Size-related premiums

THIAGO DE OLIVEIRA SOUZA*

Current Version: June 27, 2017
First Draft: November 18, 2015

ABSTRACT
The size premium only exists in high market price of risk states because ranking stocks by size is only equivalent to ranking them by risk if the differences in risk premiums overcome differences in expected cash flows among them. In these states, the size premium spans the value premium. Consequently, there is no particular risk associated with the book-to-market characteristic of the stocks, which is simply a version of size, scaled by a proxy for expected cash flows. This contradicts the characteristic based explanations of the value premium, including the ones assuming costly adjustments to installed capital.

JEL Classification: G11, G12, G14.
Keywords: Size premium, Value premium, Risk, Conditional, Adjustment costs.

*Department of Business and Economics, University of Southern Denmark, Campusvej 55, 5230 Odense M, Denmark. Email: tsouza@sam.sdu.dk. I would like to thank Stefano Giglio, John Cochrane, Paulo Maio, Jonathan Berk, Christian Flor, Linda Larsen, Francisco Gomes, Jaime de Jesus, and the participants in the World Finance Conference 2016.
Several empirical studies describe stocks in terms of some characteristics that are supposedly related to risk premiums.\(^1\) Two of these characteristics stand out following Fama and French (1996): One is the market value of equity in the firm (size). The second is the book-to-market ratio (BM), which is size divided by the book value of equity (BE). These characteristics have been linked to the size and to the value risk premiums, respectively, often overlooking how volatile the characteristics are relative to the frequency at which we expect the risk of the stocks to vary. For example, the link between risk and a price-related characteristic cannot be constant unless the risk of the stocks change every time their prices change. Thus, it can be problematic to determine this link without understanding the mechanism behind it.

The apparent association between the BM characteristic and the risk of the stocks, for example, gives empirical support to the idea that firms become riskier when they increase their tangible assets. A prolific literature on adjustment costs is broadly based on this assumption:\(^2\) Tangible assets in place (and their variants, such as fixed operating costs) are claimed to be risky because reverting an investment decision is costly once the physical capital is installed and also because this tends to happen in bad states of the economy. Considering the BE as a proxy for installed capital connects this theory to the BM characteristic of the firms. Therefore, observing a positive relation between the BM and the risk premiums in the data confirms the ideas in these models.

This paper has three goals. The first is to establish the link between the risk of the stocks and their size-related characteristics as a function of the market conditions. The second is to understand which risks are associated with the size and with the value characteristics. The third is to investigate whether the BE carries any risk information in the way predicted by the literature on costly adjustment to the capital stock.

\(^1\)See, for example, Harvey, Liu, and Zhu (2016).

I construct a theoretical framework in which the most restrictive assumption is the one of a time-varying market price of risk.\footnote{There is a myriad of models based on different assumptions that generate time variation in the market price of risk. Examples of these models include Campbell and Cochrane (1999), Shiller (2014), Rietz (1988), Barro (2006), Piazzesi, Schneider, and Tuzel (2007), Brunnermeier (2009), Hansen and Sargent (2001), Epstein and Zin (1989), Bansal and Yaron (2004), Bansal, Kiku, and Yaron (2012), Constantinides and Duffie (1996), and Garleanu and Panageas (2015).} This implies that risk and size are only related when the market price of risk is high, as the empirical evidence in Souza (2017) suggests. The intuition is similar to Berk (1995): Given two stocks with the same expected cash flows, the riskier stock has a higher required return and therefore a lower market value. So this relates small stocks to large risk premiums. But for firms with different expected cash flows, the size ranking only aligns with the risk ranking if the difference in required premiums is sufficiently large to overcome the difference in expected cash flows and dominate the size ranking. Hence, there must be a threshold for the market price of risk above which the size and the risk rankings become aligned in general.

Establishing the link between risk and the BM characteristic is less straightforward because it also involves determining the risks associated with the BE. In this respect, there are two possibilities: The BE is either a proxy for expected cash flows (Berk, 1995), or the BE is related to some risks, which could be real frictions on the firms’ investments (Zhang, 2005), for example. I test these two hypotheses.

In summary, if the BE is related to some specific risks, the value ranking should always capture these risks while the size ranking should not, regardless of the prevailing market price of risk. Alternatively, the BE may simply help aligning the value ranking with the risk ranking by controlling for the cross-sectional variation in expected cash flows (Berk, 1995). Given that controlling for cash flows is unnecessary when the market price of risk is high (because the size ranking already aligns with the risk ranking), the value ranking should only capture additional risks compared to the size ranking when the market price of risk is low under this second hypothesis.
The empirical tests reveal the absence of any premium associated with value that is not also (potentially) associated with size: The size premium spans the value premium in high market price of risk states. There is no excess return associated with the HML portfolio of Fama and French (1996) after controlling for its exposure to the SMB portfolio, and the correlation between the returns on the two portfolios is highly significant in these states. The difference is that the mean return on the HML portfolio is also positive in low market price of risk states. In these states, the mean return on the SMB portfolio is zero, there is zero correlation between the return on the two portfolios, and the positive premium associated with the HML portfolio is unexplained by the return on the SMB portfolio.

Finally, the conclusion from the findings above is that the BE and other scaling variables are proxies for expected cash flows and are unrelated to the risk of the stocks. In particular, I find no empirical support for the idea that the risks associated with adjustments to installed capital are priced in equilibrium.

In fact, this paper is not the first to reject the hypothesis that the BM characteristic is responsible for the value premium in favour of a risk based explanation. In a different context, Davis, Fama, and French (2000), replying to Daniel and Titman (1997), also reject the hypothesis that differences in the BM characteristic among the stocks generate the value premium. More recently, Chordia, Goyal, and Shanken (2015) also contribute to this literature.

This paper is also part of a long literature on the relation between stock characteristics and premiums in cross section. An incomplete list includes Banz (1981), Jegadeesh and Titman (1993), Fama and French (1996, 2015, 2016), Li, Livdan, and Zhang (2009), Novy-Marx (2013), Hou, Xue, and Zhang (2015), Harvey et al. (2016), Green, Hand, and Zhang (2017), and Light, Maslov, and Rytchkov (2017).

---

I use the state variables in Souza (2017) as proxies for the market price of risk: The median BM of all CRSP stocks (MBM), the value weighted BM of all CRSP stocks (VBM), the Dow Jones’ BM of Pontiff and Schall (1998) (DJBM), the earnings-price ratio of all S&P Composite stocks (EP), the term spread (TMS), the default spread (DFY) and the Treasury bill rate (TBL).

is to show that the value and the size premiums do not correspond to two independent risk dimensions as previously assumed. In addition, the paper provides a theoretical explanation for the conditional size premium that is empirically documented in Souza (2017).

I. Theoretical framework

Let $\zeta = (\zeta_t)$ be the unique stochastic discount factor (SDF) that follows the continuous-time stochastic process

$$d\zeta_t = -\zeta_t [r^f_t \, dt + \lambda_t \, dz_{1t}], \quad (1)$$

where $dz_{1t}$ is a one-dimensional standard Brownian motion and the stochastic processes, $r^f_t$ and $\lambda_t$, represent the risk free rate and the market price of risk process at time $t$, respectively.

Let $P_i = (P_{it})$ be the price process of portfolio $i$, such that

$$dP_{it} = P_{it} [\mu_{it} \, dt + \sigma_{it} \, dz_{1t} + \tilde{\sigma}_{it}^\top \, dz_t], \quad (2)$$

where $\mu_{it}$ and $\sigma_{it}$ are one-dimensional stochastic processes, $dz_t$ is a multi-dimensional standard Brownian motion independent of $dz_{1t}$, $\tilde{\sigma}_{it}$ is a multi-dimensional stochastic process, and $\top$ is the transposition sign. $\mu_{it}$ and $\sigma_{it}$ represent, respectively, the expected returns and the sensitivity of the returns on the portfolio to the exogenous (priced) shocks to the economy. So $\sigma_{it}$ gives the effective risk of the portfolio. $\tilde{\sigma}_{it}$ represents the unpriced return volatility of the portfolio (the sensitivity of the returns on the portfolio to the exogenous unpriced shocks). Without intermediate dividends, the expected excess rate of return on the portfolio is

$$\mu_{it} - r^f_t = \sigma_{it} \lambda_t. \quad (3)$$
So the overall market price of risk, $\lambda_t$, and the risk of the portfolio, $\sigma_{it}$, combined determine the expected returns on the portfolio at time $t$. In particular, by construction of the SMB portfolio and considering Eq. (3), we have

$$\sigma_{smb,t} = \sigma_{small,t} - \sigma_{big,t},$$  \hspace{1cm} (4)

$$\mu_{smb,t} = \sigma_{smb,t} \lambda_t,$$  \hspace{1cm} (5)

where $\sigma_{small,t}$ and $\sigma_{big,t}$ are, respectively, the risks of the portfolios of small stocks and big stocks, $\sigma_{it}$ in Eq. (2), and $\mu_{smb,t}$ is the expected return on the SMB portfolio at time $t$. Equivalently, for the HML portfolio we have

$$\sigma_{hml,t} = \sigma_{value,t} - \sigma_{growth,t},$$  \hspace{1cm} (6)

$$\mu_{hml,t} = \sigma_{hml,t} \lambda_t,$$  \hspace{1cm} (7)

where $\sigma_{value,t}$ and $\sigma_{growth,t}$ are, respectively, the risks of the portfolios of value stocks and growth stocks at time $t$, and $\mu_{hml,t}$ is the expected return on the HML portfolio at time $t$.

