Strategic Asset Allocation for Long-Term Investors: Parameter Uncertainty and Prior Information

Roy Hoevenaars[a,b,e]   Roderick Molenaar[b]   Peter Schotman[a,d,e]

Tom Steenkamp[b,c]

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Abstract: This paper considers the strategic asset allocation of long-term investors who account for prior information about expected returns. We develop a vector autoregressive model where different investors have conflicting prior views on long-run expected returns. We distinguish two types of prior information: (i) direct views on the long-term mean of the equity and bond premium, and (ii) prior views on the long-run mean of predictor variables like the dividend yield and the nominal interest rate. Both priors have a pronounced effect on optimal portfolios and term structures of risk. Even weak prior information on the unconditional mean of highly persistent time series like dividend yield and the nominal interest rate changes the estimated persistence of shocks and the predictability of excess returns. For long-term investors we find that a portfolio that is optimal given one prior, often entails large utility costs when evaluated under an alternative prior distribution. We define a robust portfolio as the portfolio of an investor with a prior that has minimal costs among all priors that we consider.

Keywords: strategic asset allocation, bayesian vector autoregression, parameter uncertainty, robust portfolio choice

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Affiliations: [a] Maastricht University, [b] ABP, [c] Vrije Universiteit Amsterdam, [d] CEPR, [e] NETSPAR.

Corresponding author: R. Hoevenaars, Department of Quantitative Economics, Maastricht University, P.O. Box 616, 6200 MD Maastricht, Netherlands, tel. +31-43-3883838, fax +31-43-3884875 email: `r.hoevenaars@ke.unimaas.nl`

1

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1   Introduction

The optimal behavior of long-run investors differs from myopic investors if asset returns are predictable. The revived interest in strategic asset allocation and the well-documented predictability in stocks and bonds indicates that optimal portfolio choice is horizon dependent.[1,2] Merton (1969, 1971) showed that under changing investment opportunities optimal portfolio decisions of long-term investors differ from those of short-term investors. Long-term investors can not only benefit from risk diversification between assets, but also from time diversification within an asset class. The optimal portfolio contains a speculative component and a hedge component. The hedge component depends on the covariance properties of returns, whereas the speculative demand depends on expected excess returns in the next period. How robust are the horizon dependent risk properties of stocks, bonds and T-bills to parameter uncertainty? How robust are these term structures of risk to prior views about the future level of predictor variables as the dividend yield?

This paper uses bayesian methods to consider the strategic asset allocation of long-term investors. The investor adapts parameter uncertainty and prior information about the level of expected asset returns in optimal portfolio choice. His investment universe consists of stocks, bonds and T-bills. First, we consider an investor who has a prior belief that the future level of expected asset returns, inflation and macro economic variables as the dividend yield and interest rates differs from the historical unconditional mean in the data. Wachter and Warusawitharana (2007), Avramov (2004) and Pastor (2000) suggest that allowing informative beliefs in the portfolio choice decision can be superior to using data alone. Furthermore, the speculative part of optimal portfolio choice in a mean variance or power utility buy-and-hold setting is extremely sensitive to small changes in the expected return assumptions (See Kan and Zhou (2006) and DeMiguel, Garlappi and Uppal (2006)). Pastor and Stambaugh (2000) and Jorion (1986) and many others have suggested priors that shrink the portfolio weights towards an asset pricing model. In this paper informative priors shrink the mean of

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[1] The predictability of asset returns is often described to be captured by valuation ratios as the dividend yield, price-earnings ratio, and also by inflation, interest rates and the term spread. A few references to this large literature are Barberis (2000) for work on the dividend yield, Campbell and Shiller (1988) for the price-earnings ratio, Lettau and Ludvigson (2001) for the consumption-wealth ratio, and see for example the enormous amount of evidence against the expectations model of the term structure reviewed in Dai and Singleton (2002, 2003). As the evidence is not uncontroversial we also refer to Goyal and Welch (2003) for a dissenting view.

[2] We refer to Campbell and Viceira (2005), Detemple, Garcia and Rindisbacher (2005), Brandt and Santa-Clara (2004), Hoevenaars, Molenaar, Schotman and Steenkamp (2007).

the future return distribution to the prior beliefs. This makes it possible to control the speculative part of the portfolio choice.

Another motivation for such a prior stems from the sensitivity of historical average returns to the choice of the sample period (unlike volatilities and correlations). Therefore investors can have good reasons to base their future return expectations not only on historical data, but also on other criteria. It is common practice for long-term investors to use historical data to estimate volatilities and correlations of the long-term future return distribution, and use economic theory and current market circumstances to form their view about the long-term mean of the future return distribution. We consider a robust portfolio which minimizes the expected utility loss when there are multiple experts with competing beliefs. In our case the robust portfolio is based on an informative prior which is rather conservative about stocks, a little optimistic about bonds and includes parameter uncertainty.

Furthermore, the persistence in the dividend yield, interest rates and yield spreads makes it hard to estimate the long-term mean of these state variables. If it turns out that the true long-term mean of the dividend yield is far above or below the mean in the sample period, the autocorrelation parameter of the dividend yield is underestimated. More persistency of the dividend yield has a direct effect on the term structures of risk of for instance stock returns. The informative prior on the future level of the state variables can therefore influence the hedge part of optimal portfolio choice. We find that imposing prior information about the mean of the future distribution of the state variables can change the term structures of risk.

Second, we account for parameter uncertainty in a bayesian setting as an additional source of uncertainty in optimal portfolio choice. We also study the robustness of the term structures to parameter uncertainty. We find that predictability dominates parameter uncertainty most of the times. Nevertheless, time diversification properties within asset classes in terms of volatilities weaken if parameter uncertainty is incorporated. The risk properties in the cross section are much more stable than the ones in the time dimension. Risk diversification properties between asset classes in terms of correlations seem robust against parameter uncertainty. Apparently, the impact of parameter uncertainty on the covariance is proportional to the impact on the variance. As a consequence the effect of parameter uncertainty cancels out to a large degree in terms of correlation.

This paper builds on previous research that applies bayesian methods in asset allo-

cation[3]. Brennan (1998) considers the role of learning about the mean return on risky assets on dynamic portfolio choice when there are constant investment opportunities. In this paper we adapt informative priors about mean returns in the presence of time varying investment opportunities and parameter uncertainty. We also derive a robust portfolio when there are competing experts with different prior beliefs. Unlike Brennan (1998), Xia (2001) and Brandt, Goyal, Santa-Clara and Stroud (2003), we ignore learning in the optimal portfolio choice. Learning induces the optimal asset allocation to be less sensitive to predictability, which reduces horizon effects.

Kandel and Stambaugh (1996) find that weak predictability of stock returns can still have large impact on the optimal portfolio choice. They use the sample evidence to update prior beliefs about regression coefficients. Wachter and Warusawitharana (2007) examine optimal portfolio choice for an investor who is skeptical about the predictability in the data. They model an informative prior on the regression coefficients as the expected improvement in the maximum Sharpe ratio from conditioning portfolio choice on the predictor variable. We specify an informative prior on the level of future asset returns and state variables, rather than on the amount of predictability as in Wachter and Warusawitharana (2007), Shanken and Tamayo (2005), Avramov (2004), Xia (2001), Kandel and Stambaugh (1996) and many others. We also focus on a long-term investor who accounts for time varying investment opportunities.

Since the early work by Klein and Bawa (1976) and Brown (1979) many studies account for parameter uncertainty in portfolio choice problems. Barberis (2000) analyzes optimal asset allocation for two asset categories: stocks and cash. He shows that incorporating parameter uncertainty can substantially reduce the horizon effect. We extend Barberis' work in a number of ways. Our investment universe includes stocks, bonds and T-bills. We do not only focus on time diversification, but particularly on risk diversification. Furthermore, we describe the return dynamics by a vector autoregression for asset returns and macro-economic state variables.

Black and Litterman (1992) develop a bayesian framework in which the optimal portfolio is the scaled market equilibrium portfolio plus a weighted sum of portfolio's representing prior views. The Black-Litterman approach is widely used for (global) tactical asset allocation. The approach in this paper focuses on the long-term strategic

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[3] A growing number of studies uses bayesian methods in finance. Apart from the references in the text we refer to Pastor and Stambaugh (1999), Lewellen and Shanken (2002), Stambaugh (1999), Pastor (2000), Avramov (2002), Cremers (2002), Kadiyala and Karlsson (1997), Hollifield, Koop and Li (2003), Fugazza, Guidolin, and Nicodano (2007).

asset allocation. It deals with parameter uncertainty, asset return predictability, and incorporates time varying investment opportunities. Furthermore, the return dynamics are based on a vector autoregression, and portfolio constraints are easily implemented. This paper also derives the robust portfolio which minimizes the expected utility loss when multiple experts have different prior beliefs about expected returns, interest rates and macro economic variables.

The remainder of this paper is organized as follows. Section 2 describes the modeling framework for the return dynamics. We consider a vector autoregression for returns and macro-economic state variables and we use bayesian methods to incorporate parameter uncertainty and prior information about the level of future returns and state variables. Section 3 describes the used data and elaborates on the prior information about the level of the future mean. Section 4 discusses the empirical results. We discuss the estimation results, and we show the robustness of the term structures to parameter uncertainty and prior information. In section 5 we show the impact of parameter uncertainty and prior information on optimal portfolio choice for a buy-and-hold investor. Section 6 defines a robust portfolio as the portfolio of an investor with a prior that has minimal costs among all priors that we consider. Finally, section 7 concludes.

2   Modeling framework

Following Campbell and Viceira (2005), among others, we describe the return dynamics by a first-order vector autoregression (VAR) model. Specifically, let

$$y_t = \begin{pmatrix} r_t \\ x_t \\ s_t \end{pmatrix}$$

where r_t is the real return on the three month T-Bill, x_t contains excess returns on stocks (x_s) and bonds (x_b), and s_t is a vector of other state variables that capture important dynamics in the data. In the empirical model s_t will have three elements: the nominal return on a 3-month T-bill, dividend yield and yield spread. Risk and return dynamics follow the first order VAR,

$$y_{t+1} = c + By_t + \epsilon_{t+1} \tag{1}$$

where ϵ_t is normally distributed with zero mean and covariance matrix Σ.

The main purpose of the analysis is the effects of parameter and model uncertainty. Parameter uncertainty is accounted for by a bayesian analysis of the VAR. Model uncertainty is represented by a series of alternative priors on the unconditional mean of the asset returns and state variables.

Our first prior is uniform prior on c and B and an invariant prior for Σ,

$$p(c, B, \Sigma) \propto |\Sigma|^{-(n+1)/2} I(B) \tag{2}$$

The indicator function $I(B)$ restricts the domain of the VAR to the stationary region $\mathcal{C} \subset R^{n \times n}$ such that the maximum eigenvalue of B is less than one. We include this prior as a benchmark for comparison with the informative priors to be specified below. The posterior mode of the flat prior coincides with the least squares estimates of the VAR. It is also a benchmark in the sense that the flat prior has previously been used by Barberis (2000) in his analysis of the effects of parameter uncertainty on long-term portfolio decisions between the riskfree asset and equity.

In order to impose prior views on the unconditional mean of asset returns and state variables, we rewrite the VAR model (1) as

$$y_{t+1} = \mu + B(x_t - \mu) + \epsilon_{t+1} \tag{3}$$

where μ is the vector of unconditional means of all elements in the VAR. Our informative prior on μ is specified as a normal distribution with mean μ_0 and covariance matrix Ω_0/κ. Both μ_0 and Ω_0 are exogenously specified. The scalar parameter κ is a shrinkage factor. It represents the investors degree of confidence in his prior information. A shrinkage factor close to zero corresponds to a dispersed prior on μ. A large shrinkage factor gives much weight to the prior information and a precision factor equal to infinity imposes the mean. For a clear interpretation of κ we set Ω_0 equal to the long-run sample covariance matrix of the time series,

$$\Omega_0 = \frac{1}{T^2} \sum_{\ell=-L}^{L} \left(1 - \frac{|\ell|}{L+1} \right) \sum_t (y_t - \bar{y})(y_t - \bar{y})' \tag{4}$$

With this choice of Ω_0 a precision factor equal to one gives equal weight to the prior information and the data in the likelihood. We vary κ to increase or decrease the precison of the prior while keeping Ω_0 fixed.

Apart from the informative prior on μ, the priors on B and Σ are as in the benchmark prior. We thus obtain the joint prior

$$p(\mu, B, \Sigma) \propto I(B)|\Sigma|^{-(n+1)/2} \exp\left(-\tfrac{1}{2}\kappa(\mu - \mu_0)'\Omega_0^{-1}(\mu - \mu_0)\right), \tag{5}$$

The non-linearity in the parameters, $c = (I - B)\mu$, makes this setting different from a uniform prior on all coefficients including the constant term c. The prior on the mean μ can be transformed back to a prior on the constant term c in the reduced form parameterization (2). Define $A = I - B$. The transformation $c = A\mu$ gives a Jacobian $|A|^{-1}$, leading to the implied joint density of c and B

$$p(c, B) \propto |A|^{-1} \exp\left(-\tfrac{1}{2}\kappa(c - A\mu_0)'(A\Omega_0 A')^{-1}(c - A\mu_0)\right) \tag{6}$$

Since the matrix A is singular at points where B has a unit root, the prior forces a singularity on the constant terms c when the dynamics of the system move towards the unit root. Prior independence of μ and B is thus very different from prior independence of c and B. This prior correlation between c and B will have a strong effect on the posterior for B. Conditional on a small constant term, in the sense that $c - A\mu_0$ is small, the prior induces a large weight on one or more unit roots in the system. Relative to the uniform prior (2) the specification of an independent prior directly on the unconditional mean will shift the posterior towards the unit root. From the results in Schotman (1994) we infer that the posterior of B will be proper with well-defined first and second moments if the prior on μ is proper in the sense that $\kappa > 0$.

