Robust Portfolio Choice with Ambiguity and Learning about Return Predictability

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The paper contains graphs in color, use color printer for best results.

Abstract

We analyze the optimal portfolio under both learning and ambiguity aversion. Returns are predictable by an observable and an unobservable predictor, and the investor has to learn about the latter. Furthermore, the investor is ambiguity-averse and relies on a robust strategy following Anderson, Hansen, \textit{and Sargent (2003)}. We find that both learning and ambiguity aversion have an impact on the level and structure of the optimal demand for the stock. If the investor refuses to learn or ignores ambiguity, he suffers utility losses. The losses from not learning are rather small and dampened even further for an ambiguity-averse investor. Utility losses from ignoring ambiguity aversion, however, are large, and can exceed 50\% of the initial wealth for an investment horizon of 20 years. It is thus much more important to take ambiguity aversion into account than to learn about an unobservable predictor.

Keywords: Return predictability, portfolio choice, ambiguity, learning, robust control

JEL subject codes: G11

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Abstract

We analyze the optimal portfolio under both learning and ambiguity aversion. Returns are predictable by an observable and an unobservable predictor, and the investor has to learn about the latter. Furthermore, the investor is ambiguity-averse and relies on a robust strategy following Anderson, Hansen, and Sargent (2003). We find that both learning and ambiguity aversion have an impact on the level and structure of the optimal demand for the stock. If the investor refuses to learn or ignores ambiguity, he suffers utility losses. The losses from not learning are rather small and dampened even further for an ambiguity-averse investor. Utility losses from ignoring ambiguity aversion, however, are large, and can exceed 50% of the initial wealth for an investment horizon of 20 years. It is thus much more important to take ambiguity aversion into account than to learn about an unobservable predictor.

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1 Introduction

Numerous empirical studies conclude that excess stock returns are predictable in the sense that average excess stock returns depend on the current value of some predictor variable. The impact of return predictability on optimal dynamic portfolios has been studied in several settings. Some papers simply assume that the expected stock (index) return is an affine function of a given predictor variable, that the predictor follows a certain stochastic process, and that all parameters involved are known. However, parameters are based on estimations and the entire modeling of expected returns might be misspecified. Several papers have extended the basic setting by incorporating either learning about the return predictability relation or model ambiguity together with an ambiguity aversion.

In this paper, we formulate a continuous-time model in which the expected excess stock return is the sum of an observable time-varying component, representing one of the known predictors, and an unobservable time-varying component. This captures the fact that any predictor is imperfect so that there are variations in expected stock returns beyond those caused by the chosen predictor. Since expected stock returns cannot be observed or estimated precisely, the second component is indeed unobservable but, by observing realized stock returns, the investor can learn about the unobservable component using Bayesian learning. Furthermore, the investor is not sure about the model and thus allows for some ambiguity. He is ambiguity-averse and takes into account that the true expected return may deviate from the expected return of the reference model. The investor follows a robust investment strategy along the lines of Anderson, Hansen, and Sargent (2003) and Maenhout (2004). Our model thus exhibits both learning and ambiguity about return predictability.

We derive the optimal investment strategy in closed form (numerical solution of simple ordinary differential equations is needed, though) in our model with both ambiguity aversion and learning. We determine the losses an investor suffers if he refuses to learn or if he does not take ambiguity into account. First, we show that both learning and ambiguity
aversion do not only have an impact on the size of the stock holdings, but also on the
structure of the optimal portfolio, with hedge terms for the uncertainty due to learning
and due to ambiguity aversion. Second, we analyze the economic importance of learning
and robust control when it comes to portfolio planning. We show that these two concepts
are not substitutes for each other, and we also show that ambiguity aversion is much more
important than learning.

In line with intuition, we find that the utility losses of an investor who refuses to learn
increase in the proportion of the expected excess return accounted for by the unobservable
predictor. Ambiguity aversion results in a more conservative strategy and lowers the losses
from not learning. Overall, however, the utility losses from refusing to learn are rather
modest. Over a 20 year horizon, the overall utility loss rarely exceeds 3% of the initial
portfolio value. This picture changes dramatically when we study the impact of ignoring
ambiguity aversion. Utility losses can now reach levels well above 50% of the initial wealth.
They decrease in risk aversion, but increase in ambiguity aversion. Our results show that
ambiguity aversion about the overall level of the expected return is much more important
in our setup than learning about the unobservable predictor of this return.

The returns on broad stock portfolios have been reported to be predictable by such
variables as the stock return in the recent past (Fama and French 1988, Jegadeesh and
Titman 1993, Rouwenhorst 1998, Asness, Moskowitz, and Pedersen 2009, Moskowitz, Ooi,
and Pedersen 2010), the price/dividend ratio (Campbell and Shiller 1988, Boudoukh,
Michaely, Richardson, and Roberts 2007), the price/earnings ratio (Campbell and Shiller
1988), the book-to-market ratio (Kothari and Shanken 1997), the short-term interest rate
(Ang and Bekaert 2007), the consumption-wealth ratio (Lettau and Ludvigson 2001), the
housing collateral ratio (Lustig and van Nieuwerburgh 2005), the ratio of stock prices
to GDP (Rangvid 2006), and the variance-risk premium (Drechsler and Yaron 2011).
However, there are various statistical challenges in measuring predictability and there is
still a lot of debate among academics about whether predictability is there or not (Ang
and Bekaert 2007, Goyal and Welch 2008, Campbell and Thompson 2008, Cochrane 2008,

Optimal dynamic portfolios with return predictability have been derived and studied
under the assumption of no parameter or model uncertainty by (Kim and Omberg 1996),
Brennan, Schwartz, and Lagnado (1997), Campbell and Viceira (1999), Campbell, Cocco,
Gomes, Maenhout, and Viceira (2001) and Wachter (2002), among others.

