Dynamic Asset Allocation

Chapter 10: Stochastic interest rates

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Motivation

- Interest rates and bond yields vary stochastically over time \(\sim\) include short-term interest rate \(r_t\) as state variable
- Obtain explicit solutions for affine short-rate models, e.g.
  - Vasicek model (1977)
  - CIR model (1985)
- Investors are concerned about **real** interest rates \(\sim\) want to invest in **real** bonds
- Determine the optimal bond/stock mix
  - assume single stock is traded (stock index); can be generalized
- Investors with non-log utility will hedge variations in interest rates
- Bonds carry a build-in hedge against interest rate risk: bond prices are inversely related to interest rates
- Look at the consequences for an investor if he follows a suboptimal investment strategy in the Vasicek world
Outline

1. The Vasicek interest rate model
2. The CIR interest rate model
3. Numerical example: Vasicek vs. CIR
4. Two-factor Vasicek model
The Vasiček model

The short rate $r_t$ follows the Ornstein-Uhlenbeck process

$$dr_t = \kappa [\bar{r} - r_t] \, dt - \sigma_r \, dz_{1t}$$

with an associated constant market price of risk $\lambda_1$.

Properties:
- Mean reversion; level-independent volatility
- Future values or $r$ are normally distributed $\sim$ negative rates
- Affine term structure model

The price of a zero-coupon bond with maturity $\bar{T}$ is given by

$$B_{t}^{\bar{T}} = e^{-a(T-t)-b(T-t)r_t},$$

$$b(\tau) = \frac{1}{\kappa} \left(1 - e^{-\kappa \tau}\right), \quad a(\tau) = y_\infty (\tau - b(\tau)) + \frac{\sigma_r^2}{4\kappa} b(\tau)^2$$

Price dynamics

$$dB_{t}^{\bar{T}} = B_{t}^{\bar{T}} \left[ (r_t + \lambda_1 \sigma_r b(\bar{T} - t)) \, dt + \sigma_r b(\bar{T} - t) dz_{1t} \right]$$
Available assets

- The dynamics of any bond price is
  \[ dB_t = B_t \left[ (r_t + \lambda_1 \sigma_B(r_t, t)) \, dt + \sigma_B(r_t, t) \, dz_{1t} \right] \]

- The price dynamics of the single stock is given by
  \[ dS_t = S_t \left[ (r_t + \psi \sigma_S) \, dt + \rho \sigma_S \, dz_{1t} + \sqrt{1 - \rho^2} \sigma_S \, dz_{2t} \right] \]

  ▶ \( \rho \) is the bond-stock correlation
  ▶ \( \sigma_S \) is the volatility of the stock
  ▶ \( \psi = \rho \lambda_1 + \sqrt{1 - \rho^2} \lambda_2 \) is the Sharpe ratio of the stock

- Complete market!

\[
\sigma_t = \begin{pmatrix} \sigma_B & 0 \\ \rho \sigma_S & \sqrt{1 - \rho^2} \sigma_S \end{pmatrix} \quad \Rightarrow \quad \sigma_t^{-1} = \frac{1}{\sigma_B \sigma_S \sqrt{1 - \rho^2}} \begin{pmatrix} \sqrt{1 - \rho^2} \sigma_S & -\rho \sigma_S \\ 0 & \sigma_B \end{pmatrix}
\]
Affine model...

The model falls in the affine framework of Sec. 7.3.2.

\[ A'_1(\tau) = 1 - \kappa A_1(\tau), \quad A_1(0) = 0 \]

has the solution

\[ A_1(\tau) = \frac{1}{\kappa} \left( 1 - e^{-\kappa \tau} \right) = b(\tau). \]

\[ A_0(\tau) = \frac{1}{2\gamma} \left( \lambda_1^2 + \lambda_2^2 \right) \tau + \left( \kappa \bar{r} + \frac{\gamma - 1}{\gamma} \sigma_r \lambda_1 \right) \int_0^\tau b(s) \, ds - \frac{\gamma - 1}{2\gamma} \sigma_r^2 \int_0^\tau b(s)^2 \, ds \]

\[ = \frac{1}{2\gamma} \left( \lambda_1^2 + \lambda_2^2 \right) \tau + \left( \bar{r} - \frac{\gamma - 1}{2\kappa^2 \gamma} \left[ \sigma_r^2 - 2\kappa \sigma_r \lambda_1 \right] \right) \left( \tau - b(\tau) \right) + \frac{\gamma - 1}{4\kappa \gamma} \sigma_r^2 b(\tau)^2, \]

using

\[ \int_0^\tau b(s) \, ds = \frac{1}{\kappa} \left( \tau - b(\tau) \right), \quad \int_0^\tau b(s)^2 \, ds = \frac{1}{\kappa^2} \left( \tau - b(\tau) \right) - \frac{1}{2\kappa} b(\tau)^2. \]
Utility from terminal wealth only

The optimal investment strategy is (see Thm. 7.7)

\[
\begin{pmatrix}
\Pi_B(W, r, t) \\
\Pi_S(W, r, t)
\end{pmatrix} = \frac{1}{\gamma} \left( \sigma(r, t)^T \right)^{-1} \lambda \\
+ \frac{\gamma - 1}{\gamma} \left( \sigma(r, t)^T \right)^{-1} \begin{pmatrix}
\sigma_r \\
0
\end{pmatrix} b(T - t)
\]

- The hedge only involves the bond, not the stock
- $\gamma \uparrow \Rightarrow$ lower investment in $\pi^{tan}$ and higher investment in $\pi^{hdg}$

We can rewrite the strategy as follows

\[
\Pi_S(W, r, t) = \frac{\lambda_2}{\gamma \sigma_S \sqrt{1 - \rho^2}}
\]

\[
\Pi_B(W, r, t) = \frac{1}{\gamma \sigma_B(r, t)} \left( \lambda_1 - \frac{\rho}{\sqrt{1 - \rho^2}} \lambda_2 \right) + \frac{\gamma - 1}{\gamma} \frac{\sigma_r b(T - t)}{\sigma_B(r, t)}
\]
Utility from consumption

The hedge term of the optimal bond investment strategy is (Thm. 7.8)

\[
\gamma - 1 \quad \frac{\sigma_r}{\gamma} \quad \frac{\int_{t}^{T} \ e^{-\frac{\delta}{\gamma} (s-t) - \frac{\gamma - 1}{\gamma} A_0 (s-t) - \frac{\gamma - 1}{\gamma} b(s-t) r b(s-t) ds}{\int_{t}^{T} e^{-\frac{\delta}{\gamma} (s-t) - \frac{\gamma - 1}{\gamma} A_0 (s-t) - \frac{\gamma - 1}{\gamma} b(s-t) r ds}
\]

The time \( t \) volatility of a coupon bond paying a continuous coupon at a deterministic rate \( K(t) \) up to time \( T \) is given by

\[
\sigma_B(r, t) = \frac{\int_{t}^{T} K(s) B_t^s \sigma_r b(s-t) ds}{\int_{t}^{T} K(s) B_t^s ds}
\]

\( \sim \) we can interpret the time \( t \) interest rate hedge as the fraction \((\gamma - 1)/\gamma\) of wealth invested in a bond with continuous coupon

\[
K(s) = e^{a(s-t) - \frac{\gamma - 1}{\gamma} A_0 (s-t) - \frac{\delta}{\gamma} (s-t) + \frac{b(s-t)}{\gamma} r}
\]
Model estimates

- Use historical estimates of mean returns, standard deviations, and correlations as representative of future
- Assume that the investor can invest in the bank account, a stock index, and a 10-year-to-maturity zero-coupon bond
- The real short-term interest rate: $\bar{r} = 1.0\%$, $\sigma_r = 5\%$
- Stock market index: $\mu_S = 8.7\%$, $\sigma_S = 20.2\%$
  - $\psi = (8.7 - 1.0) / 20.2 \approx 0.3812$
- Zero-coupon bond: $\mu_B = 2.1\%$, $\sigma_B = 10.0\%$
  - $\kappa = 0.4965$
- The correlation between stock return and bond returns is $\rho = 0.2$
- The market prices of risk: $\lambda_1 = 0.11$, $\lambda_2 = 0.3666$
- Tangency pf $\pi^{\tan} = (\underline{\sigma}^\top)^{-1} \lambda / [1^\top (\underline{\sigma}^\top)^{-1} \lambda]$ consists of 15.96% in bond and 84.04% in stock
Myopic investment strategy

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Note: speculative pf is $\frac{1}{\gamma} (\sigma_t^T)^{-1} \lambda = \frac{1}{\gamma} 1^T (\sigma_t^T)^{-1} \lambda \pi^{tan}$ in table
Mean-variance frontiers
**Utility from terminal wealth only**

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Optimal frontiers with Vasicek interest rates
Wealth loss due to myopic strategy

Computed using the methods in Larsen & Munk (2012).
Parameters: $\bar{r} = 0.01$, $\kappa = 0.63$, $\sigma_r = 0.03$, $\lambda_1 = 0.27$, $\psi = 0.35$, $\sigma_S = 0.2$, $\rho = 0.2$
Wealth loss from no hedge and no bonds
Wealth loss due near-optimal strategy, \( T - t = 2 \)

The costs of applying the investment strategy

\[
(\pi^*_S + \varphi_S, \pi^*_B(r_u, u) + \varphi_B)
\]

instead of the optimal strategy \((\pi^*_S, \pi^*_B(r_u, u))\) over \([t, T]\).
Wealth loss due to near-optimal strategy, $T - t = 20$
The CIR model

The short rate $r_t$ follows the square-root process

$$dr_t = \kappa [\bar{r} - r_t] \, dt - \sigma_r \sqrt{r_t} \, dz_{1t}$$

with an associated market price of risk $\lambda_1(r, t) = \lambda_1 \sqrt{r} / \sigma_r$.