\section{Modelling the risk of the size-related portfolios}

For tractability, assume that the risk-free rate and the market price of risk are constant between time $t$ and time $T$. So the SDF follows a one-dimensional geometric Brownian motion process,

$$d\zeta_t = -\zeta_t [r^f dt + \lambda_t dz_t],$$  \hspace{1cm} (8)

where the constants $\lambda_t$ and $r^f$ are, respectively, the market price of risk and risk free rate prevailing from time $t$ to time $T$.

As in Eq. (2), let $P_i = (P_{it})$ be the price process of the equity in the firm $i$. The firm makes only a final lump sum dividend payment, $D_{i,T}$, at time $T$, which is modelled
through \( x_{i,t} = E_t[D_{i,T}] \) following the process

\[
dx_{i,t} = x_{i,t} \left[ \sigma_{it} dt + \tilde{\sigma}_{1t} dz_{1t} + \tilde{\sigma}_{\tau} d\tilde{z}_t \right],
\]

where \( \sigma_{it} \) is a (one-dimensional) constant and \( \tilde{\sigma}_{\tau} \) is a (multi-dimensional) constant. Under these conditions, the time \( t \) value of the payoff \( D_{i,T} \) is

\[
P_{it} = E_t \left[ D_{i,T} \zeta_T \right] = E_t[D_{i,T}]e^{-\left(r_f + \tilde{\sigma}_{it} \lambda_t\right)(T-t)},
\]

which depends positively on the expected payoff, \( E_t[D_{i,T}] \), and negatively on the risk of the payoff, \( \sigma_{it} \), and on the market price of risk, \( \lambda_t \), (apart from the risk free rate, \( r_f \), and the time interval, \( T - t \)).

A.1. The size ranking

Eq. (10) summarizes the framework in Berk (1995): Given two firms with the same expected cash flows, the riskiest one, with the largest \( \sigma_{it} \), has the lowest market value, \( P_{it} \). Therefore, there is an association between a price ranking, based on \( P_{it} \), and a risk ranking, based on \( \sigma_{it} \). However, the price ranking is only imperfectly related to the risk ranking because the price also depends on the expected cash flows, \( E_t[D_{i,T}] \).

As a consequence from Eq. (10), there must be a minimum market price of risk, \( \lambda^* \), so that the price ranking and the risk ranking align. Intuitively, the risk premium must be large enough to overcome differences in expected cash flows among the firms, so that risk is the determinant of the differences in price for a large number of stocks. This translates into the first rejectable prediction of the model:

Hypothesis 1 (H1): The size premium is positive if and only if the market price of risk is

\[^6\text{Considering that the market portfolio is not on the mean-variance frontier would, thus, explain the residual CAPM excess return associated with size.}\]
above a certain threshold, $\lambda_t \geq \lambda^*$, and is zero otherwise.

In terms of Eq. (4) and Eq. (5),

\[
\sigma_{smb,t} = \begin{cases} 
0 & \lambda_t < \lambda^* \\
\lambda_t \geq \lambda^* 
\end{cases} \quad \Rightarrow \quad \mu_{smb,t} = \begin{cases} 
0 & \lambda_t < \lambda^* \\
f(\lambda_t)\lambda_t > 0 & \lambda_t \geq \lambda^* 
\end{cases}
\]

(11)

where $f(\lambda_t)$ is a non-decreasing function of the market price of risk, $\lambda_t$.

**More details about the market price of risk threshold, $\lambda^*$:** Consider a pair of stocks. Stock $r$ is riskier, with volatility term $\sigma_{rt}$, and stock $s$ is safer, with volatility term $\sigma_{st}$, such that

$$\sigma_{rt} > \sigma_{st}.$$  

(12)

The price ranking and the risk ranking are aligned for this pair of stocks if and only if the price of the risky stock is smaller than the price of the safe stock, $P_{r,t} < P_{s,t}$. In terms of the parameters in Eq. (10),

$$P_{r,t} < P_{s,t} \iff \ln \left( \frac{E_t[D_{r,T}]}{E_t[D_{s,T}]} \right) \frac{1}{(\sigma_{rt} - \sigma_{st}) (T - t)} \equiv \lambda^*,$$  

(13)

where $\lambda^*$ is actually specific to this pair of stocks, but I assume it to be the same for other pairs of stocks to simplify the argument.

Under the assumption that risk and expected cash flows are unrelated, the probabilities that a given riskier firm has either larger or smaller expected cash flows than a safer firm are the same:

$$P \left( \frac{E_t[D_{r,T}]}{E_t[D_{s,T}]} \leq 1 \right) = P \left( \frac{E_t[D_{r,T}]}{E_t[D_{s,T}]} > 1 \right).$$  

(14)

Thus, a large portfolio of small stocks should only contain a disproportionately large
number of risky stocks, so that \( \sigma_{\text{small},t} > \sigma_{\text{big},t} \), if the market price of risk is above the threshold in Eq. (13), \( \lambda_t > \lambda^* \). Otherwise, the differences in expected cash flows dominate the price ranking and the portfolios based on this ranking have the same risks, \( \sigma_{\text{small},t} = \sigma_{\text{big},t} \).

A.2. The scaled-price rankings

Consider the same pair of stocks, but ranked by a scaled-price ratio (SP). The SP ranking is aligned with the risk ranking if and only if the SP of the risky stock is smaller than the SP of the safe stock,

\[
\frac{P_{r,t}}{B_{r,t}} < \frac{P_{s,t}}{B_{s,t}},
\]

where the scaling variable for firm \( i \) at time \( t \), \( B_{i,t} \), divides the market value calculated in Eq. (10) for each stock. In terms of the parameters, the equivalent of Eq. (13) is now

\[
\frac{P_{r,t}}{B_{r,t}} < \frac{P_{s,t}}{B_{s,t}} \iff \overline{\lambda}_t > \ln \left( \frac{E_t[D_{r,T}]}{E_t[D_{s,T}]} \right) \frac{1}{(\bar{\sigma}_{rt} - \bar{\sigma}_{st}) (T - t)} \equiv \lambda_{\text{SP}}^*.
\]

This threshold is smaller than the one in Eq. (13), \( \lambda_{\text{SP}}^* < \lambda^* \), as long as the scaling variable for the risky stock is larger than the one for the safe stock, \( B_r > B_s \). In particular, the SP ranking aligns with the risk ranking for any (positive) market price of risk if the scaling variable is equal to the expected cash flow, \( B_{i,t} = E_t[D_{i,T}] \), so we have \( \overline{\lambda}_t > \lambda_{\text{SP}}^* = 0 \).

B. The BE as a proxy for expected cash flows?

Let us assume that the BE is a good proxy for the expected cash flows of at least some of the firms and use it as a scaling variable,

\[
B_{i,t} \equiv BE_{i,t} \approx E_t[D_{i,T}].
\]
This creates the price-to-book (PB) ratio that generates the HML portfolio.\textsuperscript{7} Under this assumption, there is a range for the market price of risk in which the SP ranking aligns with the risk ranking, while the price ranking does not, $\lambda^* > \lambda_t > \lambda_{SP}^*$. And depending on how well the BE proxies for the expected cash flows, the threshold in Eq. (16) can be zero, $\lambda_{SP}^* = 0$.

Hence, a large portfolio of value stocks (with low PB) should contain a disproportionately large number of risky stocks, so $\sigma_{\text{value},t} > \sigma_{\text{growth},t}$, even if the prevailing market price of risk is lower than the one necessary to generate the size premium. The equivalent to Eq. (11) for $\lambda_{SP}^* = 0$, in terms of Eq. (6) and Eq. (7), is

\[
\sigma_{\text{hml},t} = \begin{cases} 
  f(\lambda_t) > 0 & \lambda_t < \lambda^* \\
  f(\lambda_t) > 0 & \lambda_t \geq \lambda^*
\end{cases} \quad \Rightarrow \quad \mu_{\text{hml},t} = \begin{cases} 
  f(\lambda_t)\lambda_t > 0 & \lambda_t < \lambda^* \\
  f(\lambda_t)\lambda_t > 0 & \lambda_t \geq \lambda^*
\end{cases},
\]

which is the mathematical representation of the second testable prediction of the model:

Hypothesis 2 (H2): (Assuming that the BE is a proxy for expected cash flows). The value premium increases with the market price of risk, being small, but still positive, even if the market price of risk is lower than the threshold below which the size premium is zero, $\lambda_t \leq \lambda^*$.

C. Can the BE carry risk information about the stock?

The PB ranking can also capture some specific risks that the price ranking does not capture if the BE is related to these specific risks. For example, the BE can be a proxy for the installed capital in the firm which is risky according to the framework in Zhang (2005). In order to analyse this question, I must relax the assumption that the

\textsuperscript{7}The conclusions are similar for other scaling variables, such as dividends, earnings, or cash flows. This is expected because the value premium spans several other premiums associated with SP, as shown in Fama and French (1996).
fundamental uncertainty in the economy is modelled by the one-dimensional exogenous shock in Eq. (1).\(^8\) Otherwise there cannot be additional risks for the scaling variables to capture. In a multi-dimensional risk setting, the equivalents of Eq. (5) and Eq. (7) are

\[
\begin{align*}
\mu_{smb,t} &= \sigma_{smb,t}^\top \lambda_t, \\
\mu_{hml,t} &= \sigma_{hml,t}^\top \lambda_t,
\end{align*}
\]

(19) \hspace{1cm} (20)

where \(\lambda_t\) is the multi-dimensional market price of risk process, and \(\sigma_{smb,t}\) and \(\sigma_{hml,t}\) are the multi-dimensional stochastic sensitivities of the returns on the SMB and on the HML portfolios to the (independent) exogenous priced shocks to the economy, respectively.