As an illustration of the prior, figure 1 shows the marginal prior distribution of c in the univariate AR(1) case when μ is normally distributed with mean $\mu_0 = 2$ and variance $\Omega_0 = 2$, while the first order autocorrelation coefficient, the single element in B, has a uniform distribution. The density has a distinct spike at $c = 0$. Even more telling are the conditional densities $p(B|c)$ for alternative values of c. For the same prior, figure 2 shows the conditional densities of B for $c = 0$ and $c = 0.25$. Conditional on $c = 0$ the density is highly concentrated on the unit root, whereas for $c = 0.25$, the density drops to zero for $B > 0.98$. Inference on the autocorrelation is strongly correlated with inference on the constant term in the model.

Collect all parameters in $\theta = (\mu, \text{vec}(B), \text{vech}(\Sigma))$. Our posterior inference combines the prior with the conditional likelihood function

$$L(\mathbf{Y}|\theta) = \prod_{t=1}^{T} L(y_t|y_{t-1}, \theta) \tag{7}$$

which takes the initial condition y_0 as given. Inference on the parameters proceeds through a simple Gibbs sampler. Technical details are provided in the appendix.

Uhlig (1994) and Wachter and Warusawitharana (2007) derive the unconditional likelihood by combining the likelihood of the first observation with the conditional

6

likelihood of the other observations in (7). The first observation is then a draw from the unconditional distribution of y_t, which has mean μ and covariance matrix Ψ satisfying

$$\text{vec}(\Psi) = (I - B \otimes B)^{-1}\text{vec}(\Sigma)$$

This only leads to a tractable posterior in the univariate case. In that case the conditional posteriors of B, c and μ, given all other parameters, remain normal, thus enabling a simple Gibbs sampler for the posterior analysis. In our six-dimensional VAR the term $(I - B \otimes B)^{-1}$ induces such a strong deviation from conditional normality close to the unit root that a Gibbs sampler or Metropolis-Hastings algorithm are more difficult to implement.

3   Data and Priors

We consider three asset classes (stocks, bonds and T-Bills) and three state variables that help predict asset returns (inflation, dividend yield and term spread).

For our empirical analysis we use quarterly US data. All series start in 1952:I; and end in 2003:IV. The 90-days T-bill and the 10-years constant maturity yield are from the FRED website.[4] In order to generate the yield spread we obtain the zero yield data from Duffee (2002).[5] As these data are only available until 1998:IV, we have extended the series using a similar approach for the data after 1998:IV. For inflation we use the non-seasonally adjusted consumer price index for all urban consumers and all items also from the FRED website. Data on stock returns and the dividend price ratio are based on the S&P Composite and are from the "Irrational Exuberance" data of Shiller.[6] We construct the gross bond return series from 10 year constant maturity yields on US bonds using a log-linear approximation approach as in Campbell, Lo and MacKinlay (1997).

Table 1 provides summary statistics. The sample equity premium of more than 7% is much larger than most recent studies on the prospective equity premium suggest. For example, Claus and Thomas (2001) suggest a forward looking equity premium of about 3.5%. Since our post-second-world-war sample period is rather short, the sample means of the equity and bond premia might be very poor estimates of the long-run expected returns. But even with data over very long horizons, investors can

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[4] `http://research.stlouisfed.org/fred2/`
[5] `http://faculty.haas.berkeley.edu/duffee/affine.htm`
[6] `http://aida.econ.yale.edu/~shiller/data.htm`

form very different opinions about these risk premia. Dimson, Marsh and Staunton (2002) compare bond and equity premia of sixteen countries over various long horizons. Based on their evidence long-run equity and bond premia show huge cross sectional variation.

Our approach in this paper is to use multiple priors for the expected returns and the other state variables. Among these priors we implement both optimistic outlooks and more negative views on the future. We also distinguish between very confident views and highly dispersed priors on the long run expected returns. In specifying a prior for the equity premium we take into account the other variables in the system in order to define coherent long-term means. As a particular way of implementing such priors we split the historical data in two parts: NBER expansion periods and NBER contraction periods. Averages in the expansion periods represent a positive outlook, whereas the contraction period averages define a pessimistic outlook for long-term means.

The prior on the unconditional means affects both the expected returns directly as well as the other state variables. Their effect on the results is very different. The speculative part of optimal portfolio choice is extremely sensitive to small changes in the expected returns. Chopra and Ziemba (1993) argue that the primary emphasis in portfolio choice should be on obtaining superior estimates for means. Although historical data can provide robust estimates of future volatility and correlation, historical average returns are very sensitive to the choice of the data period. Investors therefore have good reasons to base their future return expectations not only on historical data, but also on other criteria. A prior on the level of future returns is already in place for some time in global tactical asset allocation at many institutional investors and asset managers (see Black and Litterman (1992)). In practice, prior information is not only used for tactical asset allocation, but also for strategic asset allocation. It is common practice for long-term investors to use historical data to estimate volatilities and correlations of the long-term future return distribution, but use economic theory and current market circumstances to form their view about the long-term mean. An informative prior on the mean of the future return distribution makes it possible to control the speculative part of the portfolio choice.

The persistence in the dividend yield, interest rates and yield spread makes it hard to estimate the long-term mean of these state variables. If it turns out that the true long-term mean of the dividend yield is far above or below the mean in the sample period, the autocorrelation parameter of the dividend yield is underestimated. More persistence of the dividend yield has a direct effect on the term structures of risk of for

instance stock returns. As a result this influences the hedge part of optimal portfolio choice.

In this paper we relate our prior information about the mean of the future distribution to business cycles. We use the NBER classification for business cycle expansion and contraction periods. The NBER Business Cycle Dating Committee chooses turning points in the economy. Their decisions are based on economic activity which is visible in macro economic variables as real (personal) income, real GDP, industrial production and employment. There is no fixed rule about the weights of various indicators or about what other measures contribute to the process. We assign each observation in our 1952:I - 2003:IV sample period to either contraction or expansion (see Figure 3). A contraction starts at the peak of a business cycle and ends at the trough, and the expansion vice versa. Nine contraction periods exist in our data sample, which have a duration between two and six quarters. Ten expansion periods exist in our sample period with a duration ranging from four to 40 quarters. Contractions appear to be much shorter than expansion, and consequently 175 out of the 208 observations are assigned to expansions, and the remaining 33 observations are contractions. We choose the closest quarter to end the contraction or expansion whenever a through occurs during a quarter. The second and third blocks in Table 1 give the summary statistics for the NBER contraction and expansion periods. The Sharpe ratios clearly reflect the different risk-return trade-offs between the periods. Bonds seem less attractive than stocks during expansion (Sharpe ratio of 0.16 versus 0.53, respectively), whereas they seem more attractive during contractions (Sharpe ratio of 0.48 versus 0.21, respectively). The difference in Sharpe ratios indicates the wide range of reasonable expectations. This range is much wider than the "good deal bounds" in Cochrane and Saa-Requejo (2000). Also the averages of state variables as the short interest rate, dividend yield, and term spread differ between the two periods. Fama and French (1989) link the dividend yield and yield spread to the business cycle. They argue that the risk premia are high in contraction periods and low in expansion periods. The opposite applies to the dividend price ratio which is high in expansion periods and low in contraction periods. Since the dividend yield adjusts very slowly over time, it describes long run business cycles. The yield spread, on the other hand, is less persistent and describes shorter business cycles. The level of the informative prior on the mean is set at either the averages over the contraction periods, or the averages over the expansion periods.

The choice of the prior information is always a debatable issue in bayesian statistics. Investors can have good reasons to base their future return expectations not only on

historical data, but also on current market circumstances (e.g. forward rates can be interpreted as the view of the market about interest rates), economic theory (see Fama and French (2002) who use dividends and earnings growth to measure the expected rate of capital gains) and human judgement (see Welch (2001) who provides a consensus forecast for the one-year equity premium). Alternatively, future expected returns could be based on a equilibrium approach as in Black and Litterman (1992). The market portfolio (or benchmark of the investor) can be seen as the optimal portfolio when investors have no explicit views regarding the expected returns and risks of the assets in his investment universe. Once the suitable benchmark or market portfolio is identified and the corresponding volatilities and correlations are defined, implied returns can be derived from for instance a (International) Capital Asset Pricing Model ((I)CAPM). These implied returns can be interpreted as equilibrium returns, and subsequently as prior information for strategic asset allocation.

4 Empirical results

4.1 Estimation results

The VAR system is estimated on the entire sample. Tables 2 and 3 summarize the OLS parameter estimates together with the correlations and standard deviations of the residuals. We highlight the most important results. First, the three state variables (nominal interest rate, dividend yield, term spread) are almost univariate AR(1) processes. The maximum eigenvalue of the coefficient matrix of 0.982 indicates that the estimated VAR is stationary. Second, the nominal interest rate and the dividend price ratio predict excess stocks returns. As in Campbell and Viceira (2005) the combination of a negative correlation of shocks to the dividend price ratio and stocks, and the positive predictive coefficient of the dividend price ratio imply mean reversion in stocks returns. The excess return on bonds is related to the yield spread, the nominal interest rate and stock returns. Third, bond returns are also mean-reverting. The nominal interest rate is a predictor of excess bond returns, which has the required opposite signs of the predictive coefficient and residual correlation. The term spread leads to a mean aversion part. The low R^2 of both stocks and bond returns of 8% implies that a large degree of the return variation remains unexplained. However, even a low degree of explanatory power on a quarterly basis can be economically meaningful at longer horizons (see Campbell and Thompson (2005)).

The priors influence the predictability of stock and bond returns, and the persistence of state variables. Tables 4 and 5 indicate that the priors substantially influence the posterior means and standard deviations of the VAR coefficients. The predictability coefficient of the dividend price ratio to stocks varies between 4.304 for the flat prior and 2.68 for the pessimist. The tables also shows that a flat prior has different implications on the posteriors than a dispersed prior on μ. If both c and B are approximately normal distributed the posterior density of $\mu = (I - B)^{-1}c$ has fat tails such that the posterior mean and standard deviation in table 4 do not exist.

The posterior distributions for a few selected parameters are shown in Figures 4 and 5. The figures summarize the differences between the optimist and pessimist priors and the effect of the precision factor. As the precision factor rises (from 0.01 to 1 to 100), the posterior distributions of the means become denser and move towards the prior means. This pattern is very evident for stocks and bonds. A high degree of prior confidence ($\kappa = 100$) drives the posterior distribution of bond returns far to the right under the pessimist view. Since the mean of bond returns is lower in the optimist view than in the data, a high precision factor shifts the posterior distribution to the left. The opposite pattern is observed for stocks. The figures also show that the impact of the precision factor is larger for more persistent series. The posterior densities of the mean of the dividend yield and nominal interest rate become extremely tight for high precision factors, whereas the densities of posterior mean stock and bond returns are less dense. Furthermore, the maximum Eigenvalue of the posterior coefficient matrix depends on the prior. This affects the restrictiveness of the non-stationarity condition. For the uninformative prior 22% of the draws are discarded, because the condition is violated.

Figure 6 illustrates the effect for the mean of the nominal interest rate. The location of the posterior densities clearly reflect that the pessimist mean (5.74 on an annualized basis) is far above the mean in the data (5.10) and the mean in the optimist view (4.98). Even $\kappa = 1$ has an enormous impact on the centrality of the distribution. $\kappa = 100$ reduces almost all uncertainty about the mean. $\kappa = 0.01$ shows that the data do not provide a lot of information about the location of the mean nominal interest rate. The posterior distribution is extremely flat in this case.

Figure 7 demonstrates that the priors also influence the persistence of state variables. The pessimist view leads to more persistency in the dividend yield process. Since the pessimist mean (-3.20) is above the data mean (-3.43), it takes longer to mean revert so that the posterior density of the autocorrelation coefficient shifts to the

11

right. The effect is already observed for $\kappa = 1$. However, the centrality around the posterior mean is hardly influenced by the confidence in the prior mean of the dividend yield.

4.2   Term structures of risk

How robust are the term structures of risk to parameter uncertainty and prior information? As an extension of Barberis (2000) we do not restrict ourselves to the time dimension, but also investigate risk properties in the cross-section.

We find that mean reversion is stocks and bonds dominates parameter uncertainty. The impact of parameter uncertainty is horizon dependent. It increases with the investment horizon. Mean uncertainty leads to a rise in annualized stock and bond volatility for investment horizons longer than 25 years. Furthermore, a pessimist prior reduces mean reversion in stock returns. In contrast to volatilities, correlations are robust against parameter uncertainty. As a consequence risk diversification between assets are much more stable than time diversification within an asset class.