Properties:
- mean reversion; volatility increases with level
- future values of $r$ non-centrally $\chi^2$-dist.; non-negative!
- affine term structure model

The price of a zero-coupon bond with maturity $\bar{T}$ is

$$B_{t}^{\bar{T}} = e^{-a(T-t)-b(T-t)r_t}, \quad b(\tau) = \frac{2 (e^{\alpha \tau} - 1)}{(\alpha + \hat{\kappa}) (e^{\alpha \tau} - 1) + 2\alpha},$$

$$a(\tau) = -\frac{2\kappa\bar{r}}{\sigma_r^2} \left( \frac{1}{2} (\hat{\kappa} + \alpha) \tau + \ln \left( \frac{2\alpha}{(\alpha + \hat{\kappa}) (e^{\alpha \tau} - 1) + 2\alpha} \right) \right)$$

where $\hat{\kappa} = \kappa - \lambda_1$ and $\alpha = \sqrt{\hat{\kappa}^2 + 2\sigma_r^2}$

Price dynamics

$$dB_{t}^{\bar{T}} = B_{t}^{\bar{T}} \left[ (r_t + b(\bar{T} - t)\lambda_1 r_t) \, dt + b(\bar{T} - t)\sigma_r \sqrt{r_t} \, dz_{1t} \right]$$
Available assets

- The dynamics of any bond price is

\[ dB_t = B_t \left[ \left( r_t + \frac{\lambda_1 \sqrt{r_t}}{\sigma_r} \sigma_B(r_t, t) \right) dt + \sigma_B(r_t, t) dz_{1t} \right] \]

- The price dynamics of the single stock is given by

\[ dS_t = S_t \left[ (r_t + \psi(r_t) \sigma_S) dt + \rho \sigma_S dz_{1t} + \sqrt{1 - \rho^2} \sigma_S dz_{2t} \right] \]

where

- \( \rho \) is the correlation between the bond market returns and stock market returns
- \( \sigma_S \) is the volatility of the stock
- \( \psi(r) = \rho \frac{\lambda_1}{\sigma_r} \sqrt{r} + \sqrt{1 - \rho^2} \lambda_2 \) is the Sharpe ratio of the stock [\( \lambda_2 \) assumed constant]

- Affine model!!!
Utility from terminal wealth only

The optimal investment strategy for a CRRA investor is

$$
\Pi_S(W, r, t) = \frac{\lambda_2}{\gamma \sigma_S \sqrt{1 - \rho^2}} 
$$

$$
\Pi_B(W, r, t) = \frac{1}{\gamma \sigma_B(r, t)} \left( \frac{\lambda_1}{\sigma_r} \sqrt{r} - \frac{\rho}{\sqrt{1 - \rho^2}} \lambda_2 \right) + \frac{\gamma - 1}{\gamma} \frac{\sigma_r \sqrt{r}}{\sigma_B(r, t)} A_1(T - t)
$$

where

$$
A_1(\tau) = \frac{2 \left( 1 + \frac{\lambda_1^2}{2 \gamma \sigma_r^2} \right) (e^{\nu \tau} - 1)}{(\nu + \bar{\kappa}) (e^{\nu \tau} - 1) + 2\nu} \neq b(\tau),
$$

$$
\bar{\kappa} = \kappa - \frac{\gamma - 1}{\gamma} \lambda_1, \quad \nu = \sqrt{\bar{\kappa}^2 + 2\sigma_r^2 \frac{\gamma - 1}{\gamma} \left( 1 + \frac{\lambda_1^2}{2 \gamma \sigma_r^2} \right)}.
$$

Again we see

- The hedge only involves the bond, not the stock
- higher $\gamma \Rightarrow$ lower investment in $\pi^{tan}$, higher investment in $\pi^{hdg}$
Utility from consumption

The hedge term of the optimal bond investment strategy is

\[
\frac{\gamma - 1}{\gamma} \frac{\sigma_r \sqrt{r}}{\sigma_B(r, t)} \left( \frac{\gamma}{\gamma} \right) \int_t^T A_1(s - t) e^{-\frac{\delta}{\gamma} (s-t) - \frac{\gamma-1}{\gamma} A_0(s-t) - \frac{\gamma-1}{\gamma} A_1(s-t) r} \, ds
\]

\[
\int_t^T e^{-\frac{\delta}{\gamma} (s-t) - \frac{\gamma-1}{\gamma} A_0(s-t) - \frac{\gamma-1}{\gamma} A_1(s-t) r} \, ds
\]

The time \( t \) volatility of a coupon bond paying a continuous coupon at a deterministic rate \( K(t) \) up to time \( T \) is given by

\[
\sigma_B(r, t) = \frac{\int_t^T K(s) B_t^s \sigma_r \sqrt{r} b(s - t) \, ds}{\int_t^T K(s) B_t^s \, ds}
\]

\( \Rightarrow \) we can interpret the time \( t \) interest rate hedge as the fraction

\[
\frac{\gamma - 1}{\gamma} \int_t^T \frac{A_1(s-t)}{b(s-t)} \, ds
\] of wealth invested in a bond with continuous coupon

\[
K(s) = e^{a(s-t) - \frac{\gamma-1}{\gamma} A_1(s-t) - \frac{\delta}{\gamma} (s-t) + \frac{1}{\gamma} b(s-t) r}
\]
Model estimates

- Compare the “CIR strategies” with the “Vasicek strategies”
- Calibrate the model to the same historical estimates
- Assume that the investor can invest in the bank account, a stock index, and a 10Y zero-coupon bond
- The real short-term interest rate: $\bar{r} = 1.0\%$, $\sigma_r = 0.5$
  - the short rate volatility is $\sigma_r \sqrt{\bar{r}} = 0.05$
- Stock market index: $\mu_S = 8.7\%$, $\sigma_S = 20.2\%$
  - $\psi = (8.7 - 1.0)/20.2 \approx 0.3812$
- Zero-coupon bond: $\mu_B = 2.1\%$, $\sigma_B = 10.0\%$
  - $\kappa = 0.7994$
- The correlation between stock return and bond returns is $\rho = 0.2$
- The market prices of risk: $\lambda_1 = 0.55$, $\lambda_2 = 0.3666$
  - the model is consistent with the estimated mean stock and bond returns
Utility from terminal wealth only

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Comparison

- The stock weights are identical
- The hedge term and hence the total bond demand do depend on the interest rate model
  - The biggest differences are for short horizons
  - For longer horizons the differences are relatively small
- The two interest rate models are comparable
  - the yield curves of the two models are almost identical
    - The long-term yield in Vasicek:
      \[
      y_\infty = \bar{r} + \frac{\sigma_r \lambda_1}{\kappa} - \frac{\sigma_r^2}{2\kappa^2} = 0.01601
      \]
    - The long-term yield in CIR
      \[
      y_\infty = \frac{2\kappa \bar{r}}{\hat{\kappa} + \nu} = 0.01600
      \]
    - With \(r_t = \bar{r} = 1\%\) both yield curves will be uniformly increasing
Model

\[ dr_t = (\varphi_r + u_t - \kappa_r r_t) \, dt - \sigma_r \, dz_{1t}, \]
\[ du_t = -\kappa_u u_t \, dt - \sigma_u \rho_r u \, dz_{1t} - \sigma_u \sqrt{1 - \rho_{ru}^2} \, dz_{2t}, \]

and constant market prices of risk \( \lambda_1, \lambda_2. \)

Zero-coupon bond prices

\[ B_{\bar{T}}(r, u, t) = e^{-a(\bar{T}-t)-b_1(\bar{T}-t)r-b_2(\bar{T}-t)u}, \]

where

\[ b_1(\tau) = \frac{1}{\kappa_r} \left(1 - e^{-\kappa_r \tau}\right), \quad b_2(\tau) = \frac{1}{\kappa_r \kappa_u} \left[1 + \frac{1}{\kappa_r (\kappa_r - \kappa_u)} e^{-\kappa_r \tau} - \frac{1}{\kappa_u (\kappa_r - \kappa_u)} e^{-\kappa_u \tau}\right] \]

\[ a(\tau) = \text{complicated and less important} \]

Bond price dynamics

\[ dB_{\bar{T}} = B_{\bar{T}} \left[(r_t + \psi_B(\bar{T} - t)) \, dt + \sigma_{B1}(\bar{T} - t) \, dz_{1t} + \sigma_{B2}(\bar{T} - t) \, dz_{2t}\right], \]

where, for all \( \tau \geq 0, \) we have defined

\[ \psi_B(\tau) = \lambda_1 \sigma_{B1}(\tau) + \lambda_2 \sigma_{B2}(\tau), \]
\[ \sigma_{B1}(\tau) = \sigma_r b_1(\tau) + \sigma_u \rho_r u b_2(\tau), \quad \sigma_{B2}(\tau) = \sigma_u \sqrt{1 - \rho_{ru}^2} b_2(\tau). \]
Asset allocation

Individual with CRRA utility of terminal wealth, who can invest in

1. the locally risk-free asset (=bank account; cash deposits) with rate of return \( r_t \),
2. a zero-coupon bond maturing at time \( T_1 \),
3. a zero-coupon bond maturing at time \( T_2 \neq T_1 \),
4. the stock with

\[
    dS_t = S_t \left[ (r_t + \psi S \sigma_S) \ dt + \sigma_S k_1 \ dz_{1t} + \sigma_S k_2 \ dz_{2t} + \sigma_S k_3 \ dz_{3t} \right].
\]

Two-dimensional affine asset allocation model

\[
    J \left( W, r, u, t \right) = \frac{1}{1 - \gamma} g(r, u, t)^\gamma W^{1 - \gamma},
\]

\[
    g(r, u, t) = \exp \left\{ -\frac{\gamma - 1}{\gamma} A_0 (T - t) - \frac{\gamma - 1}{\gamma} A_{1r} (T - t) r - \frac{\gamma - 1}{\gamma} A_{1u} (T - t) u \right\},
\]

\[
    A_{1r} (\tau) \equiv b_1 (\tau), \quad A_{1u} (\tau) \equiv b_2 (\tau).
\]