Now let the \(k^{th}\) element in \(\sigma_{smb,t}\) be equal to zero, while being different from zero in \(\sigma_{hml,t}\). This means that the PB ranking captures this risk, but the price ranking does not. Consequently, it is impossible to find a constant, \(a\), that multiplied by the risk of the SMB portfolio, \(\sigma_{smb,t}\), gives the risk of the HML portfolio, \(\sigma_{hml,t}\):

\[\nexists a \in \mathbb{R} \mid \sigma_{hml,t} = a \sigma_{smb,t}.\]

(21)

And therefore, it is also impossible to represent the expected return on the HML portfolio as a multiple of the expected return on the SMB portfolio:

\[\nexists a \in \mathbb{R} \mid \mu_{hml,t} = a \sigma_{smb,t}^\top \lambda_t = a \mu_{smb,t}.\]

(22)

If the BE is related to risk, then Eq. (22) should hold for every market price of risk, \(\lambda_t\). Alternatively, if the BE is simply a proxy for expected cash flows, Eq. (22) should still hold when the market price of risk is low because, according to Eq. (11), the size premium is zero in these states. But a constant, \(a\), could exist when the market price of risk is low.

\(^8\)Appendix A shows the adjustments to Eq. (1) and to Eq. (2) that give the results in this section.
risk is high, $\lambda_t > \lambda^*$. Thus, testing this hypothesis empirically is equivalent to testing whether such constant, $a$, exists depending on the market price of risk, $\lambda_t$:

Hypothesis 3 (H3): (Assuming that the BE is a proxy for expected cash flows, and is not related to the risk of the stock). The value premium should be a multiple of the size premium, $\mu_{hml,t} = a\mu_{smb,t}$, if and only if the market price of risk is high, $\lambda_t > \lambda^*$.

II. Empirical section

There are three main steps in order to test the three hypotheses above. The first is to find a proxy for the market price of risk, the second is to find the threshold above which the size ranking aligns with the risk ranking, and the third is to finally investigate the conditional relation between the size and the value premiums.

A. Data and variables

A.1. The market price of risk proxy

I consider the typical ICAPM state variables related to the size premium in Souza (2017) as proxies for the market price of risk. Souza (2017) shows that these variables forecast positive market conditions. This gives them a market price of risk interpretation.

The state variables that I calculate from the data in Kenneth French’s data library (time span in brackets) are the median BM of all CRSP stocks (MBM, 1926–2015) and the value weighted BM of all CRSP stocks (VBM, 1926–2015). The variables that I obtain from Goyal’s website are the Dow Jones’ BM of Pontiff and Schall (1998) (DJBM, 1921–2015), the earnings-price ratio (EP, 1871–2015), the term spread (TMS, 1920–2015), the

---

Footnotes:

9 In fact, the constant $a$ only exists in general if the size and the value rankings have the same relative exposures to the different priced shocks to the economy.

10 [http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html](http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html).
default spread (DFY, 1919–2015), and the Treasury bill rate (TBL, 1920–2015). The TBL actually forecasts negative market conditions. So I subtract it from one to obtain the same market price of risk interpretation as the other variables. The values of the state variables in year $t$ correspond the end of June of year $t$.

Table I presents the same summary statistics in Souza (2017) for the state variables, assuming that they follow a first order AR process. The state variables tend to be persistent but are stationary in line with the fact that they are essentially ratios. The correlations between innovations in the variables are generally weak. Fig. 1, also from Souza (2017), shows the evolution of the state variables over time between 1926 and 2015.

A.2. The backward looking state classification

I classify the market price of risk in each year in high or low based on how the state variable in that year compares to the variable’s own past values. This ranking is free from forward looking bias and is the exact same one implemented in Souza (2017) that also describes it in more details.

Given the historical mean of each state variable in year $t$, $\bar{z}_t$, we can calculate the difference between the value of $z_t$ and its historical mean estimated until the previous year,

$$Dev_{z,t} = z_t - \bar{z}_{t-1}. \tag{23}$$

Next, we calculate a percentile rank, $\Gamma_{z,t}$, based on how the deviation at time $t$, $Dev_{z,t}$,}

\footnote{Welch and Goyal (2008) describe these variables in more details. The variables are at http://www.hec.unil.ch/agoyal/.}
compares with the past deviations until time $t$,

$$\Gamma_{z,t} = \sum_{i=t_0}^{t} \left( I_{\{\text{Dev}_{z,i} < \text{Dev}_{z,t}\}} + 0.5 I_{\{\text{Dev}_{z,i} = \text{Dev}_{z,t}\}} \right) \frac{t - t_0 + 1}{t - t_0}$$

(24)

where $I_{\{ \cdot \}}$ is the indicator function. Intuitively, the ranking $\Gamma_{z,t}$ is large when the market price of risk is unusually far above its historical average.

Table II presents summary statistics for the percentile rankings, still assuming that they follow a first order AR process. The interpretation is similar to the one for the original state variables in Table I.

A.3. The stock returns

I obtain the return data on US stocks, described in details in Fama and French (1993), from Kenneth French’s data library. The monthly returns from July of 1926 to July of 2015 correspond to the size premium (as the return on the SMB portfolio, $R_{smb}$), the value premium (as the return on the HML portfolio, $R_{hml}$), and the market premium (as the difference between the return on the market portfolio and the risk free rate, $R_{mp}$).

B. Hypotheses tests

B.1. The conditional size premium (H1)

According to the model, the size premium should only be positive if the market price of risk is above a certain threshold, $\lambda_t \geq \lambda^*$. The biggest empirical challenge is that we do not observe the agents’ private information sets. So the market price of risk is also unobservable.
Let the (observable) threshold for the state variable, \( \Lambda^* \), be a proxy for \( \lambda^* \). The time \( t \) expectation of the size premium one period ahead is now

\[
E_t[R_{smb,t+1}] = \begin{cases} 
0 & \Lambda_t < \Lambda^* \\
R_{smb,h} & \Lambda_t \geq \Lambda^*
\end{cases},
\]

where \( \Lambda_t \) is the state variable at time \( t \), \( \Lambda^* \) is the threshold for this state variable, and \( R_{smb,h} \) is the mean return on the SMB portfolio considering only the periods in which the state variable is high (above its threshold):

\[
R_{smb,h} = \frac{\sum_{t=1}^{T} (R_{smb,t} I_{\Lambda_t \geq \Lambda^*})}{\sum_{t=1}^{T} I_{\Lambda_t \geq \Lambda^*}},
\]

where \( I_{\{\}} \) is the indicator function and \( T \) is the sample size.

The mean squared error of this forecast (MSE), which is a function of the threshold, \( \Lambda^* \), is

\[
MSE(\Lambda^*) = \frac{1}{T} \sum_{t=1}^{T} (E_t[R_{smb,t}] - R_{smb,t})^2,
\]

where \( E_t[R_{smb,t}] \) is given by Eq. (25). Figure 2 shows the MSE associated with the possible thresholds, \( \Lambda^* \), for each state variable.

My first choice for \( \Lambda^* \) is exactly the one that minimizes the MSE in Eq. (27), denoted by

\[
\Lambda^*_{MSE} \equiv \arg \min_{\Lambda^*} MSE(\Lambda^*).
\]

This choice generates the values in Table III, in which Panel A is in line with the findings of Souza (2017): The size premium is significant exclusively when the market price of risk is high, especially for the variables related to aggregate BM ratios. These results also confirm hypothesis H1 if we assume that the state variables are proxies for the market price of risk.
B.2. The conditional value premium (H2)

Under the assumption that the BE is a proxy for expected cash flows, the model also predicts that the value premium should increase with the market price of risk, but it could still be positive, even if the market price of risk is lower than the threshold below which the size premium is zero, \( \lambda_t \leq \lambda^* \).

In this respect, the evidence in Table III, Panel B, shows a significant value premium when the market price of risk is low, in line with hypothesis H2. The mean value premium also has larger point estimates when the market price of risk is high according to most state variables. This might also confirm hypothesis H2. However, the estimates contain a lot of noise and are statistically insignificant.

It would be interesting to choose \( \Lambda^* \) so that the value premium is also significant when the market price of risk is high, \( \Lambda_t \geq \Lambda^* \). So let us define the equivalent to Eq. (26) for the value premium,

\[
\overline{R}_{hml,h} = \frac{\sum_{t=1}^{T} (R_{hml,t} \cdot I_{\Lambda_t \geq \Lambda^*})}{\sum_{t=1}^{T} I_{\Lambda_t \geq \Lambda^*}},
\]

where \( R_{hml,t} \) and \( \overline{R}_{hml,h} \) are, respectively, the time \( t \) return on the HML portfolio and its mean exclusively when the market price of risk is high.

My second choice for the threshold, denoted by \( \Lambda^*_{HML} \), is the highest value for the state variable that generates a mean value premium two standard errors above zero:

\[
\Lambda^*_{HML} \equiv \arg \max_{\Lambda^*} \Lambda^* \quad \text{r.t.} \quad t(\overline{R}_{smb,h}) \geq 2, \quad t(\overline{R}_{hml,h}) \geq 2,
\]

where \( t(\overline{R}_{i,h}) \) are the t-statistics of the estimated means in Eq. (26) and Eq. (29). Figure 3
and Figure 4 show, respectively, how the t-statistics of the mean size and value premiums change with different thresholds, $\Lambda^*$. Effectively, this second threshold, $\Lambda^*_{HML}$, is lower than the first one, $\Lambda^*_{MSE}$. Therefore, the sub sample of years in which the market price of risk is classified as high is larger according to this second classification.