Figure 8 shows the annualized standard deviation of real holding period returns on stocks, bonds and T-bills for investment horizons up to 50 years (in quarters). The solid lines represent the results without parameter uncertainty, which are in line with the results of Campbell and Viceira (2005). The impact of parameter uncertainty is reflected by the dashed lines. Obviously, the p̈arameter uncertainty spread" should be positive. Parameter uncertainty increases the variance of risky asset categories over the investment horizon. This effect can be explained by the bayesian framework that updates prior probabilities of parameters. Following periods of high returns investors update their expectations for the following years upwards. The other way around, investors adjust their expectations downwards after a period of low returns. The resulting positive autocorrelation in returns causes the multi-period variance to increase, whereas the predictability reduces the multi period volatility. The graphs indicate that generally predictability dominates parameter uncertainty.

Ignoring parameter uncertainty leads to an underestimation of the annualized stock volatility by 0.5%, 1.5%, 2% and 3.75%-points at a 1, 5, 10 and 25-year horizon, respectively. The rise in annualized volatilities for investment horizons longer than 25 years indicates that parameter uncertainty dominates the mean reverting dynamics that are captured by the dividend yield in the long run. This is due to the uncertainty about the expected return.

This effect is also strong for bonds. The weight of parameter uncertainty in the total risk is almost the same for bonds as for stocks. This suggests that incorporating parameter uncertainty has a similar effect on the attractiveness of bonds as it has on the attractiveness of stocks. The VAR estimates suggest that mean reversion in bonds is captured by the nominal T-bill, while the term spread captures mean aversion in treasury returns. The first factor dominates in the case without parameter uncertainty for investment horizons up to 25 years. The latter one becomes much more important if this additional source of uncertainty is accounted for, and even dominates in the long run. The "parameter uncertainty spread", rises with the investment horizon and is for bonds around 0.5%, 1%, 1.5% and 2.5%-points at a 1, 5, 10 and 25-year horizon, respectively.

The unexpected cumulative inflation in the real T-bill and the uncertainty around the coefficient estimates of the underlying inflation process cause the mean averting pattern of the real T-bill to strengthen once parameter uncertainty is accounted for.

Prior information on the mean influences the term structures of annualized volatility. Figure 9 summarizes the annualized volatility for different priors. The solid circled line represents the term structure of risk for real stock returns according to the OLS approach thus based on fixed parameter estimates. The solid line with the plus gives the term structure according to the flat prior. The grey lines correspond to the pessimist and the black lines correspond to the optimist. The dashed lines represent a high ($\kappa = 100$) and the solid lines a low ($\kappa = 0.01$) degree of confidence in the prior information.

The location of the prior mean determines whether the annualized stock volatility increases or decreases compared to a flat prior. A pessimist prior leads to a higher equity risk. This is in line with earlier findings in this case about the more persistent dividend yield process (Figure 7) and the lower predictability of the dividend yield to stock returns (Table 4). The opposite occurs for an optimist prior. Compared to a flat prior, it leads to a downward shift of the term structure. These features are exhibited at all investment horizons, and can lead to a higher or lower equity risk of about 1.5 percentage point. Obviously, the location of the prior mean has less impact for lower precision factors. $\kappa = 0.01$ and a flat prior result result in a comparable term structures.

What explains the mean averting pattern of equity risk at investment horizons beyond 25 years? Figures 9 and 10 indicate that the uncertainty about the mean is crucial for this. Mean uncertainty is a mean averting mechanism in the holding period volatil-

ity. On the other hand, the conditional variance is the mean reverting mechanism in the unconditional volatility at almost all horizons. In order to verify this Figure 10 decomposes the annualized unconditional volatility (solid line) into two factors: the average conditional volatility (dashed line) and the volatility of the conditional mean (solid circle line).[7] The spread between the OLS volatility (solid plus line) and the average conditional volatility reflects parameter uncertainty. The distance between the average conditional volatility and the unconditional volatility represents mean uncertainty. The latter factor dominates the square root formula at longer horizons. However, the uncertainty about the mean plays a very small role in the total variance for short investment horizons.

Figure 11 demonstrates the impact on the risk diversification properties in the cross section. Risk properties between assets are much more stable than time diversification within an asset class. This can have important implications for optimal portfolio management. Apparently, the impact of parameter uncertainty on the covariance is proportional to the impact on the variance. As a result the effect of parameter uncertainty cancels out to a large degree in terms of correlation. The correlation between T-bills and bonds becomes marginally lower for horizons longer than five years, and the correlation between stocks and T-bills reduces somewhat in the very long run.

The inflation hedge qualities of (nominal) returns on T-bills, stocks and a constant maturity treasury portfolio are also robust to incorporating parameter uncertainty. Seemingly, the effect of parameter uncertainty on the covariance between these (nominal) asset returns and inflation is proportional to the effect on the corresponding variances. The inflation hedge properties in Figure 12 are in line with the ones found in Hoevenaars, Molenaar, Schotman and Steenkamp (2007) for fixed parameter estimates. Rolling over 3-month T-bills ensures that the lagged inflation is incorporated, and consequently the T-bill is the best inflation hedge among the asset classes we consider at all investment horizons. At long horizons constant maturity bonds are a good inflation hedge as well. However, due to the inverse relationship between yield changes and bond prices, the short-term inflation hedging properties are poor, before the hedging qualities improve in the long run. Stocks also seem a better inflation hedge in the long run than at short horizons, consistent with the large literature on this relationship.

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[7] Note that the unconditional variance is the sum of mean conditional variance and the variance of the conditional mean.

5   Optimal portfolio choice

We derive the optimal portfolio for a buy-and-hold investor who maximizes the utility of end-of-period wealth. At time t the investor allocates wealth to real T-bills, stocks and bonds with portfolio weights $w = (w_r, w_s, w_b)$ and holds the portfolio until time $t + k$ without rebalancing. Assuming power utility with risk aversion γ, the investor solves

$$\max_{w} \mathsf{E}_t \left[U(W_{t+k}) \right] = \max_{w} \mathsf{E}_t \left[\frac{W_{t+k}^{1-\gamma}}{1-\gamma} \right] \tag{8}$$

where wealth at $t + k$ is defined as

$$W_{t+k} = \sum w_i \exp(r_{i,t+k}^{(k)}), \tag{9}$$

normalising initial wealth at $W_t = 1$, and returns are defined as the real cumulative logarithmic return from t to $t + k$,

$$r_{i,t+k}^{(k)} = \sum_{s=1}^{k} r_{i,t+s} \tag{10}$$

We assume that short-sell restrictions restrict the weights to be non-negative, and that the weights sum to one. In the numerical optimization we use a fixed grid for the weights with stepsize 0.01. Consequently, the feasible portfolios have weights $w_i = 0, 0.01, \ldots, 0.99$ $(i = b, s)$, $w_r = 1 - w_b - w_s$, and $w_r > 0$.

In order to numerically calculate the maximum expected utility in (8) we need the distribution of future asset returns. This is either based on the OLS or bayesian approach. In the OLS case, the parameter estimates of the VAR are used to simulate the model conditional on the parameter estimates. In the bayesian case we first draw a set of parameters from the posterior distribution, and conditional on these parameters we simulate a scenario from the VAR. In both cases the unconditional sample means are chosen as the starting values to create the future scenarios.

The optimal portfolio is the one that corresponds to the maximum expected utility. In order to determine this asset allocation we begin calculating the real returns of all possible asset mix combinations in each simulation. Next, the wealth can be computed in each simulation and the expected utility is estimated as the average over all scenarios. Finally, the optimal strategic asset allocation is selected.

For $\gamma > 1$ the expectation in (8) will not always exist. Although we assume that the innovations in the VAR are normally distributed conditional on the parameters,

the predictive densities,

$$f(y_t|y_{t-1}) = \int_\theta f(y_t|y_{t-1}, \theta) \times p(\theta|\mathbf{Y}) \, \mathrm{d}\theta, \qquad (11)$$

will have fatter tails than the normal distribution. If a random variable X has a fat-tailed distribution, the expected value $\mathsf{E}[e^X]$ does not exist. For the expected utility problem (8) this implies that expected utility will be negative infinity for portfolios of assets that have fat left (negative) tails. We can guarantee that at least some portfolios have finite utility if we assume that the riskfree asset is bounded in the sense that its return is always above a lower limit r_{min}. In simulating paths of asset returns we reject draws that would violate the constraint $r_t > r_{min}$. We have set the lower bound to -15.80%, which is ten times the minimum observed in the sample. The probability that the T-bill reaches the boundary is almost zero. If the T-bill has a positive weight in the portfolio, wealth will never go to zero and expected utility is bounded away from minus infinity. This approach ensures that the maximization problem (8) is well-defined.

Table 6 compares the optimal portfolios of an investor who ignores parameter uncertainty with an investor who accounts for parameter uncertainty using the flat prior for $\tilde{B}$. The risk aversion parameter $\gamma = 5$. The OLS portfolio exhibits the standard result that the weight of stocks increases with the investment horizon. For horizons longer than 15 years the investor would prefer to be fully invested in stocks. T-bills have a high weight for short investment horizons, but this reduces quickly for longer horizons. Reinvestment risk makes them less attractive for longer horizons. The combination of the high Sharpe ratio of stocks and their mean reverting pattern makes them more attractive for long-term investors. Due to the low bond premium in the data sample (Sharpe ratio is 0.15 versus 0.44 for stocks) Treasuries are not in the optimal portfolio.

Introducing parameter uncertainty influences the long-term asset allocation substantially. However, parameter uncertainty seems not to influence the optimal asset allocation for very short investment horizons. The right part of Table 6 shows that the asset allocation for a one-year horizon hardly changes when the investor incorporates parameter uncertainty through the uninformative prior. Since correlations between assets are much more robust against parameter uncertainty than volatilities of assets, we find that risk diversification becomes much more emphasized after incorporating parameter uncertainty, typically at longer horizons. Besides stocks the optimal asset allocation also consists of T-bills, which are a good risk diversifier due to the very low correlation with stocks especially at longer horizons (see Figure 11). Bonds are not in the optimal portfolio, because their risk rises in the same proportion as the

16

risk of stocks, while the bond premium is much lower. Although ignoring parameter uncertainty leads to an overallocation of stocks at all horizons, the investor should still benefit from mean reversion in stocks by increasing their weight for investment horizons up to 20 years. On the other hand, stock holdings decrease for investment horizons beyond 25 years due to the uncertainty in the expected returns, which dominates the holding period volatility at these horizons. Consequently, ignoring parameter uncertainty can lead to an underallocation of T-bills and risk diversification.

Obviously, the optimal asset allocation varies substantially with the view and confidence the investor has about the level of the future asset returns, which is included in the future distribution through the informative prior. Table 7 shows the impact on the asset allocation for the pessimist view while accounting for parameter uncertainty. Regardless of the precision factor, risk diversification remains an obvious characteristic in the optimal portfolio typically at longer investment horizons. The strong diversification in the portfolio composition reflects the stable correlations (see Figure 8), which indicates that risk diversification remains particularly important.

The prior mean gets more weight in the posterior mean for higher precision factors. Bonds become more important in the portfolio choice when the precision factor rises, which clearly reflects the higher Sharpe ratio of bonds than stocks in the pessimist view. At a precision factor $\kappa = 100$ the investor allocates his wealth almost entirely to bonds. However, at horizons longer than 25 years the optimal portfolio becomes more diversified, and stocks and T-bills are also in the optimal asset allocation. So even an investor who is skeptical about stocks, includes them for hedging purposes. On the contrary, the investor hardly invests in bonds at a precision factor $\kappa = 0.01$. Only for very long investment horizons the optimal portfolio is diversified with bonds. A precision factor $\kappa = 1$ gives equal weight to the data and prior information. In the data sample stocks have a higher Sharpe ratio (0.44) than bonds (0.15), whereas in the pessimist view stocks have a lower Sharpe ratio (0.21) than bonds (0.48). As a consequence, the optimal asset allocation is diversified at all investment horizons. T-bills, stocks and bonds all have a substantial weight in the optimal portfolio choice.

The large weight of T-bills reflects their low correlation with stocks and bonds, which make them an interesting risk diversifier at long horizons. The low volatility of T-bills makes them attractive especially at short horizons. Other features remain observed as well. Mean reversion in stock returns still makes stocks more attractive for long-term investors. However, this effect is mitigated in the very long-term due to its rising annualized volatility.

Table 8 clearly shows that the impact on portfolio choice of the informative prior mean is much less once the investor has an optimistic view. This can be explained by the fact that the means in the optimist view differ much less from the sample mean than means in the pessimist view. The weight of stocks is positively related to κ. Just as before the weight in stocks first rises with the investment horizon, but eventually decreases for horizons longer than 25 years. Bonds only have a substantial weight in the asset allocation once there is less prior information ($\kappa = 0.01$) and typically at longer horizons. For $\kappa = 1$ or $\kappa = 100$ bonds play not a significant role. T-bills have a substantial weight in optimal portfolio choice for all horizons and for all precision factors. In contrast to the pessimist prior in Table 7, T-bills are still in the optimal portfolio even when the investor is extremely certain ($\kappa = 100$) about the prior information. This reflects the strong risk diversification properties of T-bills in a portfolio of stocks. T-bills are demanded in a portfolio with a lot of stocks for risk reduction. So even an investor who is very optimistic about stocks, invests a large part of his assets in T-bills for hedging purposes.

Figure 13 demonstrates how the impact on optimal portfolio choice of different prior information is influenced by the risk aversion of an investor. The optimal portfolios are given for a 10-year investment horizon versus the risk attitude of an investor. The pessimist prior is analyzed for three different precision factors ($\kappa = 0.01, 1, 100$). We determined the optimal asset allocations for five grid points of the risk attitude $1/\gamma$: 0.15, 0.2, 0.3, 0.4 and 0.5.