Optimal investment strategy is

\[
    \begin{pmatrix}
        \pi_{B1}(t) \\
        \pi_{B2}(t) \\
        \pi_S
    \end{pmatrix} = \frac{1}{\gamma} \left( \frac{\sigma(t)^\top}{\gamma} \right)^{-1} \lambda - \frac{\gamma - 1}{\gamma} \left( \frac{\sigma(t)^\top}{\gamma} \right)^{-1} \begin{pmatrix}
        -\sigma_r & -\sigma_u \rho_{ru} \\
        0 & -\sigma_u \sqrt{1 - \rho_{ru}^2}
    \end{pmatrix} \begin{pmatrix}
        A_{1r} (T - t) \\
        A_{1u} (T - t)
    \end{pmatrix}
\]
The optimal investment in the stock reduces to

$$\pi_S = \frac{1}{\gamma} \frac{\lambda_3}{\sigma_S k_3}. $$

The optimal investments in the two bonds can be rewritten as

$$\pi_{B1}(t) = \frac{1}{\gamma \sigma_r \sigma_u \sqrt{1 - \rho_{ru}^2} d(t)} \left( \sigma_r b_1(T_2 - t) \left[ \frac{k_2 \lambda_3}{k_3} - \lambda_2 \right] \right.$$

$$+ \sigma_u b_2(T_2 - t) \left[ \left( \lambda_1 - \frac{k_1 \lambda_3}{k_3} \right) \sqrt{1 - \rho_{ru}^2} + \left( \frac{k_2 \lambda_3}{k_3} - \lambda_2 \right) \rho_{ru} \right] \right.$$

$$+ \frac{\gamma - 1}{\gamma d(t)} (b_2(T_2 - t)b_1(T - t) - b_1(T_2 - t)b_2(T - t)), \right.$$

$$\pi_{B2}(t) = \frac{1}{\gamma \sigma_r \sigma_u \sqrt{1 - \rho_{ru}^2} d(t)} \left( \sigma_r b_1(T_1 - t) \left[ \lambda_2 - \frac{k_2 \lambda_3}{k_3} \right] \right.$$

$$+ \sigma_u b_2(T_1 - t) \left[ \left( \frac{k_1 \lambda_3}{k_3} - \lambda_1 \right) \sqrt{1 - \rho_{ru}^2} + \left( \lambda_2 - \frac{k_2 \lambda_3}{k_3} \right) \rho_{ru} \right] \right.$$

$$+ \frac{\gamma - 1}{\gamma d(t)} (b_1(T_1 - t)b_2(T - t) - b_1(T - t)b_2(T_1 - t)), \right.$$

where

$$d(t) = b_1(T_1 - t)b_2(T_2 - t) - b_1(T_2 - t)b_2(T_1 - t).$$
Optimal portfolios, $\gamma = 2$
Optimal portfolios, $\gamma = 4$
Dynamic Asset Allocation

Chapter 11: Stochastic market prices of risk

Claus Munk

August 2012
Outline

1. Stochastic market price of risk
   - The basic model
   - Distribution of future prices
   - Optimal investment strategy

2. Value/growth tilts and mean reversion
   - The model
   - Optimal strategies
   - Suboptimal strategies

3. Stochastic volatility
   - Model without hedging
   - Model with imperfect hedging (incomplete market)
   - Model with perfect hedging (complete market)
   - Other issues
Motivation

- Numerous studies have reported empirical evidence of mean reversion in stock returns
  - stock returns are high after a period of low realized returns and *vice versa*
  - average stock returns seem predictable via dividend/price, earnings/price, or short-term interest rate; still highly debated, though

- Among others Kim and Omberg (1996) and Wachter (2002) have studied the implications for portfolio decisions

- Both papers obtain closed-form solutions for the optimal investment strategies

- The investment strategy is consistent with the typical recommendation of investment advisors
  - lower variance of long-term stock returns than in the standard model
  - an investor with a long horizon should invest a larger fraction of wealth in stocks than an investor with a shorter horizon
The model

- Constant interest rates → cash/bank account
- Stock market index

\[ dP_t = P_t [(r + \sigma \lambda_t) \, dt + \sigma \, dz_t] \]

where the market price of risk (= state variable) follows

\[ d\lambda_t = \kappa \left[ \bar{\lambda} - \lambda_t \right] \, dt + \rho \sigma_{\lambda} \, dz_t + \sqrt{1 - \rho^2 \sigma_{\lambda}^2} \, d\hat{z}_t \]

- \( \rho \): correlation between realized returns and expected future returns
- \( \sigma_{\lambda} \): volatility of the market price of risk
- possibly negative \( \lambda_t \): unrealistic?
- mean reversion in stock returns if \( \rho < 0 \): then \( dz_t > 0 \) leads to
  * \( dP_t > 0 \): positive return this period \([t, t + dt]\)
  * \( d\lambda_t < 0 \) i.e. \( \lambda_{t+dt} < \lambda_t \): lower expected return next period \([t + dt, t + 2dt]\)
Distribution of future prices

From the price dynamics and Itô’s lemma it follows that

\[ P_T = P_t \exp \left\{ \left( r - \frac{1}{2} \sigma^2 \right) (T - t) + \sigma \int_t^T \lambda_s \, ds + \int_t^T \sigma \, dz_u \right\} \]

After an hour of fun...

\[ P_T = P_t \exp \left\{ \left( r - \frac{1}{2} \sigma^2 + \sigma \bar{\lambda} \right) (T - t) + \sigma b(T - t) \left( \lambda_t - \bar{\lambda} \right) \right. \]

\[ + \sigma \int_t^T (1 + \rho \sigma \lambda b(T - s)) \, dz_s + \sigma \sigma \lambda \sqrt{1 - \rho^2} \int_t^T b(T - s) \, d\hat{z}_s \]

\[ \sim P_T \text{ is lognormally distributed with} \]

\[ E_t[\ln P_T] = \ln P_t + \left( r - \frac{1}{2} \sigma^2 + \sigma \bar{\lambda} \right) (T - t) + \sigma b(T - t) \left( \lambda_t - \bar{\lambda} \right) \]

\[ \text{Var}_t[\ln P_T] = \sigma^2 \left[ \left( 1 + \frac{2 \rho \sigma \lambda}{\kappa} + \frac{\sigma^2}{\kappa^2} \right) (T - t) - \left( \frac{2 \rho \sigma \lambda}{\kappa} + \frac{\sigma^2}{\kappa^2} \right) b(T - t) - \frac{\sigma^2}{2 \kappa} b(T - t)^2 \right] \]
Distribution of future prices, cont’d

\[
\frac{\text{Var}_t[\ln P_T]}{\sigma^2(T - t)} = 1 + \frac{2\rho\sigma\lambda}{\kappa} + \frac{\sigma^2}{\kappa^2} - \left( \frac{2\rho\sigma\lambda}{\kappa} + \frac{\sigma^2}{\kappa^2} \right) \frac{b(T - t)}{T - t} - \frac{\sigma^2}{2\kappa} \frac{b(T - t)^2}{T - t}
\]

\[
\rightarrow 1 + \frac{2\rho\sigma\lambda}{\kappa} + \frac{\sigma^2}{\kappa^2} \quad \text{for } T \rightarrow \infty.
\]

The variations in the Sharpe ratio will therefore decrease the variance in the long run if

\[
\frac{2\rho\sigma\lambda}{\kappa} + \frac{\sigma^2}{\kappa^2} < 0 \iff \rho < -\frac{\sigma\lambda}{2\kappa},
\]

which is probably true.
Effects of mean reversion on the distribution, $T = 5$

Parameters: $r = 3\%$, $\sigma = 20\%$, $\kappa = 0.02$, $\bar{\lambda} = 0.3$ (excess exp. return 6\%), $\sigma_\lambda = 1\%$, $\rho = -0.8$, $\lambda_t = \bar{\lambda}$
Effects of mean reversion on the distribution, $T = 30$

Parameters: $r = 3\%$, $\sigma = 20\%$, $\kappa = 0.02$, $\bar{\lambda} = 0.3$ (excess exp. return 6\%), $\sigma_{\lambda} = 1\%$, $\rho = -0.8$, $\lambda_t = \bar{\lambda}$
Outperformance probabilities

The probability that a 100% stock investment outperforms a 100% bond investment...

Parameters: $r = 3\%$, $\sigma = 20\%$, $\kappa = 0.02$, $\bar{\lambda} = 0.3$ (excess exp. return 6%), $\sigma_\lambda = 1\%$, $\rho = -0.8$, $\lambda_t = \bar{\lambda}$
Quadratic model

It’s a quadratic model with state variable $x = \lambda$:

- $r(x) = r_0 + r_1 x + r_2 x^2$: use $r_0 = r$, $r_1 = r_2 = 0$
- $\|\lambda(x)\|^2 = \Lambda_0 + \Lambda_1 x + \Lambda_2 x^2$: use $\Lambda_0 = \Lambda_1 = 0$, $\Lambda_2 = 1$
- $m(x) = m_0 + m_1 x$: use $m_0 = \kappa \bar{\lambda}$, $m_1 = -\kappa$
- $v^2 = \rho^2 \sigma^2_{\lambda}$ is constant
- $\hat{v}^2 = (1 - \rho^2) \sigma^2_{\lambda}$ is constant
- $v(x) \lambda(x) = K_0 + K_1 x$: use $K_0 = 0$, $K_1 = \rho \sigma_{\lambda}$

Plug into general solution from Ch. 7:

$$A_2(\tau) = \frac{2 \left(2r_2 + \frac{\Lambda_2}{\gamma}\right) (e^{\nu \tau} - 1)}{\left(\nu + 2\frac{\gamma - 1}{\gamma} K_1 - 2m_1\right) (e^{\nu \tau} - 1) + 2\nu},$$

$$A_1(\tau) = \frac{r_1 + \frac{\Lambda_1}{2\gamma}}{2r_2 + \frac{\Lambda_2}{\gamma}} A_2(\tau) + \frac{4q}{\nu} \frac{(e^{\nu \tau/2} - 1)^2}{\left(\nu + 2\frac{\gamma - 1}{\gamma} K_1 - 2m_1\right)(e^{\nu \tau} - 1) + 2\nu},$$

$$\nu = 2 \sqrt{\left(m_1 - \frac{\gamma - 1}{\gamma} K_1\right)^2 + \frac{\gamma - 1}{\gamma} \left(2r_2 + \frac{\Lambda_2}{\gamma}\right) (\|v\|^2 + \hat{v}^2)},$$

$$q = \left(m_0 - \frac{\gamma - 1}{\gamma} K_0\right) \left(2r_2 + \frac{\Lambda_2}{\gamma}\right) - \left(m_1 - \frac{\gamma - 1}{\gamma} K_1\right) \left(r_1 + \frac{\Lambda_1}{2\gamma}\right).$$
Define \( \bar{\kappa} = \kappa + \frac{\gamma-1}{\gamma} \rho \sigma_\lambda \), assume \( \bar{\kappa}^2 + \sigma^2_\lambda \left( \rho^2 + \gamma(1 - \rho^2) \right) \frac{\gamma-1}{\gamma^2} > 0 \), and define \( \nu = 2 \sqrt{\bar{\kappa}^2 + \sigma^2_\lambda \left( \rho^2 + \gamma(1 - \rho^2) \right)} \).

