1) + C \\ &= (n-1)^{[n(n-1)/2+1]} - (n-1)^{m-1} \\ &> 0 \end{aligned}$$

where the last inequality follows from the fact that we defined m to be the cardinality of $\{|\rho(x,y) - 1/2| : \{x,y\} \in Y\}$ which is smaller or equal to $n(n-1)/2$.

Case 3: $u(y) \neq u(x) = u(z)$. The definitions of u and d imply

$$\begin{aligned} d(x,y) + d(y,z) - d(x,z) &= (C/2 + D_i)\,|u(x) - u(y)| + (C/2 + D_j)\,|u(y) - u(z)| - C \\ &= (C + D_i + D_j)\,|u(y) - u(z)| - C \\ &\geq 0. \end{aligned}$$

Case 4: $u(z) \neq u(x) = u(y)$. The inequality follows from the same argument as in Case 2.

Case 5: $u(x) > u(y) > u(z)$. By the definition of u we have $\{x,y\} \in Y_i$, $\{y,z\} \in Y_j$, and $\{x,z\} \in Y_\ell$, for some i, j, ℓ. The definition of d implies

$$\begin{aligned} d(x,y) + d(y,z) - d(x,z) &= (C/2 + D_i)\,|u(x) - u(y)| + (C/2 + D_j)\,|u(y) - u(z)| \\ &\quad - (C/2 + D_\ell)\,|u(x) - u(y) + u(y) - u(z)| \\ &= (D_i - D_\ell)\,|u(x) - u(y)| + (D_j - D_\ell)\,|u(y) - u(z)| \end{aligned}$$

The definition of u implies $\rho(x,y) > 1/2$ and $\rho(y,z) > 1/2$. By MST+ we have either

$\rho(x, y) = \rho(y, z) = \rho(x, z)$ or $\rho(x, z) > \min\{\rho(x, y), \rho(y, z)\}$. The first case implies $D_i = D_j = D_\ell$ above and therefore $d(x, y) + d(y, z) - d(x, z) = 0$. The second case implies $D_\ell < \max\{D_i, D_j\}$. If $D_\ell \leq \min\{D_i, D_j\}$ then both $(D_i - D_\ell)$ and $(D_j - D_\ell)$ above are positive and the desired inequality holds. It remains to show the inequality holds when $\min\{D_i, D_j\} < D_\ell < \max\{D_i, D_j\}$, which implies

$$\begin{aligned} d(x, y) + d(y, z) - d(x, z) &\geq (\max\{D_i, D_j\} - D_\ell)\, 1 + (\min\{D_i, D_j\} - D_\ell)\,(n - 2) \\ &\geq (n-1)^{\ell-1} - (n-1)^{\ell-2} + [0 - (n-1)^{\ell-2}](n-2) \\ &= 0. \end{aligned}$$

Case 6: $u(x) > u(z) > u(y)$. By the definition of u we have $\{x, y\} \in Y_i$, $\{y, z\} \in Y_j$, and $\{x, z\} \in Y_\ell$, for some i, j, ℓ. The definition of d implies

$$\begin{aligned} d(x, y) + d(y, z) - d(x, z) &= (C/2 + D_i)\,[u(x) - u(z) + u(z) - u(y)] \\ &\quad + (C/2 + D_j)\,[u(z) - u(y)] - (C/2 + D_\ell)\,[u(x) - u(z)] \\ &= (D_i - D_\ell)\,[u(x) - u(z)] + (C + D_i + D_j)\,[u(z) - u(y)] \\ &\geq \left(0 - (n-1)^{m-2}\right)(n-2) + (C + 0 + 0)\,1 \\ &= -(n-1)^{m-1} + (n-1)^{m-2} + (n-1)^{n(n-1)/2+1} \\ &> 0. \end{aligned}$$

Case 7: $u(y) > u(x) > u(z)$. By the definition of u we have $\{x, y\} \in Y_i$, $\{y, z\} \in Y_j$, and

$\{x, z\} \in Y_\ell$, for some i, j, ℓ. The definition of d implies

$$\begin{aligned} d(x,y) + d(y,z) - d(x,z) &= (C/2 + D_i)\,[u(y) - u(x)] \\ &\quad + (C/2 + D_j)\,[u(y) - u(x) + u(x) - u(z)] \\ &\quad - (C/2 + D_\ell)\,[u(x) - u(z)] \\ &= (C + D_i + D_j)\,[u(y) - u(x)] + (D_j - D_\ell)\,[u(x) - u(z)] \\ &> 0. \end{aligned}$$

Case 8: $u(y) > u(z) > u(x)$. Similarly to Case 7, we have

$$\begin{aligned} d(x,y) + d(y,z) - d(x,z) &= (C + D_i + D_j)\,[u(y) - u(z)] + (D_i - D_\ell)\,[u(z) - u(x)] \\ &> 0. \end{aligned}$$

Case 9: $u(z) > u(x) > u(y)$. Similarly to Cases 7 and 8, we have

$$\begin{aligned} d(x,y) + d(y,z) - d(x,z) &= (C + D_i + D_j)\,[u(x) - u(y)] + (D_j - D_\ell)\,[u(z) - u(x)] \\ &> 0. \end{aligned}$$

Case 10: $u(z) > u(y) > u(x)$. Since $d(x,y) + d(y,z) \leq d(x,z)$ if and only if $d(y,x) + d(z,y) \leq d(z,x)$, the inequality follows from the same argument as in Case 5.