This choice of threshold generates the values in Table VI, in which Panel A is again consistent with hypothesis H1. Panel B confirms both that the mean value premium is significant when the market price of risk is low and also that its point estimate tends to be larger in high market price of risk states. This is consistent with hypothesis H2. However, we cannot statistically reject the hypothesis that the value premium has the same mean in both states.

[Place Table VI about here]

B.3. The BE carries no risk information (H3)

Under the assumption that the BE carries no risk information about the stocks, and is simply a proxy for expected cash flows, the model predicts that the value premium should be a multiple of the size premium, $\mu_{hml,t} = a\mu_{smb,t}$, as long as the market price of risk is high enough, $\lambda_t > \lambda^*$. We can test this hypothesis empirically by considering the spanning regression

$$R_{hml,t} = \alpha_{hml} + s_{hml}R_{smb,t} + \varepsilon_{hml,t},$$

(31)

where $R_{hml,t}$ is the return on the HML portfolio at time $t$, $\alpha_{hml}$ is the constant intercept, $s_{hml}$ is the constant coefficient on the return on the SMB portfolio at time $t$, $R_{smb,t}$, and $\varepsilon_{hml,t}$ is the error term.
Under hypothesis H3, the predicted coefficients for this regression are

\[
\alpha_{hml} \begin{cases} 
> 0 & \lambda_t < \lambda^* \\
= 0 & \lambda_t \geq \lambda^* 
\end{cases} \quad \text{and} \quad s_{hml} \begin{cases} 
= 0 & \lambda_t < \lambda^* \\
> 0 & \lambda_t \geq \lambda^* 
\end{cases}.
\]  

(32)

The restriction on the intercept, \( \alpha_{hml} \), in high market price of risk states means that there is no premium associated with the value portfolios that is not associated with the size portfolios. And the restriction on the size premium coefficient, \( s_{hml} \), means that the expected premiums associated with the value and the size rankings are significantly positively correlated (given that they arise in response to the same underlying risks). But this only happens when the size ranking aligns with the risk ranking as given by Eq. (11).\(^{12}\) Otherwise, the size ranking captures no priced risks, there should be no correlation between the expected size and value premiums, and the (excess) value premium should be completely unexplained by the size premium.

The exposure to market risk can also affect the correlation between the returns on the size and the value portfolios. Including the market premium in Eq. (31) gives

\[
R_{hml,t} = \alpha_{hml} + \beta_{hml} R_{mp,t} + s_{hml} R_{smb,t} + \varepsilon_{hml,t},
\]  

(33)

where \( \beta_{hml} \) is the constant coefficient on the market premium, \( R_{mp,t} \). The predicted coefficients are the same as before, in Eq. (32), but now controlling for the market risk as well.

This exercise is more informative in the presence of a significant value premium, that I obtain with the threshold \( \Lambda^*_{HML} \) in Eq. (30). The results, in Table VII, strongly support the model’s predictions regarding the coefficients in Eq. (32). Panel A shows that the intercept, \( \alpha_{hml} \), is small and insignificant for every state variable when the market risk is large enough, \( \| \lambda_t \| > \| \lambda^* \| \).

---

\(^{12}\)In a multivariate setting, this means that the length of the market price of risk vector is large enough, \( \| \lambda_t \| > \| \lambda^* \| \).
price of risk is high (except for the TBL without controlling for the market risk). The correlation between the size and the value premiums are also highly significant for every state variable, except the TBL after controlling for the market risk. Panel B confirms the predictions for low market price of risk: Zero correlation between the size and the value premiums for any state variable, except for the DJBM, and an excess value premium which is unexplained by the size premium for every state variable.

[Place Table VII about here]

Furthermore, even the different results for the TBL and the DJBM seem to arise because I use a low threshold for the market price of risk, $\Lambda^*_{HML}$, to obtain a significant value premium. Table IV contains the results for the threshold that minimizes the MSE of the size premium estimation instead, $\Lambda^*_{MSE}$. The only coefficient in this table that is not exactly in line with the predictions of the model is the insignificant size premium coefficient, $s_{hml}$, when the market price of risk is high according to the TBL state variable. And this only happens after controlling for market risk.

Hence, I find no empirical support for the idea that a firm’s BE carries any information about its risk. In particular, firms with large shares of tangible capital are not riskier than firms that rely on intangible capital according to the data, at least with the BE as a proxy for tangible capital. This contradicts the theoretical predictions in the literature on costly adjustments to the capital stock.

[Place Table IV about here]

B.4. Are there risks associated with size but not with value?

It is also possible that the price ranking captures specific risks that the value ranking does not capture. Under the hypothesis that the BE is a proxy for expected cash flows, the value ranking can only possibly improve the price ranking when the market price of
risk is low. If Eq. (13) holds, then any ranking different from the size ranking is misaligned with the risk ranking.

Intuitively, any adjustment remotely related to the expected cash flows helps if the market price of risk is low because the price ranking is mostly unrelated to risk in this case. But a less than perfect adjustment to the price ranking should disturb the alignment with the risk ranking in case this alignment was already good. The proportion of risky firms in the portfolio of small stocks is already very high if the market price of risk is very large. Therefore, reshuffling the ranking by some variable that is only an imperfect proxy for expected cash flows is likely to make the distribution of risk more uniform among the (value) portfolios.

The equivalent of Eq. (22) for the size premium can be tested with the spanning regression

\[
R_{smb,t} = \alpha_{smb} + h_{smb} R_{hml,t} + \varepsilon_{smb,t}, \tag{34}
\]

where \(R_{smb,t}\) is the return on the SMB portfolio at time \(t\), \(\alpha_{smb}\) is the constant intercept, \(h_{smb}\) is the constant coefficient on the return on the HML portfolio at time \(t\), \(R_{hml,t}\), and \(\varepsilon_{smb,t}\) is the error term. Adding the market premium to the equation, we have

\[
R_{smb,t} = \alpha_{smb} + \beta_{smb} R_{mp,t} + h_{smb} R_{hml,t} + \varepsilon_{smb,t}, \tag{35}
\]

where \(\beta_{smb}\) is the constant coefficient on the market premium.

Under the assumptions that the BE carries no information about risk, the predicted coefficients for this regression are

\[
\begin{align*}
\alpha_{smb} & = 0 \quad \lambda_t < \lambda^* \\
& \geq 0 \quad \lambda_t \geq \lambda^*
\end{align*}
\quad \text{and} \quad
\begin{align*}
h_{smb} & = 0 \quad \lambda_t < \lambda^* \\
& > 0 \quad \lambda_t \geq \lambda^*.
\end{align*}
\tag{36}
\]
The interpretation of the coefficients are similar to the ones in Eq. (32). In low market price of risk states, the size ranking captures no priced risks. Therefore, there is no correlation between the mean return on the size and on the value portfolios and the coefficient on the value premium, $h_{smb}$, should be zero. Given that the mean size premium is zero, the intercept, $\alpha_{smb}$, should also be zero in these states.

In high market price of risk states, the size ranking captures some risks, which the value premium also captures. This gives a positive correlation between them, which shows up in the positive coefficient on the value premium, $h_{smb}$. In case the size ranking captures some risks that the value ranking does not capture, the size premium should provide an excess return and the intercept, $\alpha_{smb}$, should be positive. Otherwise, the intercept will be zero.

All the estimated coefficients based on $\Lambda^*_{MSE}$ in Table V are consistent with the predictions in Eq. (36), except the insignificant coefficient on the value premium, $h_{smb}$, in Panel A for the TBL when the market price of risk is high controlling for the market risk. The results are similar when the estimation is based on $\Lambda^*_{HML}$ in Table VIII. But there, in Panel B, there is also a negative coefficient on the value premiums, $h_{smb}$, in low risk states for the DJBM. In general, the evidence suggests that the size ranking captures risks that the value ranking does not capture, and not the other way around.

[Place Table V about here]

[Place Table VIII about here]

III. Summary

In this paper we learn how the price-related characteristics of the stocks are connected to their risks. In particular, we understand how this connection depends on the market
conditions (the market price of risk) and we learn how we can model this link.

This is important because much of our understanding about Finance is based on the estimated relation between risk premiums and these characteristics. One example is the idea that firms that increase their installed capital become riskier given that they face real investment frictions in bad economic times. This hypothesis seems to receive empirical support once we consider the BE as a proxy for invested capital and estimate the relation between the BM characteristic of the firms and their risk premiums.

However, it becomes clear that the BE actually contains no risk information about the firms once we consider how the market conditions affect the link between risk and the BM characteristic. Therefore, we also learn that the theory based on costly adjustments to tangible assets in place is much less consistent with the data than we previously thought.
Figure 1. The state variables in time series from 1926 to 2015. The panels plot the time series of the median BM (MBM), the value weighted average BM of all CRSP stocks (VBM), the BM of the Dow Jones stocks (DJBM), the default spread (DFY), the earnings-price ratio (EP), the term spread (TMS), and the T-bill rate (TBL).
Figure 2. Conditional and unconditional mean squared errors from the prediction of the size premium in 1926–2015 (from 1929 for the MBM and the VBM). The panels plot the MSE in Eq. (27) related to either 1) the forecast conditioned on different state variable thresholds to classify the market price of risk, Λ∗; or 2) the unconditional forecast (horizontal line). The state variables are the median BM (MBM), the value weighted average BM of all CRSP stocks (VBM), the BM of the Dow Jones stocks (DJBM), the earnings-price ratio (EP), the default spread (DFY), the term spread (TMS) and the T-bill rate (TBL).
Figure 3. Conditional and unconditional t-statistics of the mean size premium in 1926–2015 (from 1929 for the MBM and the VBM). The panels plot the t-statistics of the mean size premium in Eq. (26) for 1) the estimation conditioned on different state variable thresholds to classify the market price of risk, $\Lambda^*$; or 2) the unconditional estimation (horizontal line). The state variables are the median BM (MBM), the value weighted average BM of all CRSP stocks (VBM), the BM of the Dow Jones stocks (DJBM), the earnings-price ratio (EP), the default spread (DFY), the term spread (TMS) and the T-bill rate (TBL).
Figure 4. Conditional and unconditional t-statistics of the mean value premium in 1926–2015 (from 1929 for the MBM and the VBM). The panels plot the t-statistics of the mean value premium in Eq. (29) for 1) the estimation conditioned on different state variable thresholds to classify the market price of risk, $\Lambda^*$; or 2) the unconditional estimation (horizontal line). The state variables are the median BM (MBM), the value weighted average BM of all CRSP stocks (VBM), the BM of the Dow Jones stocks (DJBM), the earnings-price ratio (EP), the default spread (DFY), the term spread (TMS) and the T-bill rate (TBL).
Table I  Descriptive statistics for the state variables used as proxies for the market price of risk. The state variables are the median BM (MBM), the value weighted average BM of all CRSP stocks (VBM), the BM of the Dow Jones stocks (DJBM), the default spread (DFY), the earnings-price ratio (EP), the term spread (TMS), and the T-bill rate (TBL). The yearly data are from 1928 to 2015. The table reports the mean, standard deviation, first order autocorrelation, and correlations. The lower diagonal corresponds to the levels of the state variables and the upper diagonal corresponds to the respective AR(1) innovations.