Then the relevant \( A_i(\tau) \) functions are

\[
A_2(\tau) = \frac{2}{\gamma} \frac{e^{\nu \tau} - 1}{(\nu + 2\bar{\kappa})(e^{\nu \tau} - 1) + 2\nu},
\]

\[
A_1(\tau) = \frac{4\kappa \bar{\lambda}}{\gamma \nu} \left( \frac{e^{\nu \tau/2} - 1}{2} \right)^2 \frac{(e^{\nu \tau/2} - 1)^2}{(\nu + 2\bar{\kappa})(e^{\nu \tau} - 1) + 2\nu},
\]

\[
A_0(\tau) = r_\tau + \kappa \bar{\lambda} \int_0^\tau A_1(s) \, ds + \frac{1}{2} \sigma^2_\lambda \int_0^\tau A_2(s) \, ds
\]

\[
- \frac{\gamma - 1}{2\gamma} \sigma^2_\lambda \left( \rho^2 + \gamma(1 - \rho^2) \right) \int_0^\tau A_1(s)^2 \, ds
\]

= long and ugly expression

If \( \gamma > 1 \): \( A_1 \) and \( A_2 \) positive and increasing
Utility of terminal wealth only

\[ J(W, \lambda, t) = \frac{1}{1 - \gamma} \left( We^{A_0(T-t)+A_1(T-t)\lambda+\frac{1}{2}A_2(T-t)\lambda^2} \right)^{1-\gamma} \]

and the optimal investment strategy in the stock is

\[ \Pi(W, \lambda, t) = \frac{1}{\gamma} \lambda - \frac{\gamma - 1}{\gamma} \frac{\rho \sigma \lambda}{\sigma} (A_1(T-t) + A_2(T-t)\lambda) \]

... and \( 1 - \Pi(W, \lambda, t) \) in the bank.

If \( \lambda_t > 0 \) and \( \rho < 0 \)

- hedge term of \( \pi \) is positive for \( \gamma > 1 \)
  - stocks have a built-in hedge against bad times (low \( \lambda \)'s) when \( \rho < 0 \)
- hedge term increases with the horizon of investor for \( \gamma > 1 \)
  - an investor with a long horizon should invest a larger fraction of wealth in stocks than an investor with a shorter horizon
  - consistent with typical recommendations of investment advisors
Optimal portfolio weight

![Graph showing the relationship between portfolio weight and investment horizon]

- **Mean rev**
- **GBM**
Optimal portfolio weight
Utility from consumption

- Need complete market \( \sim \rho = 1 \) or \( \rho = -1 \): choose \( \rho = -1 \)
- Optimal stock investment:

\[
\Pi(W, \lambda, t) = \frac{1}{\gamma} \frac{\lambda}{\sigma} + \frac{\gamma - 1}{\gamma} D(\lambda, t, T) \frac{\sigma \lambda}{\sigma},
\]

where

\[
D(\lambda, t, T) = \frac{\int_t^T (A_1(s-t) + A_2(s-t) \lambda) \tilde{g}(\lambda, t; s) \, ds}{\int_t^T \tilde{g}(\lambda, t; s) \, ds},
\]

\[
\tilde{g}(\lambda, t; s) = \exp \left\{ -\frac{\delta}{\gamma} (s-t) - \frac{\gamma - 1}{\gamma} \left( A_0(s-t) + A_1(s-t) \lambda + \frac{1}{2} A_2(s-t) \lambda^2 \right) \right\},
\]

(with \( \rho = -1 \) in the expressions of the \( A_i \)'s)
- For \( \gamma > 1 \) and \( \lambda > 0 \), the hedging component is positive and increasing with the time horizon \( T \)
- Horizon effect on the stock investment is dampened due to consumption: “effective” investment horizon is lower than \( T \).
- The optimal consumption rate is \( C(W, \lambda, t) = \left( \int_t^T \tilde{g}(\lambda, t; s) \, ds \right)^{-1} W \).

Consumption/wealth increasing in \( \lambda \) when \( \lambda > 0 \) and \( \gamma > 1 \).
Value/growth tilts and mean reversion in stock returns

- Value stock: high book-to-market value
- Growth stock: low book-to-market value
- Various empirical studies show that value stocks and growth stocks have risk-return characteristics that deviate from the general stock market
  - short-term returns of value stocks have a higher average and lower standard deviation than growth stocks
  - value stocks are riskier than growth stocks in the long run
- Mean reversion in growth and value stocks?
- We follow Larsen & Munk (2012, Sec. 4)...

Stochastic market price of risk
Value/growth tilts and mean reversion
Stochastic volatility

The model
Optimal strategies
Suboptimal strategies
The model

Available assets:

- a risk-free asset with a constant rate of return
- three mutual funds of stocks:
  - the overall market portfolio
  - a portfolio of growth stocks
  - a portfolio of value stocks
- The dynamics of the price vector $S_t = (S_t^M, S_t^G, S_t^V)^	op$ is modeled as

  $$dS_t = \text{diag}(S_t) \left[ (r1 + \text{diag}(\sigma_S)K\lambda_t) \, dt + \text{diag}(\sigma_S)K \, dz_t \right]$$

- The dynamics of the market price of risk vector is described as

  $$d\lambda_t = \text{diag}(\kappa) (\bar{\lambda} - \lambda_t) \, dt + \text{diag}(\sigma_\lambda) \left[ L \, dz_t + \hat{L} \, d\hat{z}_t \right]$$
The data

- Use monthly U.S. data for the period March 1951 to December 2006 resulting in 670 observation time points
- Return data on the market, growth, and value portfolios are taken from the webpage of Professor Kenneth French
  - growth portfolio: 30% lowest book-to-market equities
  - value portfolio: 30% highest book-to-market equities
  - market portfolio: consists of all stocks in French’s data set

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<td>0.139</td>
</tr>
<tr>
<td>Growth portfolio</td>
<td>0.93</td>
<td>4.56</td>
<td>0.116</td>
</tr>
<tr>
<td>Value portfolio</td>
<td>1.29</td>
<td>4.40</td>
<td>0.202</td>
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</tbody>
</table>

- Stock return predictors: the dividend-price ratio, the short-term nominal interest rate, the yield spread between long term bonds and Treasury bills
- The model is estimated using simple OLS regressions
Mean reversion in estimated model

- Mean reversion in the returns of a given portfolio: instantaneous correlation between the current wealth return and future expected return is negative.
- With our estimates we get that:
  - Market portfolio: correlation = -0.347
  - Growth portfolio: correlation = -0.357
  - Value portfolio: correlation = -0.272
→ indicating mean reversion in all three portfolios
- Note smaller degree of mean reversion for value stocks than for growth stocks.
  - consistent with the observation of Campbell and Vuolteenaho (2004) that value stocks are riskier than growth stocks for longer horizons.
Optimal investment strategy (RRA=6)

\[ \pi^*(\lambda, t) = \frac{1}{\gamma} (K^\top \text{diag}(\sigma_S))^{-1} \lambda - \frac{1}{\gamma} (K^\top \text{diag}(\sigma_S))^{-1} L^\top \text{diag}(\sigma_\lambda) (b^*(t) + c^*(t)\lambda). \]
Optimal hedge strategy (RRA=6)
Wealth loss from ignoring the intertemporal hedge
Wealth loss from omitting assets

- Investors ignoring both value and growth stocks suffer the biggest loss
  - $\gamma = 6, T - t = 10$: loss $\approx 38\%$
- Investors ignoring value stocks suffer the second highest loss
  - $\gamma = 6, T - t = 10$: loss $\approx 28\%$
- Investors ignoring growth stocks suffer the lowest loss
  - $\gamma = 6, T - t = 10$: loss $\approx 23\%$
- Hence it is more important to take the special features of value stocks into account than the features of growth stocks
Model

- CRRA investor
- Risk-free asset with a constant interest rate
- Stock market index following the process

\[ dP_t = P_t \left[ (r + \lambda_1 \sigma_t) \ dt + \sigma_t \ dz_{1t} \right], \]

where \( \sigma_t \) can follow any stochastic process. \( \lambda_1 \) is constant.
- Then the optimal fraction of wealth invested in the stock index is

\[ \pi_t = \frac{1}{\gamma} \frac{\lambda_1}{\sigma_t} \quad \text{(no hedge!)} \]

- The dynamics of wealth is then

\[
\begin{align*}
dW_t &= \left( W_t \left[ r + \pi_t \sigma_t \lambda_1 \right] - c_t \right) \ dt + W_t \pi_t \sigma_t \ dz_{1t} \\
&= \left( W_t \left[ r + \frac{\lambda_1^2}{\gamma} \right] - c_t \right) \ dt + W_t \frac{\lambda_1}{\gamma} \ dz_{1t}
\end{align*}
\]

with the constant relative risk exposure preferred by CRRA investor.
- Move along the instantaneous mean-variance frontier.
Heston-type model, no options

- Assume Heston (1993) model often used for stock option pricing:

\[
dP_t = P_t \left[ (r + \bar{\lambda}_1 V_t) \ dt + \sqrt{V_t} \ dz_{1t} \right],
\]

\[
dV_t = \kappa [\bar{V} - V_t] \ dt + \rho \sigma \sqrt{V_t} \ dz_{1t} + \sqrt{1 - \rho^2 \sigma^2} \sqrt{V_t} \ dz_{2t}
\]

with market price of risk \( \lambda_{1t} = \bar{\lambda}_1 \sqrt{V_t} \) (= Sharpe ratio of the stock) with \( \bar{\lambda}_1 > 0 \).