By Cases 1 to 10 above, d satisfies the triangle inequality and is therefore a metric. Now, we verify that the utility u and the metric d constructed above provide an ordinal representation for ρ as in (4). First, $\rho(w,x) \geq \rho(y,z) > 1/2$ if and only if $\rho(w,x) > 1/2$, $\rho(y,z) > 1/2$, and $|\rho(w,x) - 1/2| \geq |\rho(y,z) - 1/2|$, if and only if $u(w) > u(x)$, $u(y) > u(z)$, $d(w,x) =$

$(C/2 + D_i)[u(w) - u(x)]$, $d(y, z) = (C/2 + D_j)[u(y) - u(z)]$, and $i \leq j$, if and only if

$$\frac{u(w) - u(x)}{d(w, x)} = \frac{1}{C/2 + D_i} \geq \frac{1}{C/2 + D_j} = \frac{u(y) - u(z)}{d(y, z)} > 0.$$

Second, $\rho(w, x) \geq 1/2 \geq \rho(y, z)$ if and only if $u(w) - u(x) \geq 0 \geq u(y) - u(z)$ if and only if

$$\frac{u(w) - u(x)}{d(w, x)} \geq 0 \geq \frac{u(y) - u(z)}{d(y, z)}.$$

And, finally, $1/2 > \rho(w, x) \geq \rho(y, z)$ if and only if $\rho(w, x) < 1/2$, $\rho(y, z) < 1/2$, and $|\rho(w, x) - 1/2| \leq |\rho(y, z) - 1/2|$, if and only if $u(w) < u(x)$, $u(y) < u(z)$, $d(w, x) = (C/2 + D_i)[u(x) - u(w)]$, $d(y, z) = (C/2 + D_j)[u(z) - u(y)]$, and $i \geq j$, if and only if

$$0 > \frac{u(w) - u(x)}{d(w, x)} = -\frac{1}{C/2 + D_i} \geq -\frac{1}{C/2 + D_j} = \frac{u(y) - u(z)}{d(y, z)}$$

hence the ordinal representation (4) holds. Finding a strictly increasing F such that the cardinal representation (2) holds is then straightforward and left to the reader. □

Proof of Theorem 2

Let the non-constant choice rule ρ on Δ be linear, continuous (outside the diagonal), convex, and satisfy MST+. First, we show that ρ has a unique linear extension to the $n - 1$ dimensional hyperplane H that contains Δ.

Lemma A.3. *ρ has a unique linear extension to $H = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 + \cdots + x_n = 1\}$.*

Proof. Let ρ' and ρ'' be two linear extensions of ρ and let $x, y \in \mathbb{R}^n$ with $x_1 + \cdots + x_n = y_1 + \cdots + y_n = 1$. Let $z = (1/n, \ldots, 1/n) \in \Delta$. Take $0 < \alpha < 1$ sufficiently small such that $0 < \alpha x_i + (1 - \alpha)/n < 1$ and $0 < \alpha y_i + (1 - \alpha)/n < 1$ for each i. Then $\alpha x + (1 - \alpha)z \in \Delta$,

$\alpha y + (1-\alpha)z \in \Delta$ and, by linearity,

$$\begin{aligned}\rho'(x,y) &= \rho'(\alpha x + (1-\alpha)z, \alpha y + (1-\alpha)z)\\ &= \rho(\alpha x + (1-\alpha)z, \alpha y + (1-\alpha)z)\\ &= \rho''(\alpha x + (1-\alpha)z, \alpha y + (1-\alpha)z)\\ &= \rho''(x,y)\end{aligned}$$

hence ρ' and ρ'' must be equal. $\square$

From this point on, we identify ρ with its unique linear extension. Define the relation $\succcurlyeq \subset \Delta \times \Delta$ by $x \succcurlyeq y$ if and only if $\rho(x,y) \geq 1/2$. Since ρ satisfies MST+, this $\succcurlyeq$ is complete and transitive. By linearity and continuity, $\succcurlyeq$ satisfies all the vNM axioms and admits an expected utility representation. Since ρ is non-constant, there is a unique linear function $U : \mathbb{R}^n \to \mathbb{R}$ which represents $\succcurlyeq$ with $U(\Delta) = [0,1]$.

For each lottery x, let $I(x) := \{y \in H : \rho(x,y) = 1/2\}$ denote the set of lotteries that are stochastically indifferent to x. Note that $I(x)$ is an affine subspace of dimension $n-2$. Since ρ is non-constant, there exist $\bar{x}, \bar{y} \in \Delta$ with $\rho(\bar{x}, \bar{y}) > 1/2$. By linearity, ρ is entirely determined by the values of the mapping $x \mapsto \rho(x, \bar{y})$ for $x \in I(\bar{x})$. For each $1/2 < p \leq 1$ let $B(p) := \{x \in I(\bar{x}) : \rho(x, \bar{y}) \geq p\}$ be the upper contour set of elements that are stochastically indifferent to $\bar{x}$ and that are chosen over $\bar{y}$ with probability greater or equal to p.

Lemma A.4. *$B(p)$ is convex for all $1/2 < p \leq 1$.*

Proof. Let $x, x' \in B(p)$ and let $0 < \alpha < 1$. Since $I(\bar{x})$ is an affine subspace, $\alpha x + (1-\alpha)x' \in I(\bar{x})$. Linearity implies $\rho(\alpha x + (1-\alpha)x', \alpha \bar{y} + (1-\alpha)x') = \rho(x, \bar{y}) \geq p$. Linearity also implies $\rho(\alpha \bar{y} + (1-\alpha)x', \bar{y}) = \rho(x', \bar{y}) \geq p$. Then, MST+ implies $\rho(\alpha x + (1-\alpha)x', \bar{y}) \geq p$ and therefore $\alpha x + (1-\alpha)x' \in B(p)$. $\square$