In general, the optimal weight of stocks is negatively related to the riskaversion γ: if the risk aversion decreases, the optimal weight in stocks increases. Compared to the OLS approach (the solid circled line) the weight in stocks reduces whenever the degree of confidence in the prior mean increases. This effect is independent of the risk attitude, and is explained by the lower Sharpe ratio of stocks than bonds in the pessimist view. However, observe that the weight in stocks reduces considerably due to parameter uncertainty even when there is little prior information about the location of the mean (which corresponds to $\kappa = 0.01$).

The weight in bonds solely depends on the degree of confidence in the higher bond return in the future distribution, and seems negatively related to the risk attitude of the investor. Under a very strong optimistic view ($\kappa = 100$) the investor allocates his entire wealth to bonds.

Ignoring parameter uncertainty leads to an underallocation of T-bills even when prior information about the mean is incorporated. In contrast to stocks, T-bills are

positively related to the riskaversion γ. T-bills become particularly interesting at higher risk aversions for diversification purposes. For higher risk aversions the optimal asset allocation is more balanced and diversified when there is less prior information.

Figure 14 shows how the risk attitude of the investor influences optimal portfolio choice under an optimist prior. Just as before the allocation to stocks is negatively related with the risk aversion (γ). The weight of stocks increases for higher precision factors. However, even if the investor is extremely sure about the prior information ($\kappa = 100$) the allocation to stocks is still considerable below the OLS case for higher risk aversions. This reflects some conservatism for parameter uncertainty. For this case the weight of T-bills is higher than the OLS weight. The low volatility of T-bills and the low and stable correlation with stocks at a 10-year horizon makes them an important risk diversifier. This is even more emphasized at higher risk aversions and for less prior information. The allocation to T-bills increases in these cases. The weight of bonds is negligible, regardless of the risk attitude. The lower Sharpe ratio of bonds compared to stocks in the sample data (0.15 versus 0.44), and an even lower Sharpe ratio in the optimist view (0.53 versus 0.06) explains this.

6 Multiple Priors and Expected Loss

Let $Q_i(w)$ be the utility of investor i, who uses a prior $p_i(\theta)$, when investing in portfolio w,

$$Q_i(w) = \mathsf{E}_{it}\left[U(W_{t+k}(w))\right] \tag{12}$$

The expectation operator has the additional subscript i to indicate that the expectation is taken relative to the predictive distribution of returns based on the prior of investor i. Also future wealth is explicitly expressed as a random variable that is a function of the portfolio w. The investor combines his prior $p_i(\theta)$ with the common likelihood $L(\mathbf{Y}|\theta)$ to form his posterior $p_i(\theta|\mathbf{Y})$ and predictive distribution

$$f_i(y_{t+1}|\mathbf{Y}) = \int_\theta f(y_{t+1}|y_t, \theta) \times p_i(\theta|\mathbf{Y})\, \mathrm{d}\theta, \tag{13}$$

which differs from (11) by the subscript i. Consequently, the expectation in (12) is taken with respect to (13).

According to investor j, who has prior views $p_j(\theta)$, the expected utility of the same portfolio is

$$Q_j(w) = \mathsf{E}_{jt}\left[U(W_{t+k}(w))\right] \tag{14}$$

19

The optimal portfolio according to investor i need not be optimal in the eyes of investor j. Let w_i^* and w_j^* be the optimal portfolios of investors i and j, respectively. Define the expected loss of the optimal portfolio of investor i relative to the views of investors j as

$$\ell_j(w_i^*) = Q_j(w_j^*) - Q_j(w_i^*) \qquad (15)$$

Conversely, the optimal portfolio of investor j will be evaluated as sub-optimal by investor i and ental the cost $\ell_i(w_j^*)$.

For a meaningful quantitative comparison of the expected loss, we express it in terms of a certainty equivalent. The economic loss can be interpreted as an opportunity cost. It is defined as the percentage with which the initial asset value must increase to compensate the investor for suboptimal investing. We define the economic loss C_{ij} as the percentage risk free return investor i would need to be compensated for being forced to choose asset allocation w_j^*. It is computed as

$$C_{ij} = \left(\frac{\mathsf{E}_{it}\left[W_{t+k}(w_j^*)^{1-\gamma}\right]}{\mathsf{E}_{it}\left[W_{t+k}(w_i^*)^{1-\gamma}\right]} \right)^{\frac{1}{1-\gamma}}, \qquad (16)$$

This way we construct the entire matrix C with elements giving the costs of portfolio w_j^* under prior i.

Since all priors are assumed reasonable, we assume it is impossible to put a probability on the validity of each prior. This also leads to the impossibility to define an overall weighted average prior and weighted average predictive density to evaluate the utility and costs of all portfolios as in Avramov (2002). There will thus remain some ambiguity in how to define the optimal portfolio. This situation with multiple priors has been analyzed in a variety of recent papers.[8] A robust portfolio is defined as the best worst case or minimax solution. For each portfolio the prior that gives the lowest utility is selected. The robust portfolio is the best among all these worst case evaluations.

With our limited set of portfolios and limited set of priors, the robust portfolio is selected as the minimax solution within the matrix C. As each column of C represents a portfolio, we select the worst case of portfolio w_i^* as the maximum element in column i. The portfolio with the lowest maximum is the minimax portfolio.

The results are summarized in Table 9. The asset allocation that is used for the asset allocation decision is in the columns. The prior is listed in the rows. Clearly, the

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[8] See for example Uppal and Wang (2003) and Goldfarb and Iyengar (2003).

zero diagonal reflects that the costs of portfolio w_i^* under prior i equals zero. Table 9 shows that the opportunity costs of alternative priors are economically meaningful and increase with the investment horizon. The OLS results, ignoring parameter uncertainty, are included in the table as well.

At the one year horizon all costs are low. The largest entries show the contrast between the confident optimistic view and the confident pessimistic view. The optimal portfolio for the confident optimistic view performs poorly in the eyes of the confident pessimistic investor and vice versa. The minimax solution would be the optimal portfolio according to the reasonably pessimistic view. This portfolio has the low costs for all priors. The maximum cost is 1.3% in the eyes of of the confident optimistic investor. This maximum is the lowest over all columns for the one-year horizons and is therefore the minimax solution.

An important result from table 9 is that parameter uncertainty seems not to be economically relevant on an annual investment horizon. The OLS portfolio performs reasonably well. Ignoring the parameter uncertainty does not entail economic losses except under the confident pessimistic prior.

In contrast, using the OLS approach for longer investment horizons is far from optimal for all prior views. The opportunity costs are economically significant, and are even extraordinary high at a 25-year horizon (with opportunity costs sometimes above 1000%). The OLS approach leads to a portfolio that is fully invested in equity, whereas all other portfolios have less equity. The reduction of the equity holdings is just the result of accounting for parameter uncertainty. Whatever the prior, even a very confident optimistic investor attaches a high price to being forced to hold that much equity.

An investor who is extremely skeptical about stocks and very optimistic about bonds (Pessimist ($\kappa = 100$) in the table) has very low valuations of the optimal portfolios implied by all other priors. Vice versa, the optimal portfolio of the confident pessimistic investor is very costly to all other investors. The minimax strategy is to invest according to the dispersed ($\kappa = 0.01$) optimistic prior.

7 Conclusion

In this paper we consider the strategic asset allocation of long-term investors who account for parameter uncertainty and prior information about the level of expected asset returns and state variables in optimal portfolio choice, and who can invest in

stocks, bonds and T-bills. We use bayesian methods and a vector autoregression for returns and macro-economic state variables for the empirical examination. We study both the impact of parameter uncertainty as well as the impact of prior information about the means on optimal portfolio choices.

We find that mean reversion in stocks and bonds dominates parameter uncertainty. The impact of parameter uncertainty is horizon dependent. It increases with the investment horizon. As a consequence, time diversification properties within asset classes in terms of volatilities are much weaker if parameter uncertainty is incorporated. Equities still exhibit mean reversion at short and medium investment horizons, but mean reversion is mitigated at longer horizons. The same effect also holds for bonds. Parameter uncertainty contributes as much to the risk of bonds as it does to stocks.

Priors views on the long-run mean of predictor variables influence the predictability of stock and bond returns, and the persistency of state variables. Even weak prior information on the unconditional mean of highly persistent time series like dividend yield and the nominal interest rate changes the estimated persistence of shocks and the predictability of excess returns. This has also implications for the term structures of annualized volatility. The location of the prior mean determines whether the annualized stock volatility increases or decreases compared to a flat prior. A pessimist prior for instance leads to a higher equity risk. Mean uncertainty is a mean averting mechanism in the holding period volatility. It explains the mean averting pattern of equity risk at investment horizons beyond 25 years.

Risk properties between assets are much more stable than time diversification within an asset class. Apparently, the impact of parameter uncertainty on the covariance is proportional to the impact on the variance. In the same line of reasoning inflation hedge qualities of (nominal) returns on T-bills, stocks and a constant maturity treasury portfolio are also robust to incorporating parameter uncertainty.

Parameter uncertainty alters the optimal portfolio choice. Since correlations between assets are much more robust against parameter uncertainty than volatilities of assets, we find that risk diversification becomes much more emphasized after incorporating parameter uncertainty, typically at longer horizons. Ignoring parameter uncertainty leads to an overallocation of stocks and an underallocation of T-bills and risk diversification at all horizons. On the other hand, the investor can still benefit from mean reversion in stocks by increasing their weight for longer investment horizons. However, he should take care about the rise of the annualized volatility at horizons beyond 25 years.

Optimal asset allocation varies substantially with the prior information that the investor has about long-run means. Risk diversification becomes much more important once there is less prior information, and particularly for longer investment horizons. In these cases stocks, bonds and T-bills all have a substantial weight in optimal portfolio choice. Regardless of the prior precision, T-bills are an important risk diversifier in a portfolio with stocks, which results in a substantial portfolio weight for T-bills for all horizons.

We find that an investor who is very optimistic about stocks should also invest a substantial part of his assets in T-bills for hedging purposes, regardless of the investment horizon. An investor who is skeptical about stocks still includes them in his optimal portfolio for hedging purposes, particularly for longer investment horizons. On the other hand, our findings suggest that an investor who is extremely positive about bonds invests entirely in bonds, and only includes T-bills and stocks for risk diversification at very long investment horizons.

For short (1-year) investment horizons it is justifiable to base the strategic asset allocation decision on the OLS approach that ignores parameter uncertainty. The opportunity costs are small for short horizons. In contrast, the results indicate that it is recommendable to account for parameter uncertainty for long-term strategic asset allocation. The opportunity costs are economically significant whenever the asset allocation is based on a OLS approach.

The optimal portfolio for an optimistic investor is very costly (sub-optimal) in eyes of an investor with a more negative prior view. We define a robust portfolio as the portfolio of an investor with a prior that has minimal costs among all priors that we consider. Such a robust portfolio coincides with the optimal portfolio of a moderately optimistic investor. It contains a large proportion of equity, but far less than would be implied under both more optimistic as well as very diffuse priors.

Appendix A   Properties of Posteriors

We derive the posterior for the different prior specifications in the text.

Reduced form uniform prior

The first prior is the uniform flat prior on the parameters of the VAR system in reduced from (1). Stacking observations, the model can be written in matrix notation as

$$Y = \tilde{X}\tilde{B}' + E, \tag{A1}$$

where Y is the $(T \times n)$ matrix of observations on y_t, X the $(T \times n)$ matrix of observations y_{t-1}, $\tilde{X} = (\iota\ X)$ and $\tilde{B} = (c\ B)$. Using the standard conditional likelihood,

$$p(Y|\tilde{B}, \Sigma, X) \propto |\Sigma|^{-T/2} \exp\left(-\tfrac{1}{2}\text{tr}\left(\Sigma^{-1}E'E\right)\right), \tag{A2}$$

and the prior (2) the joint posterior is given by

$$p(\tilde{B}, \Sigma|Y, X) \propto I(B)|\Sigma|^{-(T+n+1)/2} \exp\left(-\tfrac{1}{2}\text{tr}\left(\Sigma^{-1}E'E\right)\right). \tag{A3}$$

The conditional posterior of Σ is the inverted Wishart,

$$p(\Sigma|\tilde{B}, Y, X) \sim iW(E'E, T). \tag{A4}$$

The stationarity restriction implies that exact analytical integration over B is not possible, so that the marginal posterior for Σ cannot be derived analytically. An obvious simulation algorithm is available, since the conditional posterior of $\tilde{B}$ will be truncated normal and therefore easily sampled from. Using standard algebra, see Zellner (1971), we rewrite the term in the trace in (A3) as

$$
\begin{aligned}
E'E &= (Y - \tilde{X}\tilde{B}')'(Y - \tilde{X}\tilde{B}') \\
&= Y'\tilde{M}Y + (\tilde{B} - \hat{\tilde{B}})\tilde{X}'\tilde{X}(\tilde{B} - \hat{\tilde{B}})'
\end{aligned} \tag{A5}
$$

where

$$
\begin{aligned}
\tilde{M} &= I - \tilde{X}(\tilde{X}'\tilde{X})^{-1}\tilde{X}' \\
\hat{\tilde{B}} &= Y'\tilde{X}(\tilde{X}'\tilde{X})^{-1}
\end{aligned}
$$

Therefore the conditional posterior of $\tilde{B}$ is

$$p(\text{vec}(\tilde{B}')|\Sigma, Y, X) \sim N^*\left(\text{vec}(\hat{\tilde{B}}'), \Sigma \otimes (\tilde{X}'\tilde{X})^{-1}\right), \tag{A6}$$

where the superscript * denotes truncation to the stationary region. We use a Gibbs sampler to draw from the joint posterior by alternating draws for Σ from (A4) and $\tilde{B}$ from (A6), where we reject draws from the latter if B is outside the stationary range.