<table>
<thead>
<tr>
<th>State variable</th>
<th>Mean</th>
<th>Standard deviation</th>
<th>AC(1)</th>
<th>BM</th>
<th>VBM</th>
<th>DJBM</th>
<th>DFY</th>
<th>EP</th>
<th>TMS</th>
<th>TBL</th>
</tr>
</thead>
<tbody>
<tr>
<td>BM</td>
<td>-0.151</td>
<td>0.390</td>
<td>0.820</td>
<td>0.91</td>
<td>0.45</td>
<td>0.56</td>
<td>0.06</td>
<td>0.19</td>
<td>-0.22</td>
<td></td>
</tr>
<tr>
<td>VBM</td>
<td>-0.388</td>
<td>0.435</td>
<td>0.889</td>
<td>0.92</td>
<td>0.41</td>
<td>0.49</td>
<td>0.05</td>
<td>0.25</td>
<td>-0.25</td>
<td></td>
</tr>
<tr>
<td>DJBM</td>
<td>-0.661</td>
<td>0.511</td>
<td>0.876</td>
<td>0.84</td>
<td>0.43</td>
<td>0.54</td>
<td>0.12</td>
<td>-0.01</td>
<td></td>
<td></td>
</tr>
<tr>
<td>DFY</td>
<td>-4.619</td>
<td>0.500</td>
<td>0.801</td>
<td>0.57</td>
<td>0.31</td>
<td>0.05</td>
<td>0.27</td>
<td>-0.25</td>
<td></td>
<td></td>
</tr>
<tr>
<td>EP</td>
<td>-2.724</td>
<td>0.441</td>
<td>0.618</td>
<td>0.45</td>
<td>0.74</td>
<td>0.05</td>
<td>0.28</td>
<td>-0.80</td>
<td></td>
<td></td>
</tr>
<tr>
<td>TMS</td>
<td>0.018</td>
<td>0.012</td>
<td>0.481</td>
<td>0.12</td>
<td>-0.10</td>
<td>0.36</td>
<td>-0.28</td>
<td>-0.80</td>
<td></td>
<td></td>
</tr>
<tr>
<td>TBL</td>
<td>0.034</td>
<td>0.030</td>
<td>0.874</td>
<td>-0.16</td>
<td>-0.01</td>
<td>0.20</td>
<td>-0.40</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table II  Descriptive statistics for the backward looking percentile rankings of the state variables. The state variables are the median BM (MBM), the value weighted average BM of all CRSP stocks (VBM), the BM of the Dow Jones stocks (DJBM), the default spread (DFY), the earnings-price ratio (EP), the term spread (TMS), and the T-bill rate (TBL). The yearly data are from 1930 to 2015. The table reports the mean, standard deviation, first order autocorrelation, and correlations of the percentile ranking given in Eq. (24). The lower diagonal corresponds to the levels of the state variables and the upper diagonal corresponds to the respective AR(1) innovations.

<table>
<thead>
<tr>
<th>State variable</th>
<th>Mean</th>
<th>Standard deviation</th>
<th>AC(1)</th>
<th>BM</th>
<th>VBM</th>
<th>DJBM</th>
<th>DFY</th>
<th>EP</th>
<th>TMS</th>
<th>TBL</th>
</tr>
</thead>
<tbody>
<tr>
<td>BM</td>
<td>0.377</td>
<td>0.283</td>
<td>0.667</td>
<td>0.90</td>
<td>0.51</td>
<td>0.47</td>
<td>0.19</td>
<td>0.31</td>
<td>-0.44</td>
<td></td>
</tr>
<tr>
<td>VBM</td>
<td>0.363</td>
<td>0.303</td>
<td>0.756</td>
<td>0.92</td>
<td>0.56</td>
<td>0.38</td>
<td>0.26</td>
<td>0.33</td>
<td>-0.44</td>
<td></td>
</tr>
<tr>
<td>DJBM</td>
<td>0.410</td>
<td>0.326</td>
<td>0.848</td>
<td>0.68</td>
<td>0.74</td>
<td>0.26</td>
<td>0.46</td>
<td>0.12</td>
<td>-0.26</td>
<td></td>
</tr>
<tr>
<td>DFY</td>
<td>0.494</td>
<td>0.292</td>
<td>0.786</td>
<td>0.50</td>
<td>0.48</td>
<td>0.22</td>
<td>-0.04</td>
<td>0.26</td>
<td>-0.33</td>
<td></td>
</tr>
<tr>
<td>EP</td>
<td>0.405</td>
<td>0.328</td>
<td>0.679</td>
<td>0.50</td>
<td>0.55</td>
<td>0.69</td>
<td>-0.02</td>
<td>-0.13</td>
<td>0.28</td>
<td></td>
</tr>
<tr>
<td>TMS</td>
<td>0.473</td>
<td>0.339</td>
<td>0.507</td>
<td>0.34</td>
<td>0.37</td>
<td>-0.01</td>
<td>0.54</td>
<td>-0.21</td>
<td>-0.71</td>
<td></td>
</tr>
<tr>
<td>TBL</td>
<td>0.652</td>
<td>0.333</td>
<td>0.897</td>
<td>-0.02</td>
<td>-0.06</td>
<td>0.17</td>
<td>-0.26</td>
<td>0.36</td>
<td>-0.60</td>
<td></td>
</tr>
</tbody>
</table>
Table III  Size and value premiums in 1926–2015 when the market price of risk is high or low according to \( \Lambda_{MSE}^\ast \). The panels report the means of the size and the value premiums, \( \bar{R}_{smb} \) and \( \bar{R}_{hml} \), their t-statistics, and the number of monthly observations in each sub sample. Each state variable generates two sub samples in which the market price of risk is either normal (\(< \Lambda_{MSE}^\ast \)) or high (\(\geq \Lambda_{MSE}^\ast \)), according to the threshold that minimizes the MSE of the size premium estimation, \( \Lambda_{MSE}^\ast \), in Eq. (28). The state variables are: The median BM (MBM), the value weighted average BM of all CRSP stocks (VBM), the BM of the Dow Jones stocks (DJBM), the earnings-price ratio (EP), the default spread (DFY), the term spread (TMS), and the T-bill rate (TBL).

Panel A: The size premium

<table>
<thead>
<tr>
<th>MBM</th>
<th>VBM</th>
<th>DJBM</th>
<th>EP</th>
<th>DFY</th>
<th>TMS</th>
<th>TBL</th>
</tr>
</thead>
<tbody>
<tr>
<td>(&lt; \Lambda_{MSE}^\ast )</td>
<td>(\geq \Lambda_{MSE}^\ast )</td>
<td>(&lt; \Lambda_{MSE}^\ast )</td>
<td>(\geq \Lambda_{MSE}^\ast )</td>
<td>(&lt; \Lambda_{MSE}^\ast )</td>
<td>(\geq \Lambda_{MSE}^\ast )</td>
<td>(&lt; \Lambda_{MSE}^\ast )</td>
</tr>
<tr>
<td>(\bar{R}_{smb} )</td>
<td>0.03</td>
<td>1.2***</td>
<td>0.05</td>
<td>1.3***</td>
<td>0.04</td>
<td>1.5***</td>
</tr>
<tr>
<td>t-statistics in parentheses</td>
<td>(0.32)</td>
<td>(3.80)</td>
<td>(0.53)</td>
<td>(3.62)</td>
<td>(0.45)</td>
<td>(3.52)</td>
</tr>
<tr>
<td>Observations</td>
<td>864</td>
<td>180</td>
<td>888</td>
<td>156</td>
<td>960</td>
<td>120</td>
</tr>
</tbody>
</table>