Lemma A.5. *$B(p)$ is compact for all $1/2 < p \leq 1$.*

Proof. $B(p)$ is closed by continuity. Let $|\cdot|$ denote the standard Euclidean metric, not necessarily equal to the metric we are going to construct for the representation. If $B(p)$ were not bounded, there would exist a sequence $x(k)$ in $B(p)$ with $|x(k)-\bar{y}| \geq k$ for all $k \in \mathbb{N}$. For each k, by linearity $\rho(\bar{y}+(x(k)-\bar{y})/|x(k)-\bar{y}|, \bar{y}) = \rho(x(k), \bar{y}) \geq p$. By Bolzano-Weierstrass the sequence $\bar{y}+(x(k)-\bar{y})/|x(k)-\bar{y}|$ would have a subsequence converging to some $z \neq \bar{y}$. By the linearity of U we would have $U(z) = U(\bar{y})$ and $\rho(z, \bar{y}) = 1/2$, contradicting continuity. Hence $B(p)$ must be bounded and therefore compact. □

Lemma A.6. *The mapping $x \mapsto \rho(x, \bar{y})$ has a unique maximizer $\hat{x}$ on $I(\bar{x})$.*

Proof. Since $\rho(\bar{x}, \bar{y}) > 1/2$ we have $B(p) \neq \emptyset$ for some $p > 1/2$. Since ρ is continuous outside the diagonal, the mapping $x \mapsto \rho(x, \bar{y})$ is continuous on $I(\bar{x})$. $B(p)$ is compact by Lemma A.5, hence the maximum $\rho(\hat{x}, \bar{y}) = \bar{p}$ is attained at some $\hat{x} \in B(p)$. Hence $B(\bar{p})$ is not empty, and by the previous lemmas it is compact and convex. By convexity, $B(\bar{p})$ must be a singleton. □

For the rest of the proof, we denote by $\hat{x}$ the unique maximizer of $x \mapsto \rho(x, \bar{y})$ on $I(\bar{x})$.

Lemma A.7. *$x \in I(\bar{x})$ and $\rho(x, \bar{y}) = p$ implies $\rho(2\hat{x} - x, \bar{y}) = p$.*

Proof. The statement trivially holds if $x = \hat{x}$, so suppose $x \neq \hat{x}$. First note $2\hat{x} - x = \hat{x} + (\hat{x} - x) \in I(\bar{x})$. If $\rho(2\hat{x} - x, \bar{y}) < p$, since $x \mapsto \rho(x, \bar{y})$ is continuous in the segment $[\hat{x}, \hat{x} + (\hat{x} - x)]$, by the intermediate value theorem we have $\rho(x', \bar{y}) = p$ for some x' in the open segment $(\hat{x}, 2\hat{x} - x)$. But then since $\hat{x}$ is the unique maximizer in $I(\bar{x})$ it is also the unique maximizer in the segment $[x, x']$. Since $\hat{x} \neq x/2 + x'/2$ this contradicts the fact that ρ is convex. Hence we must have $\rho(2\hat{x} - x, \bar{y}) \geq p$. The same argument shows that $\rho(2\hat{x} - x, \bar{y}) \leq p$. □

Recall that $\hat{x}$ is the unique maximizer $\rho(\hat{x}, \bar{y}) = \bar{p}$ on $I(\bar{x})$. Let $B = B(p) - \hat{x}$ for some fixed $p \in (1/2, \bar{p})$. We first define an auxiliary norm $\|\cdot\|_B$ on the $n-2$ dimensional subspace $I(\bar{x}) - \hat{x}$ using B as the unit ball.

Lemma A.8. $\|x\|_B := \inf\{\lambda \geq 0 : x \in \lambda B\}$ *is a norm on* $I(\bar{x}) - \hat{x}$.

Proof. The *Minkowski functional* $\|\cdot\|_B$ defined above is a norm when B is a symmetric, convex set such that each line through zero meets B in a non-trivial, closed, bounded segment (Thompson, 1996). By definition $\|x\|_B \geq 0$ for all x. Moreover, if $\|x\|_B = 0$ then $x \in \lambda B$ for all $\lambda > 0$ and therefore $x = 0$. Now for each $\alpha \geq 0$ we have $x \in \lambda B$ if and only if $\alpha x \in \alpha\lambda B$ and therefore $\alpha\|x\|_B = \|\alpha x\|_B$. Lemma A.7 implies $x \in \lambda B$ if and only if $-x \in \lambda B$ and therefore $\|x\|_B = \|-x\|_B$. To verify the triangle inequality, note that B is closed by Lemma A.5, and therefore $x/\|x\|_B \in B$ for all x. B is also convex by Lemma A.4, and therefore

$$\frac{x + x'}{\|x\|_B + \|x'\|_B} = \left(\frac{\|x\|_B}{\|x\|_B + \|x'\|_B}\right)\frac{x}{\|x\|_B} + \left(\frac{\|x'\|_B}{\|x\|_B + \|x'\|_B}\right)\frac{x'}{\|x'\|_B} \in B.$$

Thus,

$$\left\|\frac{x + x'}{\|x\|_B + \|x'\|_B}\right\|_B \leq 1$$

and the triangle inequality $\|x + x'\|_B \leq \|x\|_B + \|x'\|_B$ holds. □

Lemma A.9. *If* $\bar{p} \geq p \geq q > 1/2$ *then* $B(p) = \hat{x} + \lambda\left[B(q) - \hat{x}\right]$ *for some* $0 \leq \lambda \leq 1$.

Proof. MST+ implies that, for any $x \neq \hat{x}$ in $B(p)$, the function $\alpha \mapsto \rho(\alpha\hat{x} + (1-\alpha)x, \bar{y})$ is strictly increasing for $0 \leq \alpha \leq 1$. It suffices to show that if $\rho(x^1, \bar{y}) = \rho(x^2, \bar{y})$ for $x^1, x^2 \in I(\bar{x})$ and $0 < \alpha < 1$, then $\rho(\alpha x^1 + (1-\alpha)\hat{x}, \bar{y}) = \rho(\alpha x^2 + (1-\alpha)\hat{x}, \bar{y})$. To see that equality must hold, suppose instead that we had $\rho(\alpha x^1 + (1-\alpha)\hat{x}, \bar{y}) < \rho(\alpha x^2 + (1-\alpha)\hat{x}, \bar{y})$. Continuity implies $\rho(\beta x^2 + (1-\beta)\hat{x}, \bar{y}) = \rho(\alpha x^1 + (1-\alpha)\hat{x}, \bar{y})$ for some $0 < \alpha < \beta < 1$.