Structural form prior on μ

For the prior on μ we rewrite the structural VAR model (3) in matrix notation as

$$Y = \iota\mu' + (X - \iota\mu')B' + E \tag{A7}$$

The joint prior on μ, B and Σ is

$$p(\mu, B, \Sigma) \propto I(B)|\Sigma|^{-(n+1)/2} \exp\left(-\tfrac{1}{2}\kappa(\mu - \mu_0)'\Omega_0^{-1}(\mu - \mu_0)\right) \tag{A8}$$

Multiplication of prior and likelihood gives the joint posterior

$$p(\mu, B, \Sigma|Y, X) \propto I(B)|\Sigma|^{-(T+n+1)/2} \exp\left(-\tfrac{1}{2}\left(\kappa(\mu - \mu_0)'\Omega_0^{-1}(\mu - \mu_0) + \text{tr}\left(\Sigma^{-1}E'E\right)\right)\right) \tag{A9}$$

As before, the conditional posterior of Σ in this case is

$$p(\Sigma|Y, X, \mu, B) \sim iW(E'E, T) \tag{A10}$$

To derive the conditional posterior of μ conditional on (B, Σ) we rewrite the terms in the exponent of (A9) as quadratic forms in μ. Let $\tilde{e} = \text{vec}(Y - XB')$, let $A = I - B$, and use

$$\begin{aligned}
\text{tr}\left(\Sigma^{-1}E'E\right) &= \text{vec}(E)'(\Sigma \otimes I_T)^{-1}\text{vec}(E) \\
&= (\tilde{e} - (A \otimes \iota)\mu)'(\Sigma \otimes I_T)^{-1}(\tilde{e} - (A \otimes \iota)\mu) \\
&= \tilde{e}'\left(\Sigma^{-1} \otimes I_T\right)\tilde{e} + T\mu'A'\Sigma^{-1}A\mu - 2\tilde{e}'\left(\Sigma^{-1}A \otimes \iota\right)\mu
\end{aligned} \tag{A11}$$

Combining the trace term in (A11) with the quadratic form in the prior (A8) gives

$$\kappa(\mu - \mu_0)'\Omega_0^{-1}(\mu - \mu_0) + \text{tr}\left(\Sigma^{-1}E'E\right) = Q + (\mu - \hat{\mu})'V^{-1}(\mu - \hat{\mu}) \tag{A12}$$

where

$$\begin{aligned}
V &= \left(TA'\Sigma^{-1}A + \kappa\Omega_0^{-1}\right)^{-1} \\
\hat{\mu} &= V\left(\left(\Sigma^{-1}A \otimes \iota\right)'\tilde{e} + \kappa\Omega_0^{-1}\mu_0\right),
\end{aligned}$$

and Q a constant independent of μ. Thus the full conditional posterior of μ is

$$p(\mu|Y, X, B, \Sigma) \sim N(\hat{\mu}, V) \tag{A13}$$

As B only appears in E, the conditional posterior of B can be derived directly from the term $\text{tr}(\Sigma^{-1}E'E)$, which for this purpose is now decomposed as a quadratic function of B. Let $\bar{Y} = Y - \iota\mu'$, $\bar{X} = X - \iota\mu'$, and use

$$\begin{aligned}
E'E &= (\bar{Y} - \bar{X}B')'(\bar{Y} - \bar{X}B') \\
&= \bar{Y}'\bar{M}\bar{Y} + (B - \hat{B})\bar{X}'\bar{X}(B - \hat{B})'
\end{aligned} \tag{A14}$$

25

where

$$\bar{M} = I - \bar{X}(\bar{X}'\bar{X})^{-1}\bar{X}'$$
$$\hat{B} = \bar{Y}'\bar{X}(\bar{X}'\bar{X})^{-1}$$

The conditional posterior of B follows directly as

$$p(\text{vec}(B')|Y, X, \mu, \Sigma) \sim N^* \left(\text{vec}(\hat{B}'), \Sigma \otimes (\bar{X}'\bar{X})^{-1} \right) \qquad \text{(A15)}$$

Again the The bayesian analysis is numerical and based on Gibbs sampling. We use the full conditional posteriors to simulate from the joint posterior by iterating over the sequence (A10), (A13) and (A15).

For the initialization of the Gibbs sampler we use the OLS estimates. We discard the first 2500 draws such that we end with a sample of 20,000 parameter estimates. Conditional on a draw for the parameters $\theta = (\mu', \text{vec}(B'), \text{vech}(\Sigma))$ we simulate two antithetic scenarios of future returns. In this way we create 40,000 scenarios of future returns for 50 years into the future.

The burn-in phase is chosen by visual inspection of the posterior draws and supported by the convergence tool of Yu and Mykland (1998). We use a standardized version of their cumsum statistic as suggested by Bauwens, Lubrano and Richard (2003). For all priors the plot of the standardized version of the cumsum statistic converges smoothly and quickly to zero, especially after the burn-in phase, which indicates the convergence of the Monte Carlo chain. Sufficient conditions for convergence of the Gibbs sampler are given in Geweke (1996). Plots of the autocorrelation function suggest that the draws do not suffer from serious autocorrelation.

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Table 1: Summary Statistics

Annualized means, standard deviations, autocorrelations and Sharpe ratios for the entire sample (1952:II - 2003:IV) and two subsamples: NBER contraction periods and NBER expansion periods. The mean log returns are adjusted by one-half their variance so that they reflect mean gross returns. Standard errors of the mean ("se") are computed using the Newey-West estimate of the long-run variance. Variables are real 3-months T-Bill return (r), excess stock returns (x_s), excess bond returns (x_b), nominal Treasury Bill return (r_{nom}), dividend yield (d_p) and term spread (S).

	r	x_s	x_b	r_{nom}	d_p	S
1952:II - 2003:IV						
Average	1.21	7.04	1.46	5.10	-3.43	1.21
Std.Dev.	1.34	15.89	9.58	1.36	0.39	0.59
se	0.23	1.05	0.68	0.32	0.19	0.10
AR(1)	0.43	0.08	-0.04	0.93	0.97	0.79
Sharpe		0.44	0.15			
NBER Contraction (33 observations)						
Average	1.47	4.75	6.50	5.74	-3.20	1.22
Std.Dev.	1.96	23.00	13.65	1.90	0.40	0.60
Sharpe		0.21	0.48			
NBER Expansion (175 observations)						
Average	1.33	7.47	0.50	4.98	-3.47	1.21
Std.Dev.	1.19	14.12	8.52	1.22	0.38	0.59
Sharpe		0.53	0.06			

Table 2: Parameter estimates

The table reports full sample (1952:II - 2003:IV) OLS parameter estimates of the VAR $y_{t+1} = c + By_t + \epsilon_{t+1}$ with variables: real 3-months T-Bill return (r), excess stock returns (x_s), excess bond returns (x_b), nominal Treasury Bill return (r_{nom}), dividend yield (d_p) and term spread (S). Standard errors are in parentheses. The last column contains the R^2.

	$r_{nom,t}$	r_t	$d_{p,t}$	S_t	$x_{s,t}$	$x_{b,t}$	R^2
$r_{nom,t+1}$	0.96	0.00	0.03	0.08	0.00	0.00	0.89
	(0.03)	(0.03)	(0.04)	(0.06)	(0.00)	(0.00)	.
r_{t+1}	0.25	0.36	-0.10	0.30	0.00	0.01	0.24
	(0.07)	(0.07)	(0.11)	(0.16)	(0.01)	(0.01)	.
$d_{p,t+1}$	0.01	0.00	0.97	-0.02	0.00	0.00	0.96
	(0.01)	(0.01)	(0.02)	(0.02)	(0.00)	(0.00)	.
S_{t+1}	0.02	0.00	-0.03	0.82	0.00	0.00	0.64
	(0.02)	(0.02)	(0.03)	(0.05)	(0.00)	(0.00)	.
$x_{s,t+1}$	-1.81	0.34	4.08	2.03	0.04	0.17	0.08
	(0.97)	(0.93)	(1.48)	(2.10)	(0.07)	(0.13)	.
$x_{b,t+1}$	0.90	-0.13	-0.36	4.71	-0.07	-0.06	0.08
	(0.58)	(0.56)	(0.89)	(1.26)	(0.04)	(0.08)	.

Table 3: Residual correlation matrix

The table reports the residual correlation matrix Σ of the VAR $y_{t+1} = c + By_t + \epsilon_{t+1}$. Diagonal entries are standard deviations; off-diagonal entries are correlations.

	$r_{nom,t}$	r_t	$d_{p,t}$	S_t	$x_{s,t}$	$x_{b,t}$
$r_{nom,t}$	0.23	—	—	—	—	—
r_t	-0.31	0.60	—	—	—	—
$d_{p,t}$	0.17	-0.26	0.08	—	—	—
S_t	-0.83	0.12	-0.09	0.18	—	—
$x_{s,t}$	-0.10	0.25	-0.95	0.02	7.77	—
$x_{b,t}$	-0.64	0.40	-0.23	0.12	0.20	4.67

Table 4: Posterior means of VAR parameters

This table shows the effect of different priors on selected important VAR parameters. Entries denote posterior means. Posterior standard deviations are in parentheses.

VAR coefficients	OLS	Flat	P01	P100	O01	O100
$r_{nom,t+1},r_{nom,t}$	0.958	0.954	0.964	0.955	0.962	0.954
	(0.029)	(0.028)	(0.027)	(0.028)	(0.027)	(0.027)
$d_{p,t+1},d_{p,t}$	0.970	0.967	0.972	0.980	0.972	0.966
	(0.015)	(0.014)	(0.014)	(0.012)	(0.014)	(0.014)
$x_{s,t+1},d_{p,t}$	4.084	4.304	3.923	2.680	3.913	4.170
	(1.483)	(1.408)	(1.379)	(1.200)	(1.371)	(1.415)
$x_{b,t+1},r_{nom,t}$	0.902	0.963	0.788	1.080	0.797	0.968
	(0.583)	(0.564)	(0.553)	(0.580)	(0.555)	(0.562)
$x_{b,t+1},S_t$	4.708	4.751	4.529	4.970	4.541	4.755
	(1.261)	(1.261)	(1.270)	(1.286)	(1.265)	(1.273)

Table 5: Posterior means of unconditional means

This table shows the effect of different priors on the unconditional means. Entries denote posterior means. Posterior standard deviations are in parentheses.

Unconditional Mean	P01	P100	O01	O100
Nominal rate	1.3	1.4	1.1	1.2
	(0.8)	(0.0)	(0.7)	(0.0)
Real T-Bill	0.4	0.4	0.3	0.3
	(0.2)	(0.0)	(0.2)	(0.0)
Dividend Yield	-3.5	-3.2	-3.7	-3.5
	(0.5)	(0.0)	(0.4)	(0.0)
Term Spread	0.3	0.3	0.3	0.3
	(0.1)	(0.0)	(0.1)	(0.0)
Equity premium	0.7	0.5	0.7	1.6
	(0.5)	(0.1)	(0.5)	(0.1)
Bond Premium	0.3	1.3	0.4	0.1
	(0.2)	(0.0)	(0.2)	(0.0)

Table 6: Optimal portfolio choice: OLS and flat prior

Optimal portfolio choice under power utility for a k-period buy and hold investor with risk aversion 5 and for different investment horizons: 1, 5, 10, 15, 25, 40, 50 years (based on 40000 simulations).

	OLS			Parameter Uncertainty		
Horizon	Bills	Equity	Bonds	Bills	Equity	Bonds
1	0.41	0.53	0.06	0.45	0.50	0.05
5	0.32	0.68	0.00	0.44	0.56	0.00
10	0.12	0.88	0.00	0.38	0.62	0.00
15	0.01	0.99	0.00	0.37	0.63	0.00
25	0.01	0.99	0.00	0.43	0.57	0.00
40	0.01	0.99	0.00	0.57	0.43	0.00
50	0.01	0.99	0.00	0.56	0.43	0.01

Table 7: Optimal portfolio choice: pessimist prior

Optimal portfolio choice under power utility for a k-period buy and hold investor with risk aversion 5 for different investment horizons: 1, 5, 10, 15, 25, 40, 50 years and for different precision factors under the pessimist prior. (based on 40000 simulations).