Panel B: The value premium

<table>
<thead>
<tr>
<th>MBM</th>
<th>VBM</th>
<th>DJBM</th>
<th>EP</th>
<th>DFY</th>
<th>TMS</th>
<th>TBL</th>
</tr>
</thead>
<tbody>
<tr>
<td>(&lt; \Lambda_{MSE}^\ast )</td>
<td>(\geq \Lambda_{MSE}^\ast )</td>
<td>(&lt; \Lambda_{MSE}^\ast )</td>
<td>(\geq \Lambda_{MSE}^\ast )</td>
<td>(&lt; \Lambda_{MSE}^\ast )</td>
<td>(\geq \Lambda_{MSE}^\ast )</td>
<td>(&lt; \Lambda_{MSE}^\ast )</td>
</tr>
<tr>
<td>(\bar{R}_{hml} )</td>
<td>0.4***</td>
<td>0.7</td>
<td>0.4***</td>
<td>0.7</td>
<td>0.4***</td>
<td>0.7</td>
</tr>
<tr>
<td>t-statistics in parentheses</td>
<td>(1.62)</td>
<td>(1.56)</td>
<td>(1.71)</td>
<td>(1.39)</td>
<td>(1.80)</td>
<td>(1.99)</td>
</tr>
<tr>
<td>Observations</td>
<td>864</td>
<td>180</td>
<td>888</td>
<td>156</td>
<td>960</td>
<td>120</td>
</tr>
</tbody>
</table>

p < 0.05, ** p < 0.01, *** p < 0.001
Table IV: Spanning tests of the value premium in 1926–2015 conditioned on the market price of risk being high or low according to $\Lambda_{MSE}^*$. The panels display the results of two models for each state variable: Column (a) reports the Constant, $\alpha_{hml}$ and the SMB coefficient, $s_{hml}$, for the regression of the value premium on the size premium in Eq. (31): $R_{hml,t} = \alpha_{hml} + s_{hml}R_{smb,t} + \epsilon_{hml,t}$. Column (b) adds the market premium (MP) to this regression with coefficient, $\beta_{hml}$, in Eq. (33): $R_{hml,t} = \alpha_{hml} + \beta_{hml}R_{mp,t} + s_{hml}R_{smb,t} + \epsilon_{hml,t}$. I estimate the coefficients conditioned on the market price of risk being either normal ($< \Lambda_{MSE}^*$) or high ($\geq \Lambda_{MSE}^*$), according to the threshold that minimizes the MSE of the size premium estimation, $\Lambda_{MSE}$, in Eq. (28). The state variables are: The median BM (MBM), the value weighted average BM of all CRSP stocks (VBM), the BM of the Dow Jones stocks (DJBM), the earnings-price ratio (EP), the default spread (DFY), the term spread (TMS), and the T-bill rate (TBL).

### Panel A: Spanning the value premium when the market price of risk is high, $\Lambda_t \geq \Lambda_{MSE}^*$

<table>
<thead>
<tr>
<th></th>
<th>MBM (a)</th>
<th>VBM (a)</th>
<th>DJBM (a)</th>
<th>EP (a)</th>
<th>DFY (a)</th>
<th>TMS (a)</th>
<th>TBL (a)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SMB</td>
<td>0.5***</td>
<td>0.3**</td>
<td>0.6***</td>
<td>0.3**</td>
<td>0.6***</td>
<td>0.2</td>
<td>0.6**</td>
</tr>
<tr>
<td></td>
<td>(5.59)</td>
<td>(3.59)</td>
<td>(4.75)</td>
<td>(1.89)</td>
<td>(3.33)</td>
<td>(2.22)</td>
<td>(5.04)</td>
</tr>
<tr>
<td>MP</td>
<td>0.4***</td>
<td>0.4***</td>
<td>0.4***</td>
<td>0.5***</td>
<td>0.4***</td>
<td>0.3***</td>
<td>0.4***</td>
</tr>
<tr>
<td></td>
<td>(8.58)</td>
<td>(6.50)</td>
<td>(7.79)</td>
<td>(6.46)</td>
<td>(9.24)</td>
<td>(11.72)</td>
<td></td>
</tr>
<tr>
<td>Constant</td>
<td>0.04</td>
<td>-0.02</td>
<td>0.09</td>
<td>-0.02</td>
<td>-0.3</td>
<td>0.5</td>
<td>-0.1</td>
</tr>
<tr>
<td></td>
<td>(0.10)</td>
<td>(-0.06)</td>
<td>(-0.33)</td>
<td>(-0.58)</td>
<td>(-0.76)</td>
<td>(-0.06)</td>
<td>(-0.45)</td>
</tr>
<tr>
<td>Observations</td>
<td>180</td>
<td>180</td>
<td>156</td>
<td>156</td>
<td>120</td>
<td>120</td>
<td>252</td>
</tr>
<tr>
<td>Adjusted $R^2$</td>
<td>0.14</td>
<td>0.39</td>
<td>0.46</td>
<td>0.15</td>
<td>0.37</td>
<td>0.16</td>
<td>0.32</td>
</tr>
</tbody>
</table>

* t statistics in parentheses
* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

### Panel B: Spanning the value premium when the market price of risk is normal, $\Lambda_t < \Lambda_{MSE}^*$

<table>
<thead>
<tr>
<th></th>
<th>MBM (a)</th>
<th>VBM (a)</th>
<th>DJBM (a)</th>
<th>EP (a)</th>
<th>DFY (a)</th>
<th>TMS (a)</th>
<th>TBL (a)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SMB</td>
<td>-0.04</td>
<td>-0.04</td>
<td>-0.02</td>
<td>-0.03</td>
<td>0.002</td>
<td>-0.02</td>
<td>0.03</td>
</tr>
<tr>
<td></td>
<td>(-1.24)</td>
<td>(-1.14)</td>
<td>(-0.53)</td>
<td>(-0.39)</td>
<td>(-0.91)</td>
<td>(-0.02)</td>
<td>(0.07)</td>
</tr>
<tr>
<td>MP</td>
<td>-0.002</td>
<td>-0.01</td>
<td>0.04</td>
<td>0.05*</td>
<td>0.04</td>
<td>0.007</td>
<td>-0.1***</td>
</tr>
<tr>
<td></td>
<td>(-0.09)</td>
<td>(-0.51)</td>
<td>(1.73)</td>
<td>(2.50)</td>
<td>(1.82)</td>
<td>(0.30)</td>
<td>(-1.60)</td>
</tr>
<tr>
<td>Constant</td>
<td>0.4***</td>
<td>0.4***</td>
<td>0.4***</td>
<td>0.3***</td>
<td>0.3***</td>
<td>0.3***</td>
<td>0.5***</td>
</tr>
<tr>
<td></td>
<td>(3.64)</td>
<td>(3.62)</td>
<td>(3.75)</td>
<td>(3.80)</td>
<td>(3.57)</td>
<td>(3.11)</td>
<td>(3.39)</td>
</tr>
<tr>
<td>Observations</td>
<td>864</td>
<td>864</td>
<td>888</td>
<td>888</td>
<td>960</td>
<td>960</td>
<td>1020</td>
</tr>
<tr>
<td>Adjusted $R^2$</td>
<td>0.00</td>
<td>-0.00</td>
<td>-0.00</td>
<td>-0.00</td>
<td>0.00</td>
<td>-0.00</td>
<td>-0.00</td>
</tr>
</tbody>
</table>

* t statistics in parentheses
* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$
The panels display the results of two models for each state variable: Column (a) reports the Constant, $\alpha_{smb}$ and the value premium coefficient, $h_{smb}$, for the regression of the size premium on the value premium in Eq. (34): $R_{smb,t} = \alpha_{smb} + h_{smb} R_{hml,t} + \epsilon_{smb,t}$. Column (b) adds the market premium (MP) to this regression with coefficient $\beta_{smb}$ in Eq. (35): $R_{smb,t} = \alpha_{smb} + \beta_{smb} R_{mp,t} + h_{smb} R_{hml,t} + \epsilon_{smb,t}$. I estimate the coefficients conditioned on the market price of risk being either normal ($< \Lambda_{MSE}^*$) or high ($\geq \Lambda_{MSE}^*$), according to the threshold that minimizes the MSE of the size premium estimation, $\Lambda_{MSE}$, in Eq. (28). The state variables are: The median BM (MBM), the value weighted average BM of all CRSP stocks (VBM), the BM of the Dow Jones stocks (DJBM), the earnings-price ratio (EP), the default spread (DFY), the term spread (TMS), and the T-bill rate (TBL).

### Panel A: Spanning the size premium when the market price of risk is high, $\Lambda_t \geq \Lambda_{MSE}^*$

<table>
<thead>
<tr>
<th></th>
<th>MBM</th>
<th>VBM</th>
<th>DJBM</th>
<th>EP</th>
<th>DFY</th>
<th>TMS</th>
<th>TBL</th>
</tr>
</thead>
<tbody>
<tr>
<td>HML</td>
<td>0.3***</td>
<td>0.3**</td>
<td>0.2**</td>
<td>0.3***</td>
<td>0.2**</td>
<td>0.3**</td>
<td>0.2**</td>
</tr>
<tr>
<td></td>
<td>(5.59)</td>
<td>(3.59)</td>
<td>(4.75)</td>
<td>(3.06)</td>
<td>(1.89)</td>
<td>(3.33)</td>
<td>(2.22)</td>
</tr>
<tr>
<td>MP</td>
<td>0.07</td>
<td>0.06</td>
<td>0.08</td>
<td>0.2**</td>
<td>0.02</td>
<td>0.08*</td>
<td>0.2**</td>
</tr>
<tr>
<td></td>
<td>(1.54)</td>
<td>(1.15)</td>
<td>(1.45)</td>
<td>(2.82)</td>
<td>(0.26)</td>
<td>(2.09)</td>
<td>(5.30)</td>
</tr>
<tr>
<td>Constant</td>
<td>1.0***</td>
<td>1.0***</td>
<td>1.1**</td>
<td>1.4**</td>
<td>0.7</td>
<td>0.5</td>
<td>0.8*</td>
</tr>
<tr>
<td></td>
<td>(3.43)</td>
<td>(3.11)</td>
<td>(3.28)</td>
<td>(3.33)</td>
<td>(1.93)</td>
<td>(2.03)</td>
<td>(2.44)</td>
</tr>
</tbody>
</table>