Figure 5 provides an illustration. Letting

$$z^1 = x^1 + \frac{\beta(1-\alpha)}{\beta-\alpha}(x^2 - x^1)$$
$$z^2 = x^1 + x^2 - z^1$$
$$z^3 = 2\hat{x} - z^1$$
$$z^4 = \alpha x^1 + \beta x^2 + (2-\alpha-\beta)\hat{x} - z^1$$

we have that the line segment $[z^1, z^2]$ contains the line segment $[x^1, x^2]$; the line segment $[z^1, z^4]$ contains the line segment $[\alpha x^1 + (1-\alpha)\hat{x}, \beta x^2 + (1-\beta)\hat{x}]$ and

$$z^1/2 + z^2/2 = x^1/2 + x^2/2$$
$$z^1/2 + z^3/2 = \hat{x}$$
$$z^1/2 + z^4/2 = (\beta x^2 + (1-\beta)\hat{x})/2 + (\alpha x^1 + (1-\alpha)\hat{x})/2$$

so that, by convexity, we must have the equalities

$$\rho(z^1, \bar{y}) = \rho(z^2, \bar{y}) = \rho(z^3, \bar{y}) = \rho(z^4, \bar{y}) = r.$$

for some $1 \geq r > 1/2$. Now note that

$$0 < \frac{2\alpha\beta}{\alpha+\beta} < \frac{2\alpha\beta}{\alpha+\alpha} = \beta < 1$$

and let

$$y = \left(\frac{2\alpha\beta}{\alpha+\beta}\right) z^2 + \left(1 - \frac{2\alpha\beta}{\alpha+\beta}\right) z^3.$$

Since $B(r)$ is convex, by Lemma A.4, we must have $\rho(y, \bar{x}) \geq r$. On the other hand, it is straightforward to verify the equality

$$z^4 = \gamma y + (1-\gamma) z^1$$

where

$$\gamma = \frac{(1-\alpha+1-\beta)(\alpha+\beta)}{2\beta(1-\alpha)+2\alpha(1-\beta)} \in (0,1)$$

and by convexity we must have $\rho(y, \bar{x}) < r$, a contradiction. □

Lemma A.10. *$\|\cdot\|_B$ is Euclidean, i.e., $\|x\|_B = \sqrt{\langle x, x\rangle_B}$ where $\langle\cdot,\cdot\rangle_B$ is an inner product.*

Proof. We use a characterization of inner product spaces by Gurari and Sozonov (1970), who showed that a normed linear space is an inner product space if and only if

$$\left\|\frac{1}{2}x + \frac{1}{2}y\right\| \leq \|\alpha x + (1-\alpha)y\| \quad \text{whenever } \|x\| = \|y\| = 1 \text{ and } 0 \leq \alpha \leq 1. \tag{14}$$

If $\|x\|_B = \|y\|_B = 1$ then x, y are on the boundary of B, hence $\rho(x+\hat{x}, \bar{y}) = \rho(y+\hat{x}, \bar{y}) = p > 1/2$ and $\rho(x+\hat{x}, y+\hat{x}) = 1/2$. Since ρ is convex, for each $0 \leq \alpha \leq 1$ we must have

$$\rho(\alpha x + (1-\alpha)y + \hat{x}, \bar{y}) \leq \rho(x/2 + y/2 + \hat{x}, \bar{y})$$

thus $\alpha x + (1-\alpha)y$ is on the boundary of $B(q) - \hat{x}$ and $x/2 + y/2$ is on the boundary of $B(q') - \hat{x}$ for some $q \leq q'$. By Lemma A.9, the norm $\|\cdot\|_B$ satisfies (14). □

Now we extend the inner product $\langle\cdot,\cdot\rangle_B$ on the $n-2$ dimensional subspace $I(\bar{x}) - \hat{x}$ obtained in the last Lemma to an inner product $\langle\cdot,\cdot\rangle$ on the $n-1$ dimensional subspace $H - \hat{x}$. Let $v_1, \ldots, v_{n-2}$ be an orthonormal base for the subspace $I(\bar{x}) - \hat{x}$ endowed with $\langle\cdot,\cdot\rangle_B$. Let $v_{n-1} := \hat{x} - \bar{y}$ and for every $1 \leq i, j \leq n-1$ let $\langle v_i, v_j\rangle = 0$ if $i \neq j$ and $\langle v_i, v_j\rangle = 1$ if $i = j$. We let the norm be induced by this inner product $\|x\| := \sqrt{\langle x, x\rangle}$ for all $x \in H - \hat{x}$.

Lemma A.11. *U and $\|\cdot\|$ provide an ordinal representation of ρ, that is, for any $w \neq x$ and $y \neq z$ we have*

$$\rho(w,x) \geq \rho(y,z) \Longleftrightarrow \frac{U(w)-U(x)}{\|w-x\|} \geq \frac{U(y)-U(z)}{\|y-z\|}.$$