	$\kappa = 0.01$			$\kappa = 1$			$\kappa = 100$		
Horizon	Bills	Equity	Bonds	Bills	Equity	Bonds	Bills	Equity	Bonds
1	0.49	0.42	0.09	0.43	0.19	0.38	0.04	0.04	0.92
5	0.57	0.43	0.00	0.56	0.11	0.33	0.01	0.00	0.99
10	0.55	0.45	0.00	0.52	0.10	0.38	0.01	0.00	0.99
15	0.54	0.46	0.00	0.51	0.12	0.37	0.01	0.02	0.97
25	0.49	0.42	0.09	0.53	0.19	0.28	0.05	0.15	0.80
40	0.46	0.36	0.18	0.61	0.23	0.16	0.06	0.31	0.63
50	0.40	0.35	0.25	0.64	0.27	0.09	0.60	0.18	0.22

Table 8: Optimal portfolio choice: optimist prior

Optimal portfolio choice under power utility for a k-period buy and hold investor with risk aversion 5 for different investment horizons: 1, 5, 10, 15, 25, 40, 50 years and for different precision factors under the optimist prior. (based on 40000 simulations).

| | $\kappa = 0.01$ | | | $\kappa = 1$ | | | $\kappa = 100$ | | |
Horizon	Bills	Equity	Bonds	Bills	Equity	Bonds	Bills	Equity	Bonds
1	0.47	0.44	0.09	0.53	0.43	0.04	0.40	0.60	0.00
5	0.55	0.45	0.00	0.51	0.49	0.00	0.33	0.67	0.00
10	0.52	0.48	0.00	0.44	0.56	0.00	0.29	0.71	0.00
15	0.51	0.49	0.00	0.41	0.59	0.00	0.34	0.66	0.00
25	0.45	0.44	0.11	0.41	0.59	0.00	0.41	0.59	0.00
40	0.39	0.45	0.16	0.37	0.49	0.14	0.53	0.47	0.00
50	0.26	0.56	0.18	0.37	0.43	0.20	0.60	0.40	0.00

Table 9: Certainty equivalent and robust portfolio choice

Certainty equivalent (in percentages of initial wealth) if the strategic asset allocation decision is based on prior $p_j(\mu)$ while the investor has prior $p_i(\mu)$. The rows refer to the prior used in the evaluation, while the columns indicate the prior used for constructing an optimal portfolio.

Prior	Portfolio							
	OLS	Flat	P01	P1	P100	O01	O1	O100
Horizon $k = 1$ year								
OLS	0.0	0.0	0.1	0.7	2.2	0.1	0.1	0.0
Flat	0.0	0.0	0.0	0.6	2.2	0.0	0.0	0.1
Pessimist ($\kappa = 0.01$)	0.1	0.0	0.0	0.4	1.9	0.0	0.0	0.2
Pessimist ($\kappa = 1$)	0.8	0.7	0.4	0.0	0.6	0.5	0.5	1.2
Pessimist ($\kappa = 100$)	2.9	2.8	2.2	0.7	0.0	2.3	2.4	3.6
Optimist ($\kappa = 0.01$)	0.1	0.0	0.0	0.5	1.9	0.0	0.0	0.2
Optimist ($\kappa = 1$)	0.1	0.0	0.0	0.5	2.0	0.0	0.0	0.2
Optimist ($\kappa = 100$)	0.1	0.1	0.3	1.3	3.4	0.2	0.2	0.0
Horizon $k = 10$ years								
OLS	0.0	2.1	5.7	21.0	33.8	4.9	3.1	0.9
Flat	4.8	0.0	1.4	14.0	32.8	1.0	0.2	0.4
Pessimist ($\kappa = 0.01$)	12.3	1.4	0.0	4.6	18.6	0.0	0.6	3.4
Pessimist ($\kappa = 1$)	33.4	10.4	4.5	0.0	7.5	5.3	7.9	15.5
Pessimist ($\kappa = 100$)	79.1	49.6	40.9	15.0	0.0	42.1	45.9	56.7
Optimist ($\kappa = 0.01$)	9.7	0.9	0.0	5.3	18.5	0.0	0.3	2.7
Optimist ($\kappa = 1$)	5.1	0.2	0.5	7.4	18.7	0.3	0.0	1.0
Optimist ($\kappa = 100$)	2.3	0.4	3.5	23.3	48.5	2.7	1.2	0.0
Horizon $k = 25$ years								
OLS	0.0	23.7	38.5	76.0	87.8	36.1	22.1	22.1
Flat	347.7	0.0	4.6	23.2	76.9	4.4	0.1	0.1
Pessimist ($\kappa = 0.01$)	134.0	1.7	0.0	3.9	29.1	0.1	2.2	2.2
Pessimist ($\kappa = 1$)	1160.	11.1	3.9	0.0	26.2	4.6	12.4	12.4
Pessimist ($\kappa = 100$)	1092.	116.3	66.7	32.6	0.0	61.6	118.7	118.7
Optimist ($\kappa = 0.01$)	177.8	1.2	0.1	4.7	31.0	0.0	1.6	1.6
Optimist ($\kappa = 1$)	133.3	0.0	1.9	11.2	27.3	1.4	0.0	0.0
Optimist ($\kappa = 100$)	115.7	0.1	7.2	34.7	82.0	6.9	0.0	0.0

Figure 1: Marginal Prior on c

The figure shows the implied marginal prior on $c = \mu(1 - b)$ with a flat prior on $b \in [0, 1]$ and $\mu \sim N(\mu_0, \omega_0^2)$.
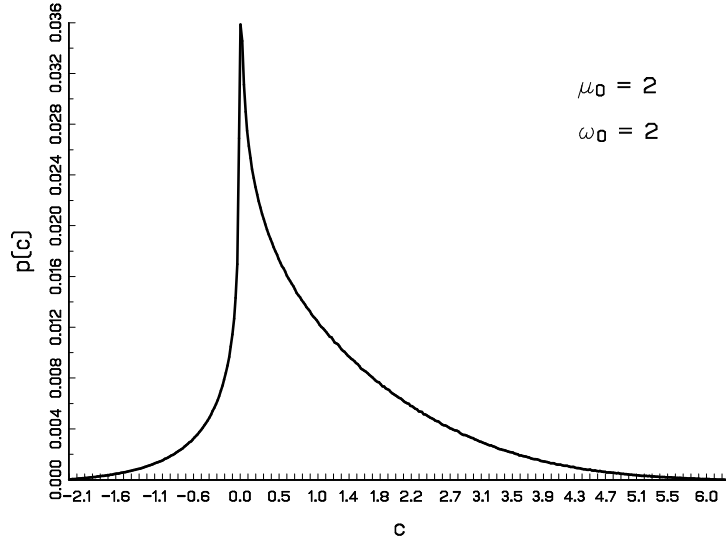

Figure 2: Conditional Prior on b

The figure shows the conditional prior on b given $c = (1 - b)\mu$ with a flat marginal prior on $b \in [0, 1]$ and $\mu \sim N(\mu_0, \omega_0^2)$.
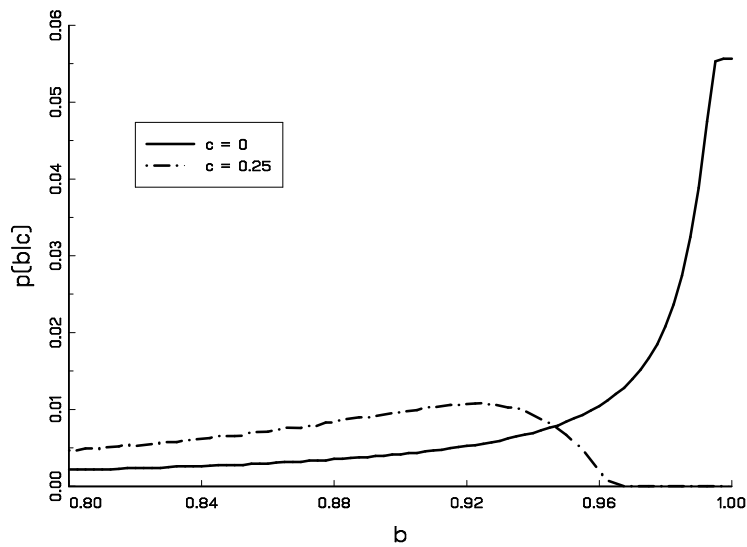

36

Figure 3: NBER expansion and contraction periods

NBER expansion periods and contraction periods in our sample are assigned 1 and 0 respectively.
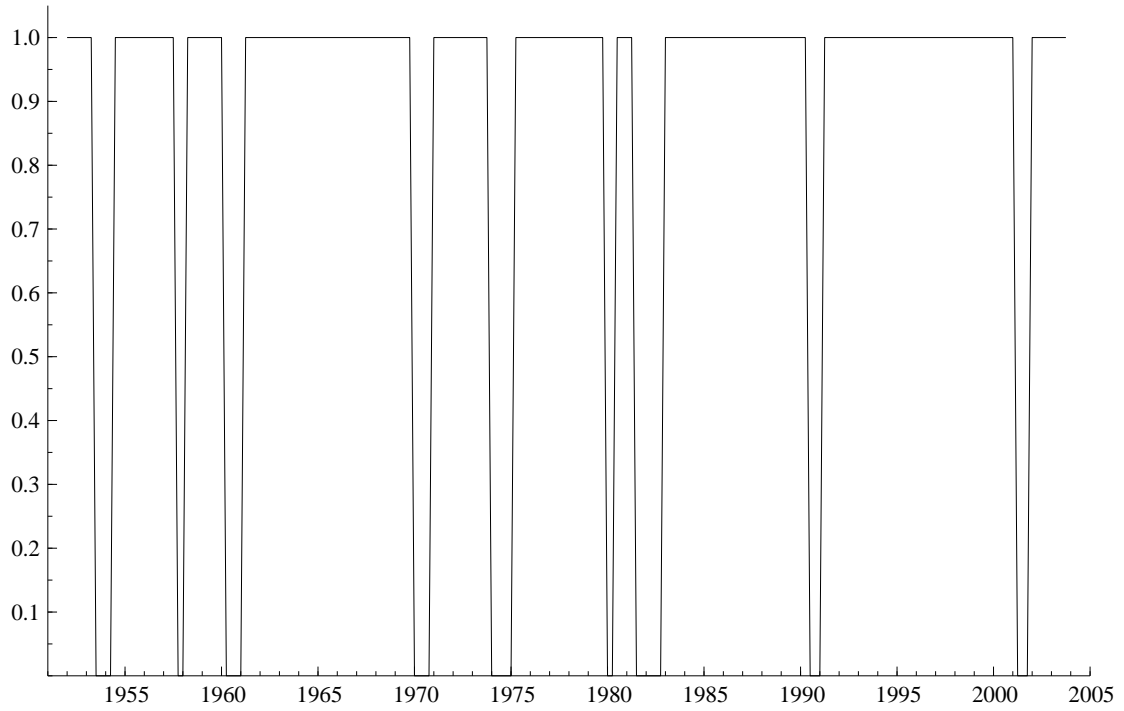
Figure 4: Posterior distributions for the pessimist prior

Posterior distributions of means, autocorrelation coefficients and maximum Eigenvalues of simulated covariance matrices and coefficient matrices of the joint posterior distribition in the Gibbs sampler under the pessimist prior. (20000 simulations). $\kappa = 0.01$: solid, $\kappa = 1$: bold solid, $\kappa = 100$: dashed

Posterior distributions parameters (excl. burn−in)
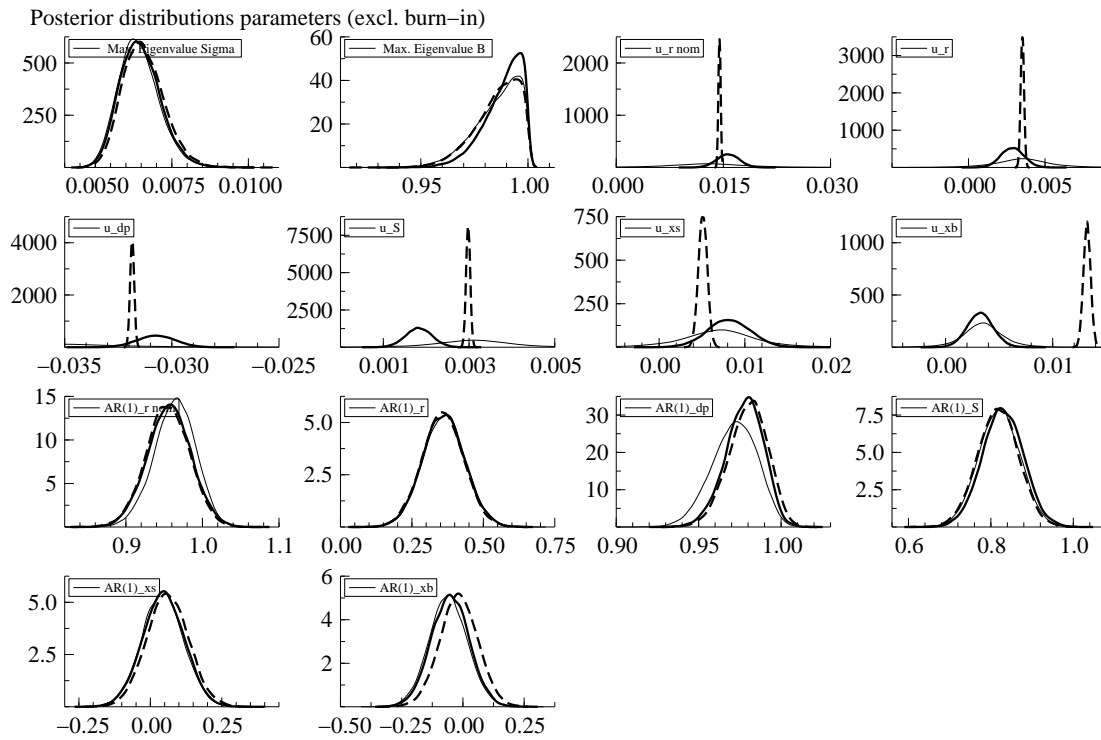
Figure 5: Posterior distributions for the optimist prior