- Low \( V \sim \) bad investment opportunities
- Constant riskfree rate \( r \).
- If \( |\rho| \neq 1 \) and the investor can only trade in the stock and the riskfree asset, the market is incomplete.
- Affine model with state variable \( x = V \). Find \( A_1 \) (and maybe \( A_0 \)).
Heston-type model, no options – cont’d

In general affine model:

\[
\begin{align*}
    r(x) &= r_0 + r_1 x, \\
    m(x) &= m_0 + m_1 x, \\
    \hat{\nu}(x)^2 &= \hat{\nu}_0 + \hat{\nu}_1 x, \\
    \|\nu(x)\|^2 &= V_0 + V_1 x, \\
    \|\lambda(x)\|^2 &= \Lambda_0 + \Lambda_1 x, \\
    \nu(x)^T \lambda(x) &= K_0 + K_1 x
\end{align*}
\]

\[
A_1(\tau) = \frac{2 \left( r_1 + \frac{\Lambda_1}{2\gamma} \right) (e^{\nu \tau} - 1)}{\left( \nu + \frac{\gamma - 1}{\gamma} K_1 - m_1 \right) (e^{\nu \tau} - 1) + 2\nu},
\]

\[
\nu = \sqrt{\left( m_1 - \frac{\gamma - 1}{\gamma} K_1 \right)^2 + 2 \frac{\gamma - 1}{\gamma} \left( r_1 + \frac{\Lambda_1}{2\gamma} \right) \left( V_1 + \gamma \hat{\nu}_1 \right)}
\]

Assume \( \gamma > 1 \) and define

\[
\bar{\kappa} = \kappa + \frac{\gamma - 1}{\gamma} \rho \sigma \sqrt{\lambda_1}, \quad \nu = \sqrt{\bar{\kappa}^2 + \frac{\gamma - 1}{\gamma^2} \bar{\lambda}^2 \sigma_V^2 (\rho^2 + \gamma [1 - \rho^2])}
\]

Then

\[
A_1(\tau) = \frac{\bar{\lambda}_1^2}{\gamma} \frac{e^{\nu \tau} - 1}{(\nu + \bar{\kappa})(e^{\nu \tau} - 1) + 2\nu}.
\]
Optimal investment strategy

\[ A_1(\tau) = \frac{\bar{\lambda}^2_1}{\gamma} \frac{e^{\nu \tau} - 1}{(\nu + \bar{\kappa})(e^{\nu \tau} - 1) + 2\nu}. \]

\( A_1 \) is positive and increasing in \( \tau \)

For CRRA investor with utility of time \( T \) wealth only (Thm. 7.7):

\[ \pi(t) = \frac{1}{\gamma} \frac{\lambda_{1t}}{\sqrt{V_t}} - \gamma – 1 \frac{\rho \sigma \nu \sqrt{V_t}}{\gamma} A_1(T - t) \]

\[ = \frac{\bar{\lambda}_1}{\gamma} - \gamma – 1 \frac{\rho \sigma \nu A_1(T - t)}{\gamma} \]

\[ = \frac{\bar{\lambda}_1}{\gamma} - \gamma – 1 \frac{\rho \sigma \nu}{\gamma^2} \frac{\lambda^2_1}{(\nu + \bar{\kappa})} \frac{e^{\nu(T - t) - 1}}{(e^{\nu(T - t) - 1} + 2\nu)}. \]

Empirically \( \rho < 0 \) so positive hedge demand: stocks have a built-in hedge against low stochastic volatility (= low Sharpe ratio).
Heston-type model with option

- Add any asset/portfolio with non-zero exposure to $dz_2$: a “stock option”
- Option price $O_t = f(P_t, V_t, t)$ with dynamics
  \[
  dO_t = \ldots dt + f_P(P_t, V_t, t) P_t \sqrt{V_t} \, dz_{1t} \\
  + f_V(P_t, V_t, t) \left( \rho \sigma \sqrt{V_t} \, dz_{1t} + \sqrt{1 - \rho^2 \sigma} \sqrt{V_t} \, dz_{2t} \right) \\
  \Rightarrow \\
  \frac{dO_t}{O_t} = \ldots dt + \left( f_P(P_t, V_t, t) \lambda_1 \sqrt{V_t} + f_V(P_t, V_t, t) \rho \sigma \sqrt{V_t} \right) \sqrt{O_t} \, dz_{1t} \\
  + f_V(P_t, V_t, t) \sqrt{1 - \rho^2 \sigma} \sqrt{O_t} \, dz_{2t}.
  \]
- Assume that $\lambda_2 t = \overline{\lambda}_2 \sqrt{V_t}$.
- Now complete market model with two risky assets
  \[
  \sigma_t = \begin{pmatrix}
  \sqrt{V_t} \\
  (f_P(P_t, V_t, t) P_t + f_V(P_t, V_t, t) \rho \sigma) \sqrt{O_t} \\
  \end{pmatrix}, \\
  \lambda(V_t) = \begin{pmatrix}
  \overline{\lambda}_1 \sqrt{V_t} \\
  \overline{\lambda}_2 \sqrt{V_t} \\
  \end{pmatrix}, \\
  \nu(V_t) = \begin{pmatrix}
  \rho \sigma \sqrt{V_t} \\
  \sqrt{1 - \rho^2 \sigma} \sqrt{V_t} \\
  \end{pmatrix}, \\
  \hat{\nu}(V_t) = 0.
  \]
- Still affine model!
Assume $\gamma > 1$ and define

$$\hat{\kappa} = \kappa + \frac{\gamma - 1}{\gamma} \sigma_V (\rho \bar{\lambda}_1 + \sqrt{1-\rho^2} \bar{\lambda}_2), \quad \hat{\nu} = \sqrt{\hat{\kappa}^2 + \frac{\gamma - 1}{\gamma^2} \sigma^2_V (\bar{\lambda}_1^2 + \bar{\lambda}_2^2)}.$$

Now relevant $A_1$-function is

$$\hat{A}_1(\tau) = \frac{\bar{\lambda}_1^2 + \bar{\lambda}_2^2}{\gamma} \frac{e^{\hat{\nu} \tau} - 1}{(\hat{\nu} + \hat{\kappa})(e^{\hat{\nu} \tau} - 1) + 2\hat{\nu}} (\neq A_1(\tau)).$$

$\hat{A}_1$ is also positive and increasing.

Optimal portfolio for an investor with CRRA utility of time $T$ wealth

$$\left(\begin{array}{c}
\pi_{St} \\
\pi_{Ot}
\end{array}\right) = \frac{1}{\gamma} \left(\sigma_t^{-T}\right)^{-1} \left(\frac{\bar{\lambda}_1 \sqrt{V_t}}{\bar{\lambda}_2 \sqrt{V_t}} - \frac{\gamma - 1}{\gamma} \left(\frac{\rho \sigma_V \sqrt{V_t}}{\sqrt{1-\rho^2} \sigma_V \sqrt{V_t}}\right) \hat{A}_1(T-t),
\right)$$

so that

$$\pi_{Ot} = \frac{O_t}{f_V(P_t, V_t, t)} \left(\frac{\bar{\lambda}_2}{\gamma \sigma_V \sqrt{1-\rho^2}} - \frac{\gamma - 1}{\gamma} \hat{A}_1(T-t)\right),$$

$$\pi_{St} = \frac{1}{\gamma} \left(\bar{\lambda}_1 - \bar{\lambda}_2 \left[\frac{\rho}{\sqrt{1-\rho^2}} + \frac{f_P(P_t, V_t, t)P_t}{\sigma_V \sqrt{1-\rho^2} f_V(P_t, V_t, t)}\right]\right) + \frac{\gamma - 1}{\gamma} \frac{f_P(P_t, V_t, t)P_t}{f_V(P_t, V_t, t)} \hat{A}_1(T-t)$$

$$= \bar{\lambda}_1 - \frac{\rho \bar{\lambda}_2}{\gamma \sqrt{1-\rho^2}} - f_P(P_t, V_t, t)P_t \frac{\pi_{Ot}}{O_t}.$$
Assume $f_V(P_t, V_t, t) > 0$. 
Option investment

$$
\pi_{Ot} = \frac{O_t}{f_V(P_t, V_t, t)} \left( \frac{\bar{\lambda}_2}{\gamma \sigma_V \sqrt{1 - \rho^2}} - \frac{\gamma - 1}{\gamma} \hat{A}_1(T - t) \right).
$$

- Speculative demand: same sign as $\bar{\lambda}_2$
  - Negative, according to empirical studies. Magnitude debated.
  - Negative exposure to volatility risk $\Rightarrow$ positive risk premium.
- Hedge term negative
  - Hedge instrument should increase in value when volatility drops (= bad inv. opp’s)
  - For option with $f_V > 0$, this requires short position.
- Short options!
Liu & Pan (2003) choose a “delta-neutral straddle”

- Straddle: long call + long put, identical strike and maturity

$$O_t = \text{Call}(P_t, V_t; K, \tau) + \text{Put}(P_t, V_t; K, \tau) \equiv f(P_t, V_t, t)$$

Call/put prices “easily” computed from Heston’s results.

- Straddle has high “vega”, i.e., $f_V(P_t, V_t, t) >>> 0$.
- Delta-neutral means $f_P(P_t, V_t, t) = 0$.
  For a given $\tau$ this determines $K$.

- Numerical examples: see Liu & Pan’s Figure 1 (next page)
  - Relatively little hedging: see bottom-left or top-right (for $\bar{\lambda}_2 = 0$)
  - For $\bar{\lambda}_2 = -6$ (conservative estimate):
    -54% in straddle, 24% in stock $\rightsquigarrow$ 130% in bank!!!
Fig. 1. Optimal portfolio weights. The $y$-axes are the optimal weight $\psi^*$ on the “delta-neutral” straddle (solid line), $\phi^*$ on the risky stock (dashed line), and $1 - \psi^* - \phi^*$ on the riskfree bank account (dashed-dot line). The base-case parameters are as described in Section 2, and the volatility-risk premium coefficient is fixed at $\xi = -6$. The base-case investor has risk aversion $\gamma = 3$ and investment horizon $T = 5$ years. The riskfree rate is fixed at $r = 5\%$, and the base-case market volatility is fixed at $\sqrt{V} = 15\%$. 

Liu & Pan (2003), Figure 1
Losses from suboptimal strategies

See Liu & Pan (2003) and Larsen & Munk (2012)

• Important to include options in the speculative portfolio because of the apparently very attractive risk-return tradeoff
  ▶ highly depending on the estimate of $\bar{\lambda}_2$!