Proof. First, suppose $\rho(w,x) \geq \rho(y,z) > 1/2$. Then $w \succ x$, $y \succ z$ and since U represents $\succcurlyeq$ we have $U(w) > U(x)$ and $U(y) > U(z)$. Let

$$\begin{aligned} w' &= \bar{y} + \frac{U(\hat{x})-U(\bar{y})}{U(w)-U(x)}(w-x) \\ y' &= \bar{y} + \frac{U(\hat{x})-U(\bar{y})}{U(y)-U(z)}(y-z) \end{aligned}$$

and note that $w', y' \in H$. Since U is linear, $U(w') = U(y') = U(\bar{x})$ and hence $w', y' \in I(\bar{x})$. By the linearity of ρ, $\rho(w', \bar{y}) = \rho(w,x) \geq \rho(y,z) = \rho(y', \bar{y})$. Hence $\|w' - \hat{x}\|_B \leq \|y' - \hat{x}\|_B$. By construction, $\hat{x} - \bar{y}$ is orthogonal to $I(\bar{x}) - \hat{x}$, and therefore

$$\begin{aligned} \|w' - \bar{y}\|^2 &= \|w' - \hat{x} + \hat{x} - \bar{y}\|^2 \\ &= \|w' - \hat{x}\|^2 + \|\hat{x} - \bar{y}\|^2 \\ &\leq \|y' - \hat{x}\|^2 + \|\hat{x} - \bar{y}\|^2 \\ &= \|y' - \bar{y}\|^2 \end{aligned}$$

Thus

$$\left\| \frac{U(\hat{x})-U(\bar{y})}{U(w)-U(x)}(w-x) \right\| = \|w' - \bar{y}\| \leq \|y' - \bar{y}\| = \left\| \frac{U(\hat{x})-U(\bar{y})}{U(y)-U(z)}(y-z) \right\|$$

which implies

$$\frac{U(w)-U(x)}{\|w-x\|} \geq \frac{U(y)-U(z)}{\|y-z\|}.$$

Next, suppose $\rho(w,x) \geq 1/2 \geq \rho(y,z)$ with $w \neq x$ and $y \neq z$. Then $U(w) \geq U(x)$ and

$U(z) \geq U(y)$ which implies

$$\frac{U(w) - U(x)}{\|w - x\|} \geq 0 \geq \frac{U(y) - U(z)}{\|y - z\|}.$$

Finally, suppose $1/2 > \rho(w, x) \geq \rho(y, z)$. Then $\rho(z, y) \geq \rho(x, w) > 1/2$ and the desired inequality follows from the first step.

Reversing the argument to show that

$$\frac{U(w) - U(x)}{\|w - x\|} \geq \frac{U(y) - U(z)}{\|y - z\|} \implies \rho(w, x) \geq \rho(y, z)$$

is straightforward and left to the reader. $\square$

Lemma A.12. *The image of ρ is an interval* $[1 - \bar{p}, \bar{p}]$.

Proof. As we noted before, linearity implies ρ is entirely determined by the values of the mapping $x \mapsto \rho(x, \bar{y})$ for $x \in I(\bar{x})$. Hence, ρ achieves its maximum at $\bar{p} = \rho(\hat{x}, \bar{y})$. By Lemma A.5 each $B(p) \subset I(\bar{x})$ is compact, hence $\inf\{\rho(x, \bar{y}) : x \in I(\bar{x})\} = 1/2$. Finally by continuity of ρ outside the diagonal, the mapping $x \mapsto \rho(x, \bar{y})$ is continuous. The result then follows from the intermediate value theorem. $\square$

To construct F, we first define an auxiliary function $f : [1 - \bar{p}, \bar{p}] \to \mathbb{R}$. Let $f(1/2) = 0$. For each $t \neq 1/2$, let $f(t) = [U(x) - U(y)]/\|x - y\|$ for any x, y such that $\rho(x, y) = t$. By Lemma A.11 and Lemma A.12, the function f is well defined. To see that the image of f must be a compact interval in $\mathbb{R}$, take any $x \neq \hat{x}$ with $U(x) = U(\hat{x})$. Then we have $U(\hat{x} + t(x - \hat{x})) - U(\bar{y}) = U(\hat{x}) - U(\bar{y})$ for all $t > 0$ and $\|\hat{x} + t(x - \hat{x}) - \bar{y}\| \geq t\|x - \hat{x}\| - \|\hat{x} - \bar{y}\|$ which goes to infinity when t goes to infinity. Hence $U(\hat{x} + t(x - \hat{x})) / \|\hat{x} + t(x - \hat{x}) - \bar{y}\|$ goes to zero and the image of f is the compact interval $[-T, T]$, where $T = [U(\hat{x}) - U(\bar{y})]/\|\hat{x} - \bar{y}\|$. By Lemma A.11 f is strictly increasing and has an inverse. We let $F = f^{-1}$ be the inverse of f. It immediately follows that $U, \|\cdot\|$ and F are a MEM representation of ρ as in (5). $\square$

Proof of Proposition 3

Let $\bar{x}, \bar{y}, \hat{x}$ be defined exactly as in the proof of Theorem 2. Let $(U, \|\cdot\|, F)$ be a MEM representation of ρ as in (5), and let $\langle \cdot, \cdot \rangle$ be the inner product that induces the norm.

Lemma A.13. *$\langle x - \hat{x}, \hat{x} - \bar{y} \rangle = 0$ for all x with $\rho(x, \hat{x}) = 1/2$.*

Proof. This holds by construction for the particular representation obtained in the proof of Theorem 2, and now we show it holds for every representation. When $x = \hat{x}$ the statement is obviously true. Suppose $x \neq \hat{x}$. By Lemma A.7 $\rho(x, \bar{y}) = \rho(2\hat{x} - x, \bar{y})$. By the representation (5) it must be $\|x - \bar{y}\| = \|2\hat{x} - x - \bar{y}\|$. Hence

$$\begin{aligned}
\|x - \hat{x}\|^2 + 2\langle x - \hat{x}, \hat{x} - \bar{y} \rangle + \|\hat{x} - \bar{y}\|^2 &= \langle x - \bar{y}, x - \bar{y} \rangle \\
&= \langle 2\hat{x} - x - \bar{y}, 2\hat{x} - x - \bar{y} \rangle \\
&= \|x - \hat{x}\|^2 + 2\langle x - \hat{x}, \bar{y} - \hat{x} \rangle + \|\hat{x} - \bar{y}\|^2
\end{aligned}$$

which implies $4\langle x - \hat{x}, \hat{x} - \bar{y} \rangle = 0$ and we are done. □