### Panel B: Spanning the size premium when the market price of risk is normal, $\Lambda_t < \Lambda_{MSE}^*$

<table>
<thead>
<tr>
<th></th>
<th>MBM</th>
<th>VBM</th>
<th>DJBM</th>
<th>EP</th>
<th>DFY</th>
<th>TMS</th>
<th>TBL</th>
</tr>
</thead>
<tbody>
<tr>
<td>HML</td>
<td>-0.04</td>
<td>-0.04</td>
<td>-0.01</td>
<td>-0.03</td>
<td>0.002</td>
<td>-0.02</td>
<td>0.007</td>
</tr>
<tr>
<td></td>
<td>(-1.24)</td>
<td>(-1.14)</td>
<td>(-0.53)</td>
<td>(-0.39)</td>
<td>(0.07)</td>
<td>(-0.62)</td>
<td>(0.22)</td>
</tr>
<tr>
<td>MP</td>
<td>0.2***</td>
<td>0.2***</td>
<td>0.2**</td>
<td>0.2***</td>
<td>0.2***</td>
<td>0.2***</td>
<td>0.2***</td>
</tr>
<tr>
<td></td>
<td>(10.37)</td>
<td>(10.53)</td>
<td>(10.17)</td>
<td>(8.82)</td>
<td>(10.84)</td>
<td>(10.67)</td>
<td>(8.14)</td>
</tr>
<tr>
<td>Constant</td>
<td>0.05</td>
<td>-0.07</td>
<td>-0.06</td>
<td>-0.04</td>
<td>0.10</td>
<td>0.04</td>
<td>0.006</td>
</tr>
<tr>
<td></td>
<td>(0.47)</td>
<td>(-0.72)</td>
<td>(-0.64)</td>
<td>(-0.67)</td>
<td>(1.01)</td>
<td>(0.38)</td>
<td>(1.16)</td>
</tr>
</tbody>
</table>

### Notes
- *$p < 0.05$, **$p < 0.01$, ***$p < 0.001$
- Adjusted $R^2$ and $t$ statistics in parentheses
Table VI: Size and value premiums in 1926–2015 when the market price of risk is high or low according to $\Lambda_{HML}^*$. The panels report the means of the size and the value premiums, $\bar{R}_{smb}$ and $\bar{R}_{hml}$, their t-statistics, and the number of monthly observations in each sub sample. Each state variable generates two sub samples in which the market price of risk is either normal ($<\Lambda_{HML}^*$) or high ($\geq\Lambda_{HML}^*$), according to the threshold, $\Lambda_{HML}^*$ in Eq. (30), which is the highest value for the state variable, such that the value premium has a mean at least two standard errors away from zero. The state variables are: The median BM (MBM), the value weighted average BM of all CRSP stocks (VBM), the BM of the Dow Jones stocks (DJBM), the earnings-price ratio (EP), the default spread (DFY), the term spread (TMS), and the T-bill rate (TBL).

### Panel A: The size premium

<table>
<thead>
<tr>
<th>State Variable</th>
<th>$&lt;\Lambda_{HML}^*$</th>
<th>$\geq\Lambda_{HML}^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>MBM</td>
<td>$\bar{R}_{smb}$</td>
<td>0.05</td>
</tr>
<tr>
<td></td>
<td>t-statistics</td>
<td>(0.51)</td>
</tr>
<tr>
<td>VBM</td>
<td>$\bar{R}_{smb}$</td>
<td>0.07</td>
</tr>
<tr>
<td></td>
<td>t-statistics</td>
<td>(0.66)</td>
</tr>
<tr>
<td>DJBM</td>
<td>$\bar{R}_{smb}$</td>
<td>-0.05</td>
</tr>
<tr>
<td></td>
<td>t-statistics</td>
<td>(-0.43)</td>
</tr>
<tr>
<td>EP</td>
<td>$\bar{R}_{smb}$</td>
<td>0.1</td>
</tr>
<tr>
<td></td>
<td>t-statistics</td>
<td>(1.13)</td>
</tr>
<tr>
<td>DFY</td>
<td>$\bar{R}_{smb}$</td>
<td>0.09</td>
</tr>
<tr>
<td></td>
<td>t-statistics</td>
<td>(0.86)</td>
</tr>
<tr>
<td>TMS</td>
<td>$\bar{R}_{smb}$</td>
<td>0.09</td>
</tr>
<tr>
<td></td>
<td>t-statistics</td>
<td>(0.81)</td>
</tr>
<tr>
<td>TBL</td>
<td>$\bar{R}_{smb}$</td>
<td>0.05</td>
</tr>
<tr>
<td></td>
<td>t-statistics</td>
<td>(0.50)</td>
</tr>
</tbody>
</table>

Observations: 828, 216, 864, 180, 720, 360, 924, 156, 768, 312, 792, 288, 564, 516

### Panel B: The value premium

<table>
<thead>
<tr>
<th>State Variable</th>
<th>$&lt;\Lambda_{HML}^*$</th>
<th>$\geq\Lambda_{HML}^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>MBM</td>
<td>$\bar{R}_{hml}$</td>
<td>0.3***</td>
</tr>
<tr>
<td></td>
<td>t-statistics</td>
<td>(2.86)</td>
</tr>
<tr>
<td>VBM</td>
<td>$\bar{R}_{hml}$</td>
<td>0.9</td>
</tr>
<tr>
<td></td>
<td>t-statistics</td>
<td>(2.41)</td>
</tr>
<tr>
<td>DJBM</td>
<td>$\bar{R}_{hml}$</td>
<td>0.3***</td>
</tr>
<tr>
<td></td>
<td>t-statistics</td>
<td>(1.30)</td>
</tr>
<tr>
<td>EP</td>
<td>$\bar{R}_{hml}$</td>
<td>0.8</td>
</tr>
<tr>
<td></td>
<td>t-statistics</td>
<td>(2.02)</td>
</tr>
<tr>
<td>DFY</td>
<td>$\bar{R}_{hml}$</td>
<td>0.3***</td>
</tr>
<tr>
<td></td>
<td>t-statistics</td>
<td>(2.80)</td>
</tr>
<tr>
<td>TMS</td>
<td>$\bar{R}_{hml}$</td>
<td>1.0*</td>
</tr>
<tr>
<td></td>
<td>t-statistics</td>
<td>(2.25)</td>
</tr>
<tr>
<td>TBL</td>
<td>$\bar{R}_{hml}$</td>
<td>0.3***</td>
</tr>
<tr>
<td></td>
<td>t-statistics</td>
<td>(2.36)</td>
</tr>
</tbody>
</table>

Observations: 828, 216, 864, 180, 720, 360, 924, 156, 768, 312, 792, 288, 564, 516

* p < 0.05, ** p < 0.01, *** p < 0.001
Table VII Spanning tests of the value premium in 1926–2015 conditioned on the market price of risk being high or low according to $\Lambda^*_HML$. The panels display the results of two models for each state variable: Column (a) reports the Constant, $\alpha_{hml}$ and the SMB coefficient, $s_{hml}$, for the regression of the value premium on the size premium in Eq. (31): $R_{hml,t} = \alpha_{hml} + s_{hml}R_{smb,t} + \varepsilon_{hml,t}$. Column (b) adds the market premium (MP) to this regression with coefficient, $\beta_{hml}$, in Eq. (33): $R_{hml,t} = \alpha_{hml} + \beta_{hml}R_{mp,t} + s_{hml}R_{smb,t} + \varepsilon_{hml,t}$. I estimate the coefficients conditioned on the market price of risk being either normal ($< \Lambda^*_HML$) or high ($\geq \Lambda^*_HML$), according to the threshold, $\Lambda^*_HML$ in Eq. (30), which is the highest value for the state variable, such that the value premium has a mean at least two standard errors away from zero. The state variables are: The median BM (MBM), the value weighted average BM of all CRSP stocks (VBM), the BM of the Dow Jones stocks (DJBM), the earnings-price ratio (EP), the default spread (DFY), the term spread (TMS), and the T-bill rate (TBL).

**Panel A: Spanning the value premium when the market price of risk is high, $\Lambda_t \geq \Lambda^*_HML$**

<table>
<thead>
<tr>
<th>MBM</th>
<th>VBM</th>
<th>DJBM</th>
<th>EP</th>
<th>DFY</th>
<th>TMS</th>
<th>TBL</th>
</tr>
</thead>
<tbody>
<tr>
<td>SMB</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5***</td>
<td>0.3**</td>
<td>0.2**</td>
<td>0.5***</td>
<td>0.2*</td>
<td>0.2*</td>
<td>0.2*</td>
</tr>
<tr>
<td>(5.27)</td>
<td>(3.20)</td>
<td>(2.50)</td>
<td>(7.95)</td>
<td>(2.50)</td>
<td>(5.49)</td>
<td>(4.91)</td>
</tr>
<tr>
<td>MP</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.3***</td>
<td>0.4***</td>
<td>0.3***</td>
<td>0.4***</td>
<td>0.3***</td>
<td>0.3***</td>
<td>0.3***</td>
</tr>
<tr>
<td>(8.35)</td>
<td>(8.90)</td>
<td>(9.51)</td>
<td>(7.34)</td>
<td>(7.84)</td>
<td>(9.20)</td>
<td>(10.61)</td>
</tr>
<tr>
<td>Constant</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.4</td>
<td>0.4</td>
<td>0.2</td>
<td>0.6</td>
<td>0.6</td>
<td>0.4</td>
</tr>
<tr>
<td>(1.30)</td>
<td>(1.31)</td>
<td>(0.92)</td>
<td>(1.29)</td>
<td>(0.75)</td>
<td>(0.27)</td>
<td>(1.42)</td>
</tr>
</tbody>
</table>