• Relatively little is gained by the intertemporal hedge
  ▶ highly depending on the functional form assumed for the market prices of risk
Larsen & Munk (2012), Figure 4

(i): no hedging; options for spec., (ii): no hedging; no options, (iii) no options; hedge with stock
(  ) $\gamma = 2$; (  ) $\gamma = 4$; (  ) $\gamma = 6$; (  ) $\gamma = 10$
Jumps

- Liu & Pan allow for jumps of given size (large negative, “crash”)
  - Jump arrival intensity proportional to $V_t$
  - Need estimate of jump risk premium (high?)
  - Need another jump-sensitive option to complete the market, e.g., an out-of-the-money put
  - Affine jump-diffusion type model $\leadsto$ closed-form solution
  - Their Table 1 shows that the optimal put position is very sensitive to assumed jump parameters

- Jumps of many possible sizes $\leadsto$ need more options to complete the market

- Jumps in volatility?
  Liu, Longstaff & Pan (2003), Branger, Schlag & Schneider (2008)

- Need to learn more about theoretical asset pricing with stochastic volatility and jumps to understand the relevant market prices of risk!
Related issues

- Correlation risk: correlations vary stochastically
  Markets move together in bad times ⇒ diversification less effective
  See Buraschi, Porchia & Trojani (2009, JF)

- Contagion: events in one market may directly influence other markets
  See Branger, Kraft & Meinerding (2009, InsMathEcon)

- Unspanned stochastic bond volatility
  See Trolle (2009, wp)
Dynamic Asset Allocation

Chapter 13: Labor income

Claus Munk

August 2012
Outline

1. Introduction
2. Merton with income
3. Constraints and better income model
4. Interest rate risk and labor income
Labor income

- Primary source of funds for savings/investments for most individuals
- Total wealth = financial wealth + *human capital*
- Human capital is LARGE for young individuals
  With a real interest rate of 1% and a constant annual income of 300,000 for 30 years, human capital equals 7.7 millions
- The optimal financial investment will depend on
  - the magnitude of human capital,
  - the uncertainty of labor income, and
  - the correlation between labor income and asset prices
- Obvious horizon effects in human capital — life-cycle perspective
Modeling issues

- exogenous or endogenous income ($\sim$ labor supply decision)
- correlations with asset returns; spanned or unspanned income
- variations in labor income over the life-cycle
- variations in labor income over the business cycle
- restrictions on borrowing with future income as implicit collateral (moral hazard, adverse selection)
- disability risk, unemployment risk, mortality risk
A simple example

- **Constant investment opportunities**
  - Constant risk-free rate $r = 4\%$
  - A single stock (index) with $\mu = 10\%$ and $\sigma = 20\%$

- **Intuitive solution procedure:**
  1. find desired riskiness of total wealth
  2. adjust for risk profile of human capital in order to find desired riskiness of financial wealth
  3. find the portfolio with desired riskiness of financial wealth

- **With RRA $\gamma = 2$, the optimal fraction of total wealth in the risky asset is**

\[
\pi = \frac{\mu - r}{\gamma \sigma^2} = 75\%
\]
Investor with a relatively short horizon

Financial wealth: 500.000. Human capital: 500.000.

<table>
<thead>
<tr>
<th>Income Type</th>
<th>Stocks</th>
<th>Risk-free asset</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Risk-free income</strong></td>
<td>0 (0%)</td>
<td>500.000 (100%)</td>
</tr>
<tr>
<td>Financial inv.</td>
<td>750.000 (150%)</td>
<td>-250.000 (-50%)</td>
</tr>
<tr>
<td>Total position</td>
<td>750.000 (75%)</td>
<td>250.000 (25%)</td>
</tr>
<tr>
<td><strong>Modestly risky income</strong></td>
<td>250.000 (50%)</td>
<td>250.000 (50%)</td>
</tr>
<tr>
<td>Financial inv.</td>
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<td>0 (0%)</td>
</tr>
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<td>250.000 (25%)</td>
</tr>
<tr>
<td><strong>Very risky income</strong></td>
<td>500.000 (100%)</td>
<td>0 (0%)</td>
</tr>
<tr>
<td>Financial inv.</td>
<td>250.000 (50%)</td>
<td>250.000 (50%)</td>
</tr>
<tr>
<td>Total position</td>
<td>750.000 (75%)</td>
<td>250.000 (25%)</td>
</tr>
</tbody>
</table>
Investor with a relatively long horizon

Financial wealth: 500.000. Human capital: 1.500.000.

<table>
<thead>
<tr>
<th></th>
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<th>Risk-free asset</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Risk-free income</strong></td>
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<td>1.500.000 (100%)</td>
</tr>
<tr>
<td>Financial inv.</td>
<td>1.500.000 (300%)</td>
<td>-1.000.000 (-200%)</td>
</tr>
<tr>
<td>Total position</td>
<td>1.500.000 (75%)</td>
<td>500.000 (25%)</td>
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<tr>
<td><strong>Modestly risky income</strong></td>
<td>750.000 (50%)</td>
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<td>500.000 (25%)</td>
</tr>
</tbody>
</table>

- No/modest income risk: stock weight increasing in horizon
- Opposite result if labor income resembles stock returns
- Large effects of labor income on optimal portfolio
Assumptions and human capital

- Constant interest rate $r$
- Risky asset prices:

\[ dP_t = \text{diag}(P_t) \left[ (r1 + \sigma \lambda) \ dt + \sigma \ dz_t \right] \]

- Annualized income rate:

\[ dy_t = y_t[\alpha \ dt + \xi^T \ dz_t] \]

- spanned by traded assets
- human capital $\sim$ valuation of a stream of dividends (=incomes)

- With a single risky asset with $\sigma > 0$:
  - if $\xi > 0$: perfect positive correlation
  - if $\xi < 0$: perfect negative correlation

- No portfolio or borrowing constraints (except non-negative terminal wealth)
Human capital with spanned income

Computation of human capital (on the black-board?) gives

\[ H_t = H(y_t, t) = y_t M(t), \]

where

\[ M(t) = \begin{cases} 
\frac{1}{r - \alpha + \xi^\top \lambda} \left(1 - e^{-(r - \alpha + \xi^\top \lambda)(T - t)}\right), & \text{if } r - \alpha + \xi^\top \lambda \neq 0, \\
T - t, & \text{if } r - \alpha + \xi^\top \lambda = 0.
\end{cases} \]

- \( M(t) \) increasing in \( \alpha \)
- \( M(t) \) decreasing in \( r \)
- One risky asset, \( \lambda > 0 \): \( M(t) \) decreasing in \( \xi \), thus higher for \( \xi < 0 \) than for \( \xi > 0 \)
Human capital multiplier with correlation $+1$

Note: $\alpha$ varies from 1% (lowest) to 6% (highest)
Human capital multiplier with correlation $-1$

Note: $\alpha$ varies from 1% (lowest) to 6% (highest)
The optimal portfolio

- Intuitive derivation: on the black-board?
- Formal derivation: solve HJB-equation

Optimal amounts:

$$\theta_t = \frac{1}{\gamma} (W_t + H(y_t, t)) (\sigma^\top)^{-1} \lambda - H(y_t, t) (\sigma^\top)^{-1} \xi$$

Portfolio weights:

$$\pi_t = \frac{1}{\gamma} (\sigma^\top)^{-1} \lambda + \frac{H(y_t, t)}{\mathcal{W}} (\sigma^\top)^{-1} \left( \frac{1}{\gamma} \lambda - \xi \right)$$
The optimal portfolio

- With a single risky asset

\[ \pi_t = \frac{1}{\gamma} \frac{\mu - r}{\sigma^2} + \frac{H(y_t, t)}{W_t} \frac{1}{\sigma} \left( \frac{1}{\gamma} \frac{\mu - r}{\sigma} - \xi \right) \]

- \( H(y_t, t) \) increasing in horizon
  \( \Rightarrow \pi_t \) increasing in horizon if \( \mu - r > \xi \gamma \sigma \)

- Numerical example:
  - Assume \( \sigma = 0.2, \mu - r = 0.06, \xi = 0.1, \) and \( H/W = 20. \)
  - For \( \gamma = 2: \pi = 0.75 + 5 = 5.75; \) increasing in \( T - t \)
  - For \( \gamma = 4: \pi = 0.375 - 2.5 = -2.125; \) decreasing in \( T - t \)
Optimal portfolio, stock/income correlation=1

<table>
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<tr>
<th>$W/y$</th>
<th>$T - t = 10, M = 9.52$</th>
<th>$T - t = 30, M = 25.92$</th>
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<tbody>
<tr>
<td>stock</td>
<td>cash $c/y$</td>
<td>stock $c/y$</td>
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<tr>
<td>0.2</td>
<td>12.65 -11.65 1.07</td>
<td>33.15 -32.15 1.40</td>
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<tr>
<td>1</td>
<td>3.13 -2.13 1.16</td>
<td>7.23 -6.23 1.45</td>
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<tr>
<td>5</td>
<td>1.23 -0.23 1.60</td>
<td>2.05 -1.05 1.66</td>
</tr>
<tr>
<td>50</td>
<td>0.80 0.20 6.55</td>
<td>0.88 0.12 4.08</td>
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</table>

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<tr>
<td>0.2</td>
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<td>-8.92 9.92 1.25</td>
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<td>5</td>
<td>-0.52 1.52 1.51</td>
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<tr>
<td>50</td>
<td>0.08 0.92 6.18</td>
<td>-0.03 1.03 3.52</td>
</tr>
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</table>
Optimal portfolio, stock/income correlation=$\pm 1$

<table>
<thead>
<tr>
<th>$W/y$</th>
<th>$\rho = +1, M = 9.52$</th>
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<tr>
<td></td>
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<td>-0.23</td>
</tr>
<tr>
<td>50</td>
<td>0.80</td>
<td>0.20</td>
</tr>
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</table>
With multiple financial assets

- Spanning assumption less extreme. With $n$ similar assets with cross-correlations $\rho_{PP}$, the asset-income correlation has to be

$$\rho_{Py}^2 = \rho_{PP} + \frac{1 - \rho_{PP}}{n}.$$ 

- Individual investors should under-invest in stocks that are highly correlated with their labor income
  - 58% of Enron’s 401k pension fund was invested in Enron stocks prior to the 98.8% drop in the Enron stock price in 2001
Model by Bodie, Merton & Samuelson (1992, JEDC)