Lemma A.14. *$\rho(x, \hat{x}) = \rho(x', \hat{x}) = 1/2$ and $\rho(x, \bar{y}) = \rho(x', \bar{y})$ implies $\|x - \hat{x}\| = \|x' - \hat{x}\|$.*

Proof. By the representation (5) we must have $\|x - \bar{y}\| = \|x' - \bar{y}\|$. By Lemma A.13, $\langle x - \hat{x}, \hat{x} - \bar{y} \rangle = \langle x' - \hat{x}, \hat{x} - \bar{y} \rangle = 0$. Thus,

$$\|x - \hat{x}\|^2 + \|\hat{x} - \bar{y}\|^2 = \|x - \bar{y}\|^2 = \|x' - \bar{y}\|^2 = \|x' - \hat{x}\|^2 + \|\hat{x} - \bar{y}\|^2$$

and therefore $\|x - \hat{x}\| = \|x' - \hat{x}\|$ as desired. □

The expected utility function U is unique by the requirement that it is linear and that $U(\Delta) = [0, 1]$. Now suppose both $\|x\|_1 = \sqrt{\langle x, x \rangle_1}$ and $\|x\|_2 = \sqrt{\langle x, x \rangle_2}$ are norms induced by their respective inner products and that each norm together with U represents ρ. Fix a lottery $z \neq \hat{x}$ with $\rho(z, \hat{x}) = 1/2$ and let $A := \|z - \hat{x}\|_1 / \|z - \hat{x}\|_2 > 0$. Now take any x with $U(x) = U(\hat{x})$. To show that $\|\cdot\|_1 = A\|\cdot\|_2$ on the null space of U, it suffices to show that $\|x - \hat{x}\|_1 = A\|x - \hat{x}\|_2$. This clearly holds if $x = \hat{x}$ so suppose $x \neq \hat{x}$. Let $q = \rho(x, \bar{y})$ and $p = \rho(z, \bar{y})$. Lemma A.6 implies $p, q < \bar{p} = \rho(\hat{x}, \bar{y})$. Suppose wlog $p \geq q$. By Lemma A.9 $B(p) = \hat{x} + \lambda[B(q) - \hat{x}]$ for some $0 \leq \lambda \leq 1$. Since $p < \bar{p}$ it must be $0 < \lambda \leq 1$. Then $\rho(\lambda x + (1 - \lambda)\hat{x}, \bar{y}) = p = \rho(z, \bar{y})$. By Lemma A.14 we have

$$\|\hat{x} + \lambda(x - \hat{x}) - \hat{x}\|_1 = \|z - \hat{x}\|_1 = A\|z - \hat{x}\|_2 = A\|\hat{x} + \lambda(x - \hat{x}) - \hat{x}\|_2$$

hence $\lambda\|x - \hat{x}\|_1 = \lambda A\|x - \hat{x}\|_2$ and since $\lambda > 0$ we obtain $\|x - \hat{x}\|_1 = A\|x - \hat{x}\|_2$ as desired.

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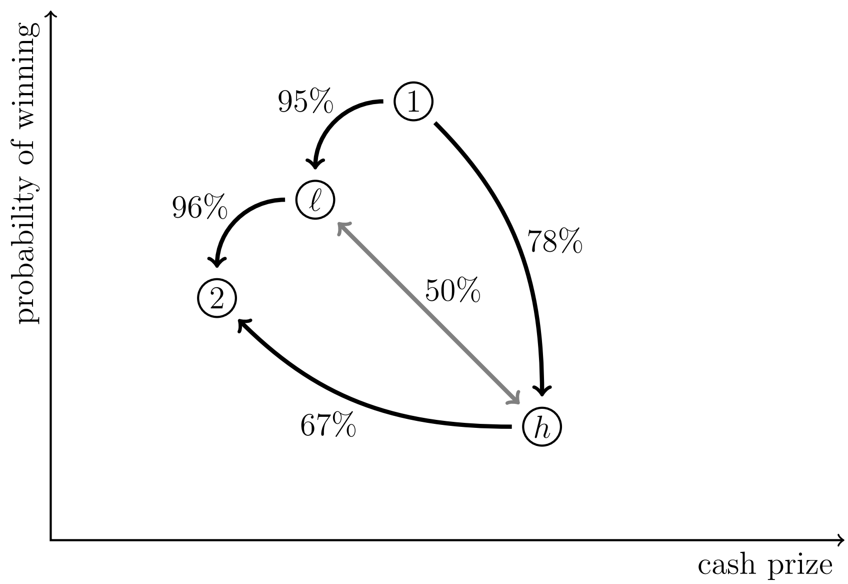

Figure 1: Binary choice frequencies violate SST but satisfy MST in the experimental choice data of Soltani, De Martino and Camerer (2012). Each choice option is a simple lottery (m, p) that pays m dollars with probability p. A high risk lottery h and a low risk lottery ℓ were calibrated for each subject to be chosen equally often.