Posterior distributions of means, autocorrelation coefficients and maximum Eigenvalues of simulated covariance matrices and coefficient matrices of the joint posterior distribition in the Gibbs sampler under the optimist prior, (20000 simulations). $\kappa = 0.01$: solid, $\kappa = 1$: bold solid, $\kappa = 100$: dashed

Posterior distributions parameters (excl. burn–in)
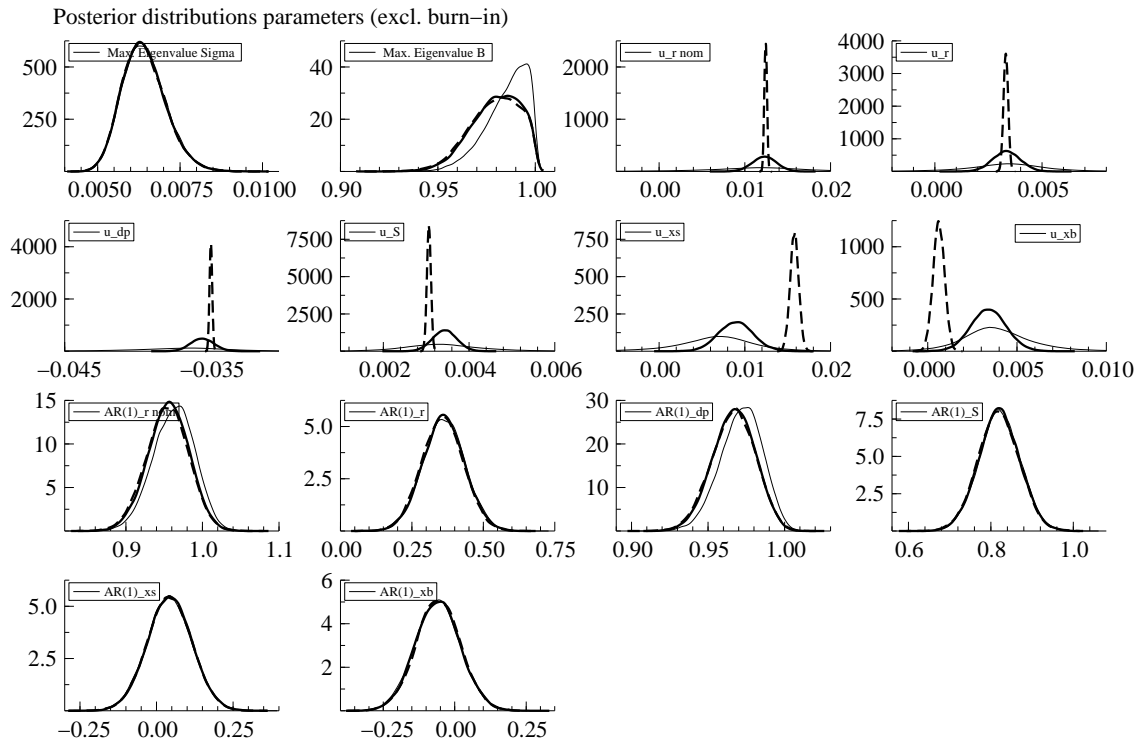
Figure 6: Mean nominal interest rate

Posterior distributions of the mean of the nominal interest rate for the optimist and pessimist priors and for three precision factors $\kappa = 0.01$: solid, $\kappa = 1$: bold solid, $\kappa = 100$: dashed (20000 simulations).
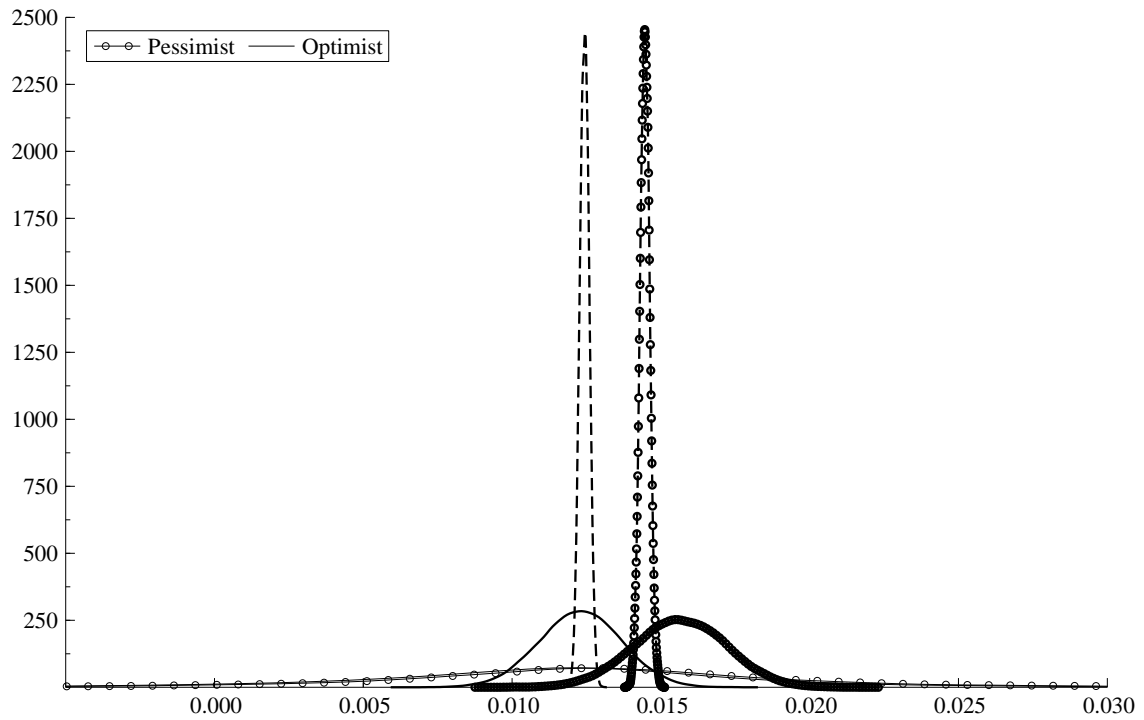
Figure 7: Persistency of dividend yield

Posterior distributions of the autocorrelation coefficient of the dividend yield for the optimist and pessimist priors and for three precision factors $\kappa = 0.01$: solid, $\kappa = 1$: bold solid, $\kappa = 100$: dashed (20000 simulations).
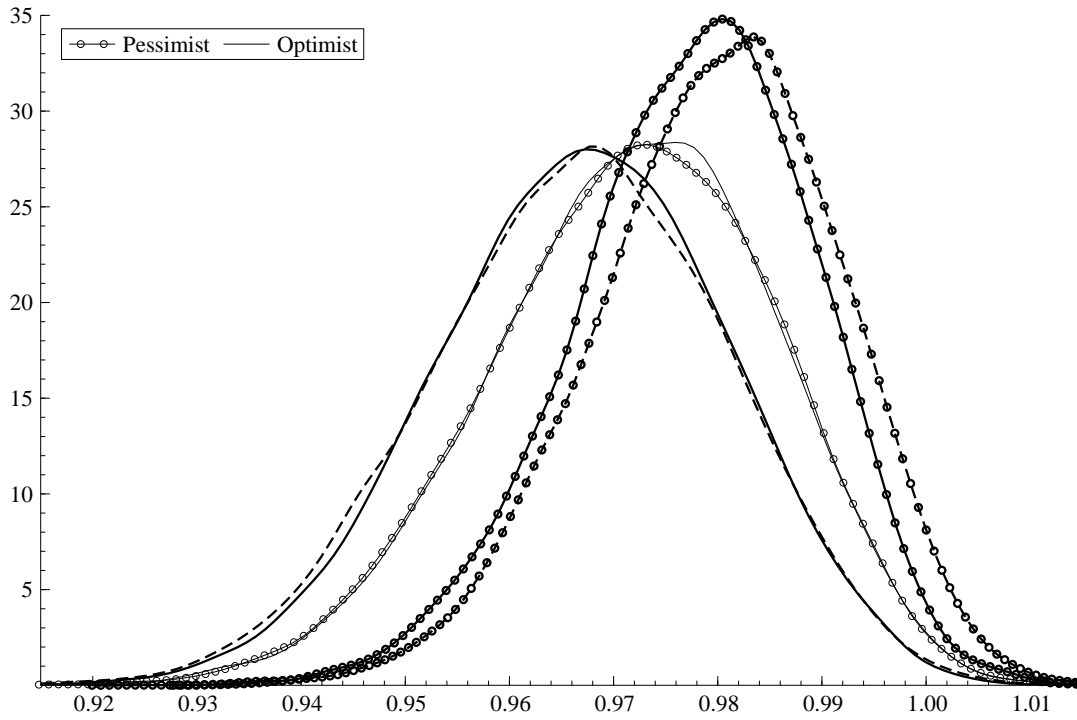
Figure 8: Annualized volatilities and parameter uncertainty
This graph shows the effect of parameter uncertainty on the annualized holding period volatilities of real returns on equities, bonds and the real T-bill (40000 simulations). OLS: solid, Parameter uncertainty: dashed
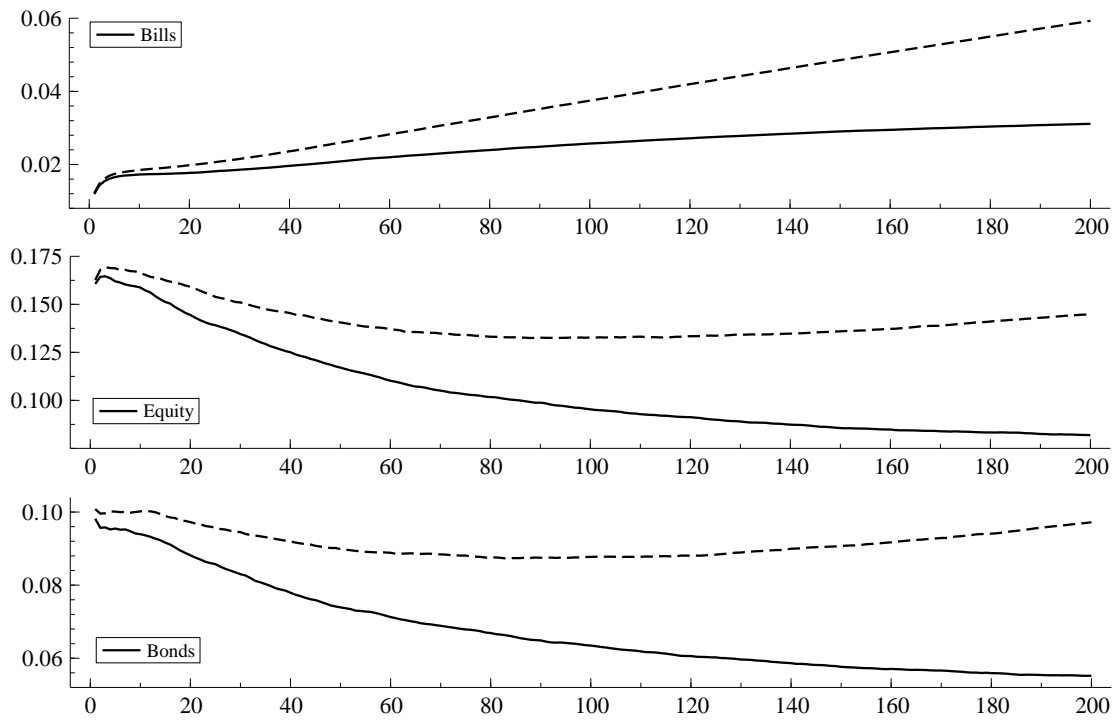
Figure 9: Annualized volatilities: parameter uncertainty and prior information
This graph shows the effect of different priors on the annualized holding period volatilities of real equity returns (40000 simulations). Two precision factors are chosen for either the optimist or pessimist prior: $\kappa = 0.01$: solid, $\kappa = 100$: dashed, OLS: solid circle
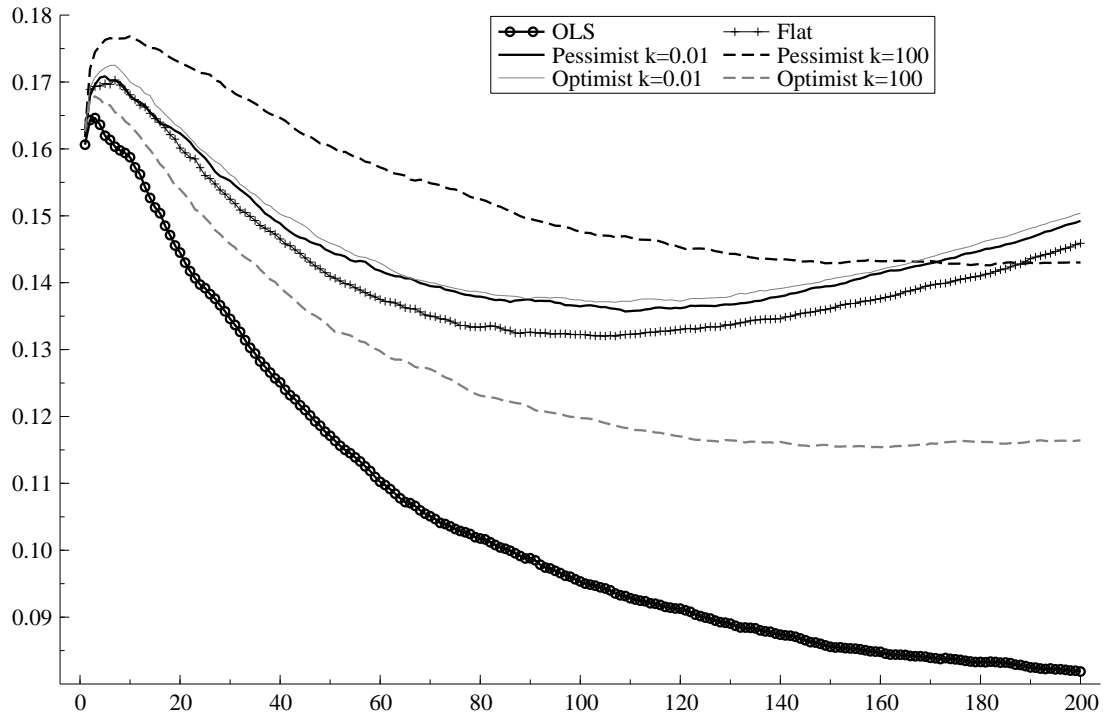
Figure 10: Mean uncertainty vs. parameter uncertainty