- Spanned periodic wage rate:
  \[ d\omega_t = \omega_t [m \, dt + v^\top \, dz_t] \]
- \( \varphi_t \in [0, 1] \): fraction of time working \( \Rightarrow \) income rate \( \varphi_t \omega_t \)
- \( l_t = 1 - \varphi_t \): leisure time
- Utility of consumption and leisure:
  \[ u(c, l) = \frac{1}{1 - \gamma} \left( c^\beta l^{1-\beta} \right)^{1-\gamma} \]
  - leisure is a second cons. good with relative price \( \omega_t \)
- Constant investment opportunities
- Solve problem:

\[
J(W, \omega, t) = \max_{(c_s, \theta_s, l_s)_{s \in [t, T]}} E_{W, \omega, t} \left[ \int_t^T e^{-\delta(s-t)} \frac{1}{1 - \gamma} \left( c_s^\beta l_s^{1-\beta} \right)^{1-\gamma} ds \right]
\]

\( \rightsquigarrow \) solve HJB-equation (ignoring constraint \( l_t \leq 1 \))
Solution

- Indirect utility function:

\[ J(W, \omega, t) = \frac{1}{1 - \gamma} K \ G(t)^{\gamma} \omega^{-(1-\beta)(1-\gamma)} (W + \omega F(t))^{1-\gamma} \]

- Optimal consumption and investment strategies:

\[ c_t = \frac{\beta}{G(t)} (W_t + \omega_t F(t)), \]
\[ \ell_t = \frac{1 - \beta}{G(t)} \frac{W_t + \omega_t F(t)}{\omega_t}, \]
\[ \theta_t = \frac{1}{\gamma} (W_t + \omega_t F(t)) \left( \sigma^\top \right)^{-1} \lambda - F(t) \omega_t \left( \sigma^\top \right)^{-1} \nu \]
\[ - \frac{(1 - \beta)(1 - \gamma)}{\gamma} (W_t + \omega_t F(t)) \left( \sigma^\top \right)^{-1} \nu. \]
Fixed labor supply, $\bar{\varphi}$

- Exogenous income $y_t = \bar{\varphi} \omega_t$
- Indirect utility (black-board?):
  \[
  J(W, \omega, t; \bar{\varphi}) = \frac{1}{1 - \gamma} (1 - \bar{\varphi})^{(1 - \beta)(1 - \gamma)} g(t)^{1 - \beta(1 - \gamma)} (W + \bar{\varphi} \omega F(t))^{\beta(1 - \gamma)}
  \]
- Optimal investment strategy:
  \[
  \theta_t = \frac{1}{1 - \beta(1 - \gamma)} (W_t + \bar{\varphi} \omega_t F(t)) (\sigma^T)^{-1} \lambda - \bar{\varphi} \omega_t F(t) (\sigma^T)^{-1} \nu.
  \]
- Can maximize $J(W_0, \omega_0, 0; \bar{\varphi})$ to find $\varphi^* = \beta - \frac{(1 - \beta)W_0}{\omega_0 F(0)}$
Labor supply flexibility and optimal risk-taking

- Assume a single risky asset and a deterministic wage rate
- Optimal investment with flexible labor supply:
  \[ \theta_t^{\text{flex}} = \frac{1}{\gamma} \left( W_t + \omega_t F(t) \right) \frac{\lambda}{\sigma} \]
- Optimal investment with fixed labor supply:
  \[ \theta_t^{\text{fix}} = \frac{1}{1 - \beta(1 - \gamma)} \left( W_t + \bar{\omega}_t F(t) \right) \frac{\lambda}{\sigma} \]
- If \( \gamma < 1 \), then \( \theta_t^{\text{flex}} > \theta_t^{\text{fix}} \)
- “Take more risk when you can adjust labor supply”
- Intuition: compensate for bad returns by working harder
- Same conclusion for \( \gamma \) “somewhat higher than” 1, in particular when \( \omega F(t)/W \) is high (young investors)
- Conclusions unclear with risky (maybe unspanned) wage
Borrowing constraints and market incompleteness

- Impossible to perfectly hedge or insure against all income changes — unspanned income; incomplete market (including real estate helps)
- Significant borrowing against future income impossible in real life: moral hazard, adverse selection
- Human capital investor-specific
- No-borrowing reduces human capital considerably. See, e.g., Munk (2000, JEDC) with GBM income and infinite horizon.
- Constraints and/or unspanned income require numerical solution
Income variations over life cycle and business cycle

- **Life cycle [discrete time]:**

- **Life and business cycle [interest rates, cont. time]:**

- **Life and business cycle [dividend yield, discrete time]:**
Calibrated life-cycle income profiles

Cocco, Gomes & Maenhout (RFS 2005) estimate the typical life-cycle pattern in (broad) labor income for different education groups based on US data from the Panel Study of Income Dynamics. They assume constant retirement income given as a fraction (“the replacement rate”) of (permanent) income just before retirement.

Figure 1
Labor income processes estimated from the PSID for the three different education groups: households without high school education, households with high school education but without a college degree, and college graduates.
For each group, the figure plots the estimated age dummies and a fitted third-order polynomial.
Model

- Discrete-time; annual decisions
- CRRA utility of consumption [and, possibly, wealth at death (bequest)]
- Uncertain life-time; max age 100; empirical survival probabilities
- Labor income of individual \( i \) is \( Y_{it} \)

\[
\ln Y_{it} = \begin{cases} 
    f(t, Z_{it}) + \nu_{it} + \varepsilon_{it}, & \text{for } t \leq K \\
    \ln \lambda + f(K, Z_{iK}) + \nu_{iK}, & \text{for } t > K
\end{cases}
\]

with deterministic part \( f(t, Z_{it}) \), permanent income shock \( \nu_{it} = \nu_{i,t-1} + \xi_t + \omega_{it} \) and transitory income shock \( \varepsilon_{it} \), where \( \xi_t, \omega_{it}, \varepsilon_{it} \sim N(0, \cdot) \)

- Riskfree bond with constant return [Amount \( B_{it} \)]
- Risky stock with iid returns [Amount \( S_{it} \)], possibly correlated with \( \xi_t \)
- Borrowing and short-selling prohibited: \( B_{it}, S_{it} \geq 0 \)
Solution method

Solve the Bellman equation of individual $i$

$$J_i(t, X_{it}, \nu_{it}) = \max_{c_{it} \geq 0; \alpha_{it} \in [0, 1]} \left\{ u(c_{it}) + e^{-\delta} p_t E_t [J_i(t + 1, X_{i,t+1}, \nu_{i,t+1})] \right\}.$$  

- $X_{it}$: cash-on-hand at time $t$ (value of previous pf + income)
- $\alpha_{it} = S_{it}/(X_{it} - c_{it})$: portfolio weight of stock
- $p_t$: conditional survival probability
- terminal condition at $T$ reflecting possible utility of bequest
- value function is homogeneous wrt. $\nu_{it}$  
  $\leadsto$ only one effective state variable

Solve numerically on a grid using backward induction...  
Illustrated results based on 10,000 simulations using optimal strategies...  
Benchmark parameters include $\rho = 0$, $\gamma = 10$, no bequest...
Consumption, income, and wealth over the life cycle (Fig3A)
Varying importance of human capital over the life cycle (Fig3B)
Stock weight over the life cycle (Fig3C)

Some young people should have $c = y \rightleftharpoons$ no savings $\rightleftharpoons$ not participate in stock market
Importance of income risk (Fig4A)

“Income risk crowds out asset-holding risk”
This is for $\gamma = 10$, but for $\gamma = 3$ differences are very small
Importance of income-stock correlation (Fig4B)

High correlation $\leadsto$ retire with lower wealth $\leadsto$ need higher $\pi$ in retirement to obtain overall desired risk exposure
With possible disastrous income shock (Fig5)

![Graph showing simulated portfolio share invested in stocks with a 0.5% probability of zero-income realization, and for the benchmark case.](image)

**Figure 5**
Simulated portfolio share invested in stocks with a 0.5% probability of zero-income realization, and for the benchmark case.

Note: jump to zero income is very extreme!
Importance of risk aversion (Fig10A)
With Epstein-Zin preferences (Fig10B)

High EIS \rightarrow less concerned about consumption smoothing \rightarrow saves less for retirement
Labor income and interest rate uncertainty

- Reference: Munk & Sørensen (JFE 2010)
- The paper combines two important features of long-term portfolio choice
  - labor income
  - business cycle variations in investment opportunities: real interest rates (fixed market prices of risk)
- The paper studies optimal stock/bond/cash allocation with labor income
  - bonds and stocks correlate differently with labor income
  - invest in bonds to hedge total wealth (including human wealth) against interest rate risk
  - labor income growth related to interest rate (business cycle): does human wealth replace a long-term coupon bond or cash?
Financial assets

- Real short rate (=return on cash):
  \[ dr_t = \kappa (\bar{r} - r_t) \, dt - \sigma_r \, dz_{rt}, \quad \lambda_r \text{ constant} \]

- Price \( B_t = B(r_t, t) \) of real bond:
  \[ dB_t = B_t \left[ (r_t + \sigma_B(r_t, t) \lambda_r) \, dt + \sigma_B(r_t, t) \, dz_{rt} \right] \]

  Zero-coupon bond: \( B^s(r, t) = \exp\{-a(s - t) - b(s - t)r\} \)

- Real stock price dynamics
  \[ dS_t = S_t \left[ (r_t + \psi) \, dt + \sigma_S \left( \rho_{SB} \, dz_{rt} + \sqrt{1 - \rho_{SB}^2} \, dz_{St} \right) \right] \]

  where \( \psi = \rho_{SB} \sigma_S \lambda_r + \sqrt{1 - \rho_{SB}^2} \sigma_S \lambda_S. \)
**Labor income**

\[ dy_t = y_t \left[ (\xi_0(t) + \xi_1(t)r_t) \ dt + \sigma_y \left\{ \rho_{yB} \, dz_{rt} + \hat{\rho}_{yS} \, dz_{st} + \sqrt{1 - \|\rho_{yP}\|^2} \, dz_{yt} \right\} \right] \]

- exogenously given
- for now: zero income in retirement phase \([\tilde{T}, T]\)
- drift is interest rate dependent: business cycle fluctuations
  \(\implies\) expect \(\xi_1 > 0\) on average
- ignore variations in income volatility over the business cycle (and the life-cycle)
- only permanent shocks
- no jumps (lay-offs)
Model calibration