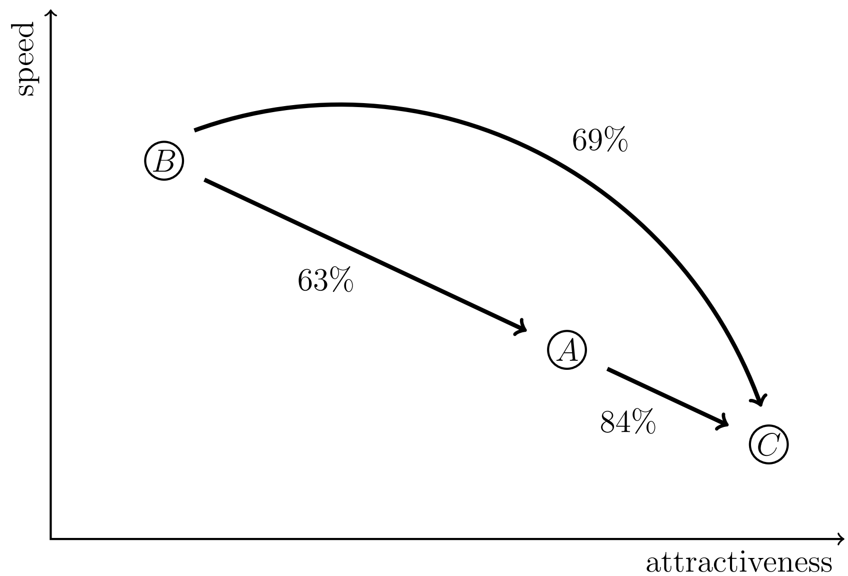

Figure 2: Binary choice frequencies violate SST but satisfy MST in the experimental choice data of Lea and Ryan (2015). Female túngara frogs choose mating partners based on the sound of their call. Three mating options A, B and C were differentiated along two desirable dimensions. The horizontal axis represents a measure of static attractiveness, and the vertical axis represents speed measured in calls per second.

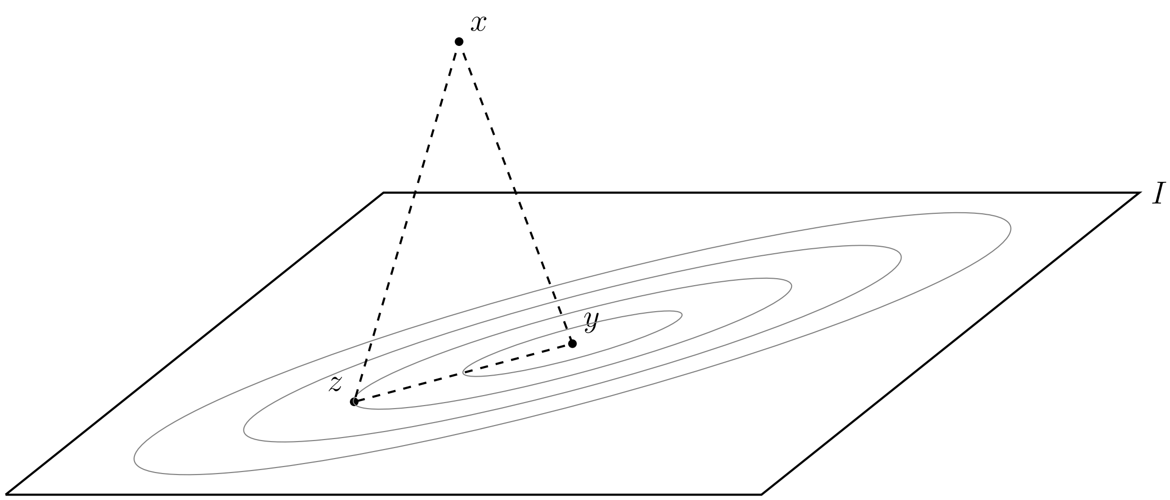

Figure 3: Illustration of the construction of the norm in the proof of Theorem 2. Any two lotteries in the affine subspace I are equally likely to be chosen in binary comparisons. Lottery x is chosen with probability strictly larger than 1/2 against any lottery in I, with maximum choice probability $\rho(x, \cdot)$ obtained at lottery y. The depicted contour sets in I given by $\{z \in I : \rho(x, z) \geq \alpha\}$ are concentric ellipsoids centered at y. We take one of these ellipsoids to be the unit ball that defines the norm on $\ker(U)$, the null space of U, which is a $n - 2$ dimensional subspace parallel to I.