This graph decomposes the annualised unconditional volatility of equities (solid) ($\kappa = 0.01$) into two factors: the average conditional volatility (dashed) and the volatility of the conditional mean (solid circle). The OLS volatility is added as a reference (solid plus).
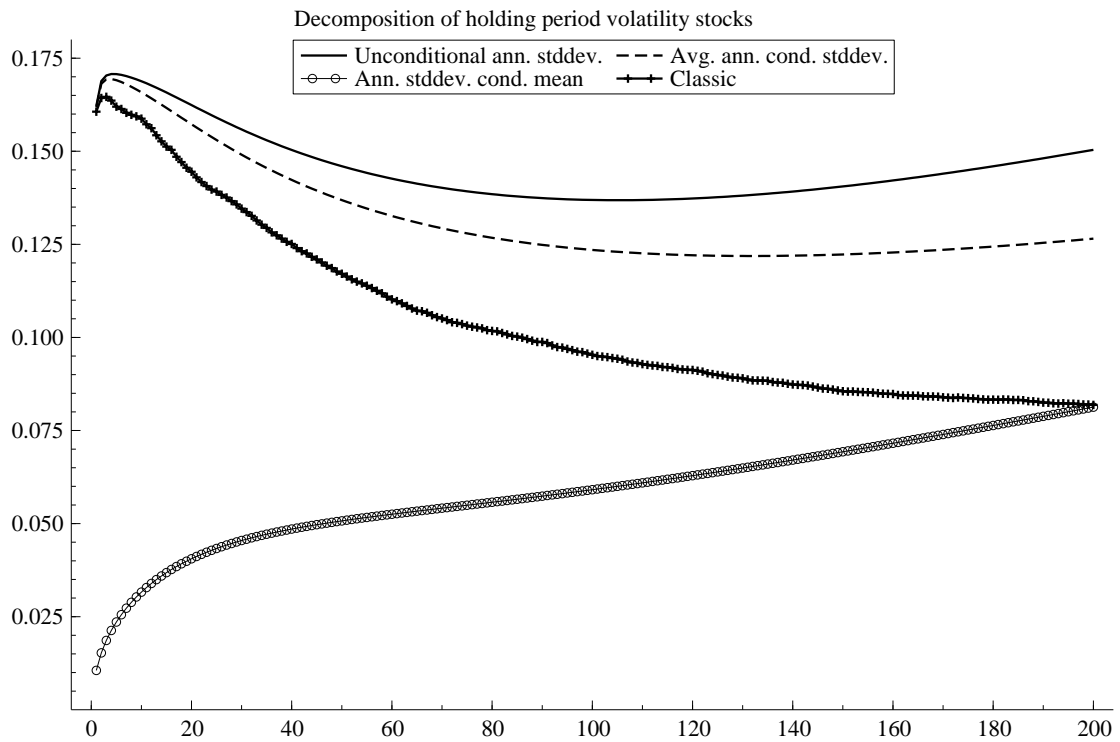
Figure 11: Correlations and parameter uncertainty

This graph shows the effect of parameter uncertainty on the correlations of real holding period returns between equities and T-bill, bonds and equities, and the T-bill and bonds (40000 simulations). OLS: solid, Parameter uncertainty: dashed
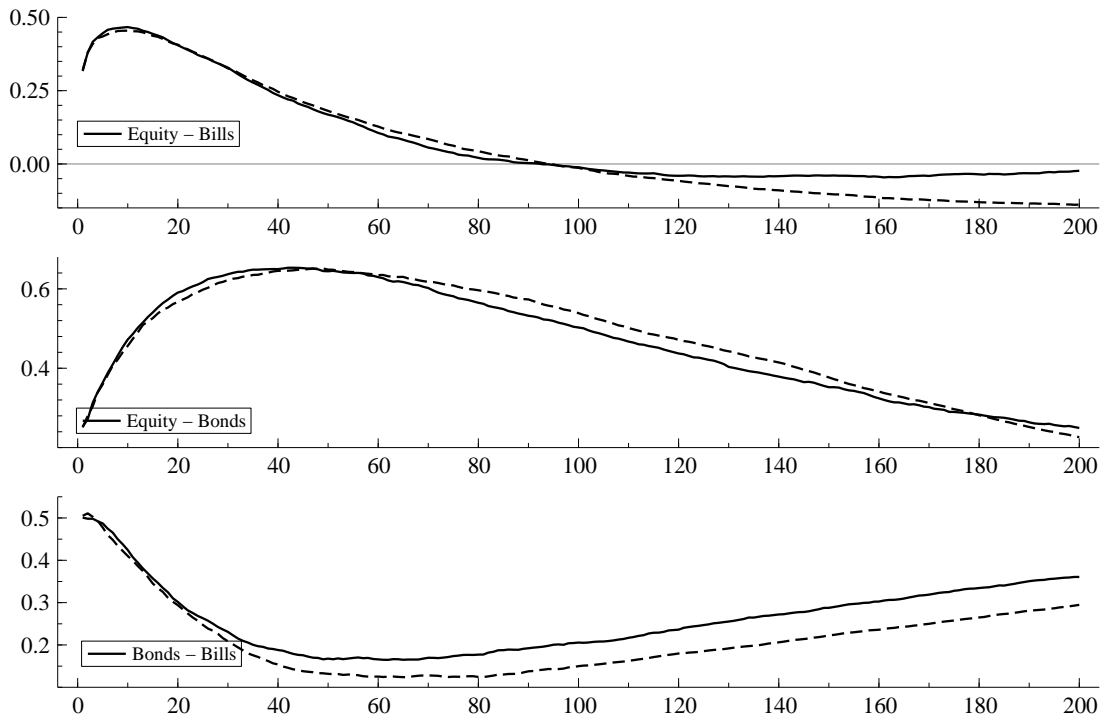
Figure 12: Inflation hedge qualities and parameter uncertainty
This graph shows the effect of parameter uncertainty on the inflation hedge qualities of nominal holding period returns of equities, bonds, and the T-bill (40000 simulations). OLS: solid, Parameter uncertainty: dashed
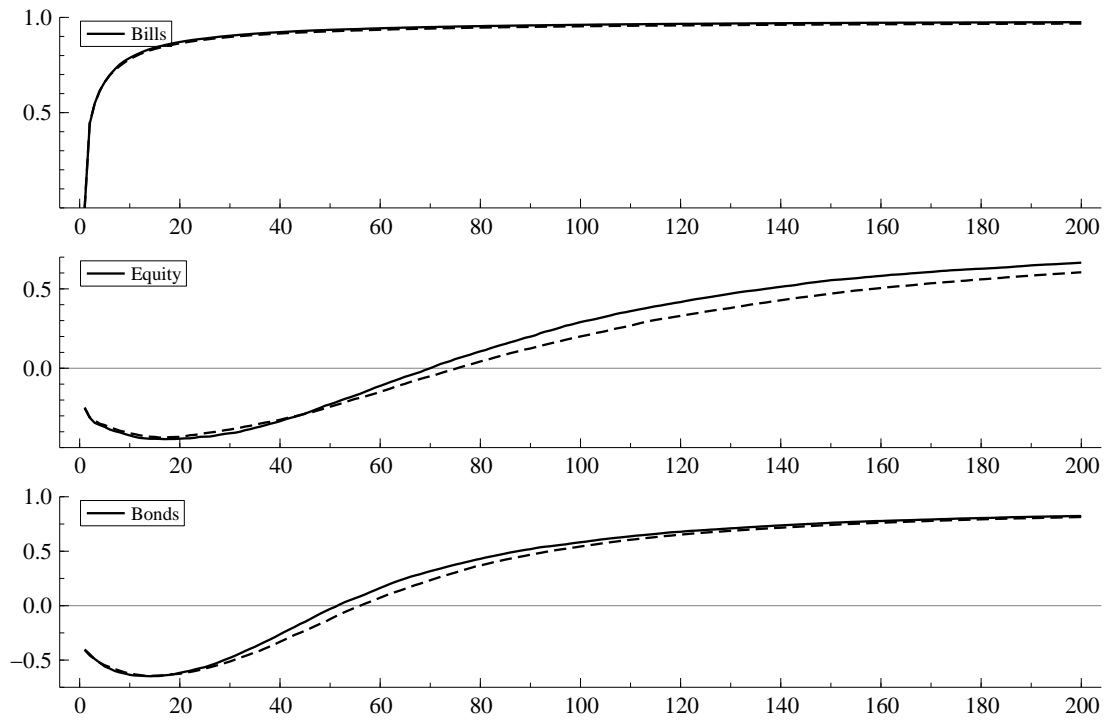
Figure 13: Buy-and-Hold asset allocation versus risk aversion
Optimal portfolio weights under power utility for a buy and hold investor with a
10-year investment horizon versus (1/risk aversion) (we choose 0.15, 0.2, 0.3, 0.4,
and 0.5 as grid for 1/risk aversion) and for different precision factors under the
pessimist prior (based on 40000 simulations). $\kappa = 0.01$: solid, $\kappa = 1$: bold solid,
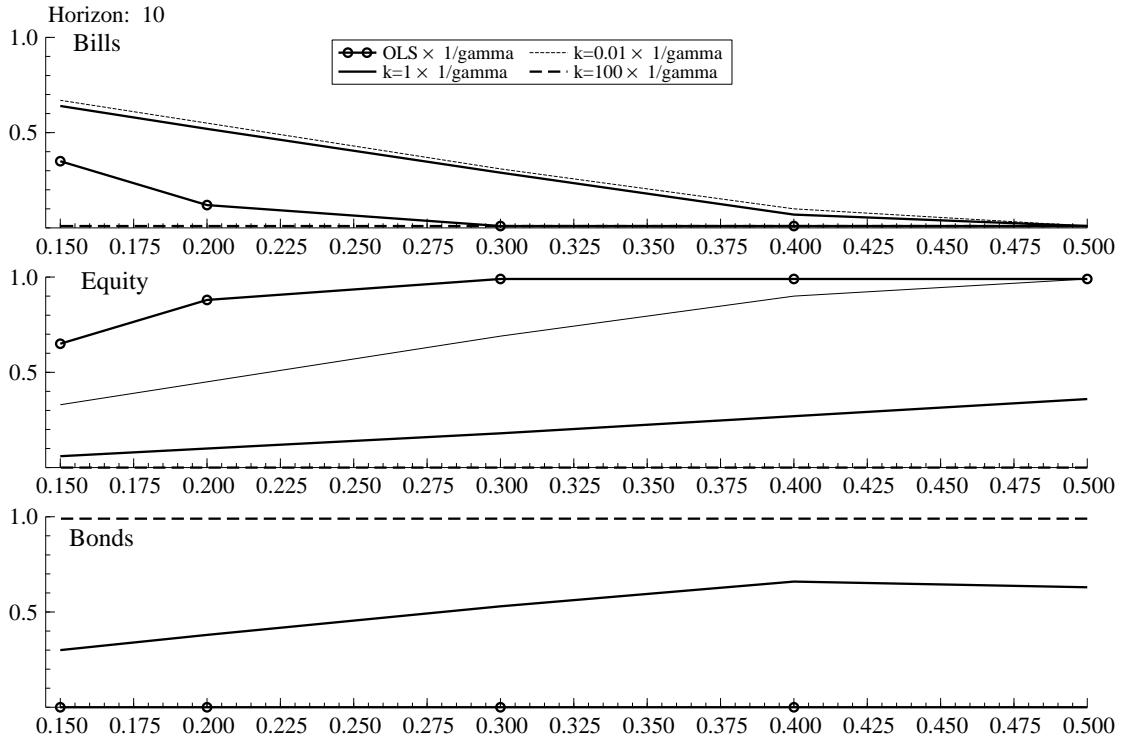$\kappa = 100$: dashed, OLS: solid circle

Figure 14: Buy-and-Hold asset allocation versus risk aversion

Optimal portfolio weights under power utility for a buy and hold investor with a 10-year investment horizon versus (1/risk aversion) (we choose 0.15, 0.2, 0.3, 0.4, and 0.5 as grid for 1/risk aversion) and for different precision factors under the optimist prior (based on 40000 simulations). $\kappa = 0.01$: solid, $\kappa = 1$: bold solid, $\kappa = 100$: dashed, OLS: solid circle