- Asset parameters estimated from quarterly data, 1951-2003, U.S.; fixed when estimating income parameters

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<th>Parameter</th>
<th>Benchmark</th>
<th>PSID College graduates</th>
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<tr>
<td>$\xi_1$</td>
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<th>Value (S.E.)</th>
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<td>0.0322 (0.2124)</td>
<td>0.3589 (0.1584)</td>
<td>0.4861 (0.1727)</td>
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<td>0.4912 (0.2124)</td>
<td>5.4803 (0.1584)</td>
<td>0.4861 (0.1727)</td>
</tr>
</tbody>
</table>
Decision problem

- Consumption $c_t$, fractions of financial wealth invested $\pi_t = (\pi_{Bt}, \pi_{St})^\top$
- Real financial wealth:

$$dW_t = \left[ W_t (r_t + \pi_{Bt}\sigma_B(r_t, t)\lambda_r + \pi_{St}\psi) - c_t + y_t \right] dt$$

$$+ W_t \left[ (\pi_{Bt}\sigma_B(r_t, t) + \pi_{St}\sigma_S\rho_{SB}) dz_{rt} + \pi_{St}\sigma_S \sqrt{1 - \rho_{SB}^2} dz_{St} \right]$$

- CRRA utility function $U(c) = \frac{c^{1-\gamma}}{1-\gamma}$
- Indirect utility function

$$J(W, r, y, t) = \max_{(c, \pi) \in A_t} E_t \left[ \int_t^T e^{-\delta(s-t)} U(c_s) ds + \varepsilon e^{-\delta(T-t)} U(W_T) \right]$$

$A_t$: admissible strategies (possibly constrained)
General solution approach

Solve the HJB-equation

\[
\delta J = \max_{c, \pi} \left\{ U(c) + J_t + J_W \left( rW + W\pi^\top \Sigma \lambda - c + y \right) + \frac{1}{2} W^2 J_{WW} \pi^\top \Sigma \Sigma^\top \pi \\
+ J_r \kappa [\bar{r} - r] + \frac{1}{2} J_{rr} \sigma_r^2 + J_y y (\xi_0 + \xi_1 r) + \frac{1}{2} J_{yy} y^2 \sigma_y^2 \\
- W J_{Wr} \pi^\top \Sigma e_1 \sigma_r + W J_{Wy} y \sigma_y \pi^\top \Sigma \rho_p + J_{ry} y \rho_y \sigma_y \sigma_r \right\}
\]

with \( J(W, r, y, T) = \varepsilon U(W) = \varepsilon W^{1-\gamma}/(1-\gamma) \).

- Spanned income risk + no portfolio constraints: explicit solution
- Otherwise: numerical solution
  - Unspanned income risk
  - Liquidity constraint: When financial wealth reaches zero...
    - invest nothing in risky assets
    - consume less than current income
  - Short-sales constraints: \( \pi_B, \pi_S, 1 - \pi_B - \pi_S \in [0, 1] \)
Spanned income and no portfolio constraints

- Requires unrealistic income-asset correlations
- Human wealth $H_t = y_t \int_t^T h(s - t)(B^s(r, t))^{1-\xi_1} \, ds$
  - $\xi_1 = 0$: human wealth $\sim$ long-term coupon bond
  - $\xi_1 = 1$: human wealth $\sim$ cash
- Indirect utility: $J(W, r, y, t) = \frac{1}{1-\gamma} g(r, t)\gamma (W + H(y, r, t))^{1-\gamma}$
- Optimal consumption: $c_t = (W_t + H_t)/g(r_t, t)$
- Stock weight:
  $$\pi_{St} = \frac{1}{\gamma} \frac{W+H}{W} \frac{\lambda_s}{\sigma_s\sqrt{1-\rho_{SB}^2}} - \frac{H}{W}\sigma_y \frac{\rho_{YS}-\rho_{SB}\rho_{YB}}{\sigma_s(1-\rho_{SB}^2)}$$
- Bond weight:
  $$\pi_{Bt} = \frac{1}{\gamma \sigma_B} \frac{W+H}{W} \left( \lambda_r - \frac{\rho_{SB}\lambda_s}{\sqrt{1-\rho_{SB}^2}} \right) + \left( 1 - \frac{1}{\gamma} \right) \frac{\sigma_r}{\sigma_B} \frac{W+H}{W} G(r_t, t)$$
  $$- \frac{H}{W}\sigma_y \frac{\rho_{YS}-\rho_{SB}\rho_{YB}}{\sigma_s(1-\rho_{SB}^2)} - (1 - \xi_1) \frac{\sigma_r}{\sigma_B} \frac{\gamma}{W} \int_t^T b(s - t) h(t, s) (B^s)^{1-\xi_1} \, ds$$
- Low financial wealth: invest more in risky assets to obtain desired overall risk-return trade-off
Special case: locally riskfree income

Now let $\sigma_y = 0$.

Furthermore, illustrations assume

- Benchmark parameters unless otherwise mentioned
- Bond = 10-year zero-coupon bond
- Investor has...
  - time preference rate $\delta = 0.03$,
  - relative risk aversion $\gamma = 4$,
  - horizon $T = 30$ years
  - age-independent income parameters, no retirement phase
- Assume $r = \bar{r}$ and make sure $\xi_0 + \xi_1 r$ is fixed (at 2%) when varying $\xi_0, \xi_1$
- Without income: 25% in stock, 70% in bond (to hedge $r$-risk), 5% in bank
Investment/total wealth as function of human/total wealth

![Graph showing investment/total wealth as function of human/total wealth. The graph includes lines representing different asset distributions: Stock, Bond (x1=0), Bank (x1=0), Bond (x1=1), Bank (x1=1). The x-axis represents human wealth/total wealth, and the y-axis represents investment/total wealth.]
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Investment/financial wealth as function of human/total wealth

Investment / Financial wealth vs Human wealth / Total wealth

- Stock
- Bond (xi1=0)
- Bank (xi1=0)
- Bond (xi1=1)
- Bank (xi1=1)
Investment/total wealth over life
Expected wealth over life

Value of $\xi_1$ not so important for life-cycle pattern of wealth, but very important for allocation between cash and long-term bonds.
Numerical solution approach

CRRA utility allows reduction of the dimension:

\[ J(W_t, r_t, y_t, t) = y_t^{1-\gamma} F(x_t, r_t, t), \quad x_t = e^{-\beta t} \frac{W_t}{y_t} \]

... leads to “HJB-equation” for \( F \), which we solve numerically using a backward iterative finite difference procedure.

With no income and no binding portfolio constraints in retirement:

\[ J(W, r, y, \tilde{T}) = g(r, \tilde{T})^\gamma W^{1-\gamma}/(1-\gamma) \quad \Rightarrow \]

\[ F(x, r, \tilde{T}) = \frac{1}{1-\gamma} e^{-\beta(\gamma-1)\tilde{T}} g(r, \tilde{T})^\gamma x^{1-\gamma}. \]

Solve backwards from \( \tilde{T} \).

We ensure non-negative financial wealth at all times.
Bond investment for different constraints

Thick curves: $\xi_1 = 0$. Thin curves: $\xi_1 = 1$. 

\[ \text{Bond inv./Financial wealth} \] 

\[ \text{Time in years} \]
Stock investment for different constraints

Thick curves: $\xi_1 = 0$. Thin curves: $\xi_1 = 1$. 
Sensitivity to $\sigma_y$: stock investment

Thick curves: $\xi_1 = 0$. Thin curves: $\xi_1 = 1$.
Black, diamonds: low $\sigma_y = 0.1$. Gray, boxes: high $\sigma_y = 0.3$.
Income risk crowds out stock market risk.
Sensitivity to $\sigma_y$: bond investment

**Thick** curves: $\xi_1 = 0$. **Thin** curves: $\xi_1 = 1$.

Black, diamonds: low $\sigma_y = 0.1$. Gray, boxes: high $\sigma_y = 0.3$. 

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Sensitivity to $\rho_{yr}$: stock investment

**Thick** curves: $\xi_1 = 0$. **Thin** curves: $\xi_1 = 1$.
Black, diamonds: low $\rho_{yr} = -0.25$. Gray, boxes: high $\rho_{yr} = 0.25$. 
Sensitivity to $\rho_{yr}$: bond investment

**Thick** curves: $\xi_1 = 0$. **Thin** curves: $\xi_1 = 1$.
Black, diamonds: low $\rho_{yr} = -0.25$. Gray, boxes: high $\rho_{yr} = 0.25$.
High $\rho_{yr} \Rightarrow H$ less like bond $\Rightarrow$ higher direct inv in bond
Sensitivity to $\rho_{yr}$: bank investment

**Thick** curves: $\xi_1 = 0$. **Thin** curves: $\xi_1 = 1$.
Black, diamonds: low $\rho_{yr} = -0.25$. Gray, boxes: high $\rho_{yr} = 0.25$. 
Sensitivity to $\xi_1$: stock investment

Thick unmarked curve: high $\xi_1 = 4$. Thin unmarked curve: $\xi_1 = -4$. 
Sensitivity to $\xi_1$: bond investment

**Thick** unmarked curve: high $\xi_1 = 4$. **Thin** unmarked curve: $\xi_1 = -4$.

High $\xi_1 \sim H$ like short position in long-term bond
Sensitivity to $\xi_1$: bank investment

**Thick** unmarked curve: high $\xi_1 = 4$. **Thin** unmarked curve: $\xi_1 = -4$. 
Calibrated life-cycle income profiles

Source: Cocco, Gomes & Maenhout, RFS 2005.

Corresponds to

\[
\xi_0(t) = \begin{cases} 
\bar{\xi}_0 + b^i + 2c^i t + 3d^i t^2 & \text{if } 20 \leq t \leq 65, \\
-(1 - \bar{P}^i) & \text{if } 65 < t < 66, \\
0 & \text{if } t \geq 66,
\end{cases}
\]

and a constant income in retirement.
Optimal portfolio: no high school

Note: based on $\xi_1 = 0.03$ (median value for no high school).
Optimal portfolio: college degree

Note: based on $\xi_1 = 0.49$ (median value for college graduates)
Optimal portfolio: college degree, high $\xi_1 = 3$

Optimal portfolio: college degree, low $\xi_1 = -3$