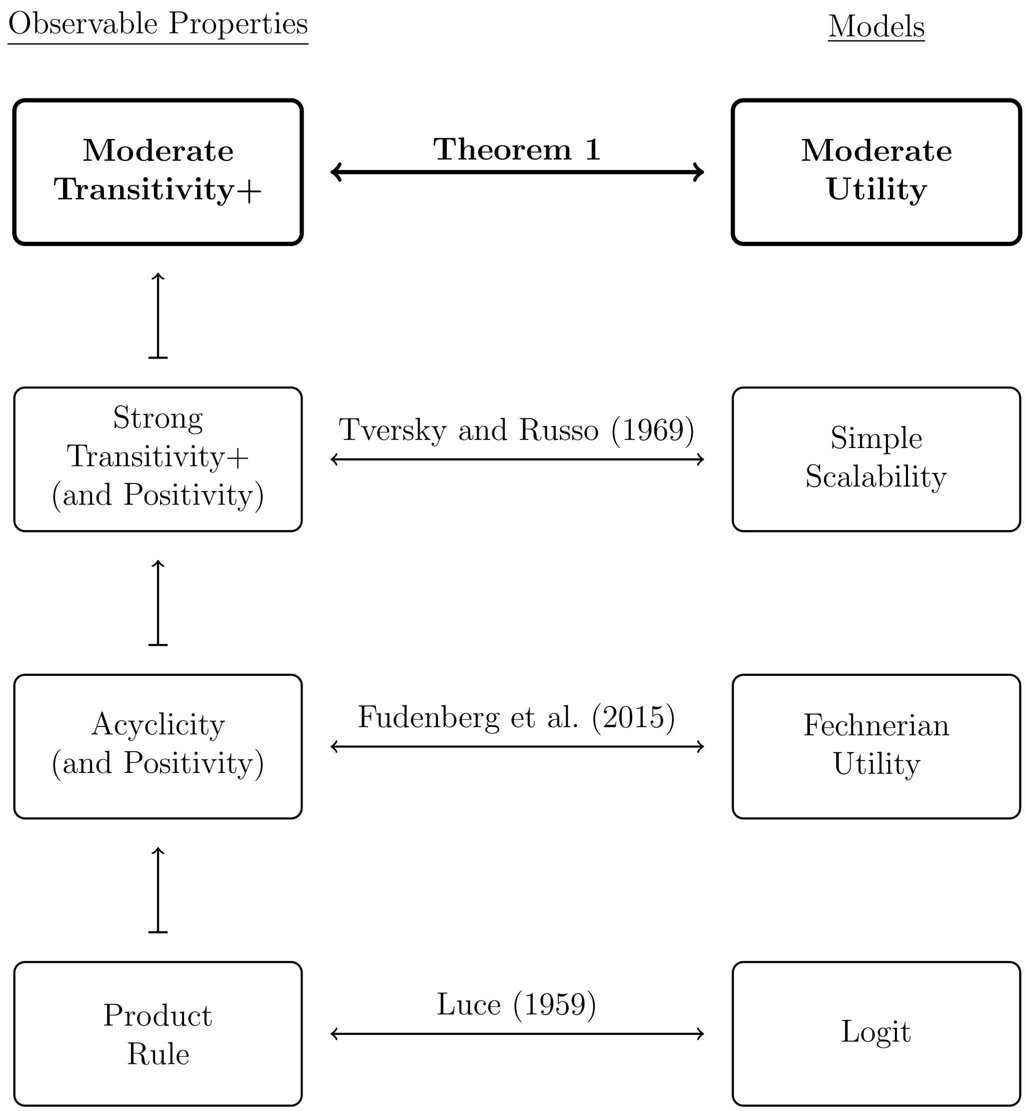

Figure 4: Relationship between models and postulates on choice probabilities for binary stochastic choice over a finite set of options. A double arrow ($\leftrightarrow$) indicates equivalence while an arrow ($\mapsto$) indicates implication in the direction of the arrow and failure of implication in the opposite direction.

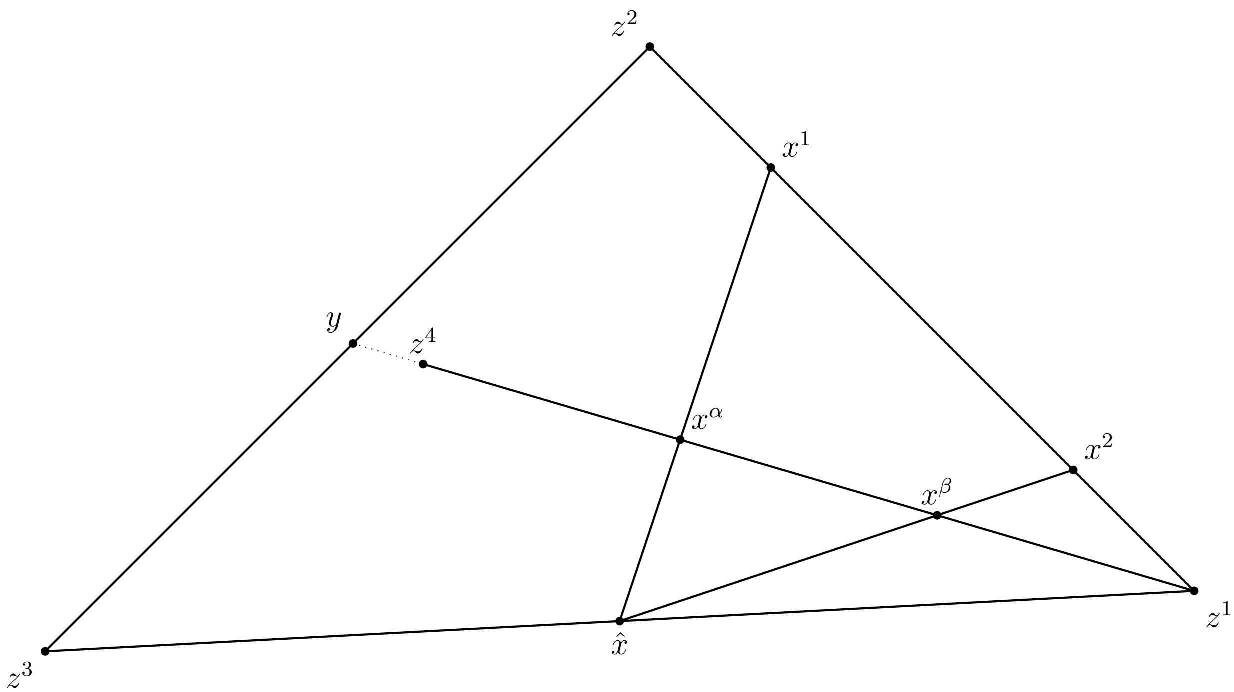

Figure 5: Illustration of the proof of Lemma A.9. All lotteries pictured above are chosen fifty-fifty against each other in binary comparisons. All lotteries pictured above are also chosen with probability strictly larger than one-half against a lottery $\bar{x}$ (not shown). Lotteries x^1 and x^2 are each chosen with probability p against $\bar{x}$. Lotteries $x^\alpha = \alpha x^1 + (1-\alpha)\hat{x}$ and $x^\beta = \beta x^2 + (1-\beta)\hat{x}$ are each chosen with probability q against $\bar{x}$. Since $\alpha < \beta$, the line through x^1 and x^2 crosses the line through x^α and x^β at a point z^1. The convexity postulate implies z^1, z^2, z^3, z^4 must all be chosen against $\bar{x}$ with the same probability r. Lottery z^4 is in the interior of the triangle formed by z^1, z^2, z^3. Lottery y is on the line through z^1 and z^4 and by convexity must be chosen against $\bar{x}$ with probability strictly smaller than r. But lottery y is also a convex combination of z^1 and z^2, and by Lemma A.4 must be chosen against $\bar{x}$ with probability larger or equal to r, a contradiction.