**Panel B: Spanning the value premium when the market price of risk is normal, $\Lambda_t < \Lambda^*_HML$**

<table>
<thead>
<tr>
<th>MBM</th>
<th>VBM</th>
<th>DJBM</th>
<th>EP</th>
<th>DFY</th>
<th>TMS</th>
<th>TBL</th>
</tr>
</thead>
<tbody>
<tr>
<td>SMB</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-0.03</td>
<td>-0.03</td>
<td>-0.007</td>
<td>-0.008</td>
<td>-0.2***</td>
<td>-0.2***</td>
<td>0.005</td>
</tr>
<tr>
<td>(-0.90)</td>
<td>(-0.92)</td>
<td>(-0.22)</td>
<td>(-5.77)</td>
<td>(-4.85)</td>
<td>(0.14)</td>
<td>(0.62)</td>
</tr>
<tr>
<td>MP</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.005</td>
<td>0.0009</td>
<td>-0.06**</td>
<td>0.06**</td>
<td>0.02</td>
<td>0.01</td>
<td>-0.1***</td>
</tr>
<tr>
<td>(0.23)</td>
<td>(0.04)</td>
<td>(-2.73)</td>
<td>(2.77)</td>
<td>(0.98)</td>
<td>(0.49)</td>
<td>(-5.89)</td>
</tr>
<tr>
<td>Constant</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.3**</td>
<td>0.3**</td>
<td>0.3**</td>
<td>0.3**</td>
<td>0.3**</td>
<td>0.3**</td>
<td>0.3**</td>
</tr>
<tr>
<td>(2.88)</td>
<td>(2.82)</td>
<td>(3.21)</td>
<td>(3.17)</td>
<td>(3.26)</td>
<td>(3.69)</td>
<td>(2.87)</td>
</tr>
</tbody>
</table>

**Notes:**
- $t$ statistics in parentheses
- * $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$
Table VIII Spanning tests of the size premium in 1926–2015 conditioned on the market price of risk being high or low according to $\Lambda_{HML}^*$. The panels display the results of two models for each state variable: Column (a) reports the Constant, $\alpha_{smb}$ and the value premium coefficient, $h_{smb}$, for the regression of the size premium on the value premium in Eq. (34): $R_{smb,t} = \alpha_{smb} + h_{smb}R_{hml,t} + \epsilon_{smb,t}$. Column (b) adds the market premium (MP) to this regression with coefficient $\beta_{smb}$ in Eq. (35): $R_{smb,t} = \alpha_{smb} + \beta_{smb}R_{mp,t} + h_{smb}R_{hml,t} + \epsilon_{smb,t}$. I estimate the coefficients conditioned on the market price of risk being either normal ($< \Lambda_{HML}^*$) or high ($\geq \Lambda_{HML}^*$), according to the threshold, $\Lambda_{HML}^*$ in Eq. (30), which is the highest value for the state variable, such that the value premium has a mean at least two standard errors away from zero. The state variables are: The median BM (MBM), the value weighted average BM of all CRSP stocks (VBM), the BM of the Dow Jones stocks (DJBM), the earnings-price ratio (EP), the default spread (DFY), the term spread (TMS), and the T-bill rate (TBL).

Panel A: Spanning the size premium when the market price of risk is high, $\Lambda_t \geq \Lambda_{HML}^*$

<table>
<thead>
<tr>
<th></th>
<th>MBM</th>
<th>VBM</th>
<th>DJBM</th>
<th>EP</th>
<th>DFY</th>
<th>TMS</th>
<th>TBL</th>
</tr>
</thead>
<tbody>
<tr>
<td>HML</td>
<td>0.3***</td>
<td>0.2**</td>
<td>0.2***</td>
<td>0.3***</td>
<td>0.1*</td>
<td>0.2***</td>
<td>0.1**</td>
</tr>
<tr>
<td></td>
<td>(5.27)</td>
<td>(3.20)</td>
<td>(4.77)</td>
<td>(7.95)</td>
<td>(4.77)</td>
<td>(5.20)</td>
<td>(2.09)</td>
</tr>
<tr>
<td>MP</td>
<td>0.09*</td>
<td>0.1*</td>
<td>0.1***</td>
<td>0.2***</td>
<td>0.1***</td>
<td>0.09*</td>
<td>0.2***</td>
</tr>
<tr>
<td></td>
<td>(2.28)</td>
<td>(2.21)</td>
<td>(4.03)</td>
<td>(3.83)</td>
<td>(4.01)</td>
<td>(2.57)</td>
<td>(6.29)</td>
</tr>
<tr>
<td>Constant</td>
<td>0.7***</td>
<td>0.7***</td>
<td>0.8***</td>
<td>0.6**</td>
<td>0.5**</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td></td>
<td>(2.72)</td>
<td>(2.74)</td>
<td>(2.79)</td>
<td>(2.90)</td>
<td>(2.98)</td>
<td>(2.72)</td>
<td>(1.57)</td>
</tr>
<tr>
<td>Observations</td>
<td>216</td>
<td>216</td>
<td>180</td>
<td>180</td>
<td>360</td>
<td>360</td>
<td>156</td>
</tr>
<tr>
<td>Adjusted $R^2$</td>
<td>0.11</td>
<td>0.11</td>
<td>0.13</td>
<td>0.13</td>
<td>0.15</td>
<td>0.15</td>
<td>0.14</td>
</tr>
<tr>
<td></td>
<td>0.04</td>
<td>0.04</td>
<td>0.06</td>
<td>0.06</td>
<td>0.07</td>
<td>0.07</td>
<td>0.08</td>
</tr>
</tbody>
</table>

Panel B: Spanning the size premium when the market price of risk is normal, $\Lambda_t < \Lambda_{HML}^*$

<table>
<thead>
<tr>
<th></th>
<th>MBM</th>
<th>VBM</th>
<th>DJBM</th>
<th>EP</th>
<th>DFY</th>
<th>TMS</th>
<th>TBL</th>
</tr>
</thead>
<tbody>
<tr>
<td>HML</td>
<td>-0.03</td>
<td>-0.03</td>
<td>-0.008</td>
<td>-0.007</td>
<td>-0.2**</td>
<td>-0.2***</td>
<td>0.005</td>
</tr>
<tr>
<td></td>
<td>(-0.90)</td>
<td>(-0.92)</td>
<td>(-0.22)</td>
<td>(-0.22)</td>
<td>(-5.77)</td>
<td>(-4.85)</td>
<td>(0.14)</td>
</tr>
<tr>
<td>MP</td>
<td>0.2***</td>
<td>0.2***</td>
<td>0.2***</td>
<td>0.2***</td>
<td>0.2***</td>
<td>0.2***</td>
<td>0.2***</td>
</tr>
<tr>
<td></td>
<td>(10.41)</td>
<td>(10.15)</td>
<td>(6.76)</td>
<td>(8.60)</td>
<td>(9.23)</td>
<td>(10.54)</td>
<td>(7.57)</td>
</tr>
<tr>
<td>Constant</td>
<td>0.06</td>
<td>-0.06</td>
<td>0.07</td>
<td>-0.06</td>
<td>0.03</td>
<td>-0.07</td>
<td>0.1</td>
</tr>
<tr>
<td></td>
<td>(0.59)</td>
<td>(-0.64)</td>
<td>(0.68)</td>
<td>(-0.64)</td>
<td>(0.26)</td>
<td>(-0.67)</td>
<td>(1.11)</td>
</tr>
<tr>
<td>Observations</td>
<td>828</td>
<td>828</td>
<td>864</td>
<td>864</td>
<td>720</td>
<td>720</td>
<td>924</td>
</tr>
<tr>
<td>Adjusted $R^2$</td>
<td>-0.00</td>
<td>0.11</td>
<td>-0.00</td>
<td>0.11</td>
<td>0.04</td>
<td>0.10</td>
<td>-0.00</td>
</tr>
</tbody>
</table>

1 statistics in parentheses
*p < 0.05, **p < 0.01, ***p < 0.001
Appendix A. Multivariate derivation

In a multidimensional setting, the SDF in Eq. (1) becomes

\[ d\zeta_t = -\zeta_t[r^f_t \, dt + \lambda^T_t \, dz_{1t}], \]  

where \( dz_{1t} \) is a multi-dimensional standard Brownian motion and \( \lambda_t \) is the multi-dimensional stochastic process representing the market price of risk at time \( t \) associated to each risk source. We also adjust the price process in Eq. (2) to

\[ dP_{it} = P_{it}[\mu_{it} \, dt + \sigma^T_{it} \, dz_{1t} + \tilde{\sigma}_{it} \, dz_t], \]  

where \( \sigma_{it} \) is now a multi-dimensional stochastic process, and \( dz_t \) is independent of \( dz_{1t} \).

The multi-dimensional equivalent to the expected excess rate of return in Eq. (3) is now

\[ \mu_{it} - r^f_t = \sigma^T_{it} \lambda_t, \]  

and the multi-dimensional equivalent of Eq. (5) gives the return on the SMB and HML portfolios, respectively, as

\[ \mu_{smb,t} = \sigma^T_{smb,t} \lambda_t, \]  
\[ \mu_{hml,t} = \sigma^T_{hml,t} \lambda_t. \]  

The main difference with the one-dimensional formulation is that with a single risk source the premiums on small and value stocks (and indeed all risk premiums) must be perfectly correlated given that there is only one priced risk in the economy.
REFERENCES


37


Souza, Thiago de O., 2017, The size premium is not a puzzle, but its predictability is, *Unpublished working paper. University of Southern Denmark*.