The effects on optimal portfolios of learning about a constant expected return have been investigated by Brennan (1998). In a model with return predictability Barberis (2000) incorporates parameter uncertainty, but does not allow for dynamic learning. Xia (2001) assumes that the expected stock return is linearly related to a certain predictor and studies the optimal portfolio choice of an investor learning about the slope of this relation (where the slope is either constant or follows an Ornstein-Uhlenbeck process). Xia finds a substantial welfare cost of ignoring predictability or learning, in terms of a reduced certainty equivalent wealth. In her model, all variations in expected returns are due to the observable predictor, whereas we allow for additional variations via an unobservable predictor and also incorporate model uncertainty. As reported above, the welfare cost of not learning is modest in our model. Brandt, Goyal, Santa-Clara, and Stroud (2005) consider learning about other parameters of the return processes in addition to the predictive relation.

On the other hand, some papers investigate the effects on portfolio choice of an aversion against ambiguity about the return process. Ambiguity aversion can be modeled in various ways. We take the robust control approach suggested by Anderson, Hansen, and Sargent (2003). Maenhout (2004) adapts the idea to dynamic portfolio choice problems with constant relative risk aversion by imposing a homothetic specification of ambiguity aversion which renders the problem tractable and ensures that the optimal amounts invested in the different assets are proportional to wealth. He considers the simple Merton setting with a single stock and a risk-free asset with constant investment opportunities and assumes ambiguity about the expected rate of return on the stock. In an extension, Maenhout (2006) explores the role of ambiguity aversion when the expected stock return varies over time according to an Ornstein-Uhlenbeck process, as in the Kim and Omberg (1996) setting. Liu (2010) generalizes the analysis of Maenhout (2006) to Epstein-Zin preferences. We extend the model of Maenhout (2006) to the case where the expected stock return also

1 Alternatives include a maximin specification of preferences proposed by Gilboa and Schmeidler (1989) and Epstein and Schneider (2003) and a smooth ambiguity aversion specification suggested by Klibanoff, Marinacci, and Mukerji (2005). Portfolio problems under such forms of ambiguity aversion have been studied by Cao, Wang, and Zhang (2005), Garlappi, Uppal, and Wang (2007), Boyle, Garlappi, Uppal, and Wang (2009), Campanale (2011), and Peijnenburg (2011) among others. The literature on ambiguity aversion has recently been surveyed by Epstein and Schneider (2010) and Guidolin and Rinaldi (2010).
has an unobservable component and the investor learns about this component based on observed stock returns and the observable component of the expected stock return. Our setting allows us to study the interactions between learning and ambiguity about stock return predictability.

The remainder of the paper is organized as follows. Section 2 describes the model setup and gives the best estimate of the true data-generating process. The optimal strategy with learning and ambiguity aversion is derived in Section 3. Section 4 focuses the utility losses that arise due to following suboptimal strategies. A numerical example for the optimal solution and the consequences of ignoring learning and ambiguity are give in Section 5. Section 6 concludes.

2 Model setup

Consider an investor who can invest continuously in a risk-free asset with a constant rate of return $r$ and a single risky stock. The stock price dynamics is described by the stochastic process:

$$dS_t = S_t \left[(r + a + b_x x_t + b_y y_t) dt + \sigma_S dS_t \right],$$
$$dx_t = \kappa_x (\mu_x - x_t) dt + \sigma_x dz_{x,t},$$
$$dy_t = \kappa_y (\mu_y - y_t) dt + \sigma_y dz_{y,t},$$

where $x_t$ is an observable state variable, $y_t$ is an unobservable state variable, and $z_S, z_x, z_y$ are one-dimensional correlated Brownian motions under the reference measure $\mathbb{P}$. The expected excess return on the stock is given by

$$\mu_t = a + b_x x_t + b_y y_t.$$

Hence, the expected excess return is the sum of a constant, an observable state variable, representing one of the known predictors mentioned in the Introduction, and an unobservable state variable. This captures the fact that any predictor is imperfect so that there are variations in the expected excess returns beyond those caused by the predictor. The predictive power of the observable and unobservable state variable is given by the constants.

\footnote{We assume that the stock pays no dividends, however the analysis also holds for a dividend paying stock when the dividends is reinvested in the stock.}
$b_x$ and $b_y$, respectively. If $b_y \neq 0$, the investor cannot observe the expected excess return but, from observing realized stock returns and the observable predictor, the investor can learn about the unobservable state variable using Bayesian learning. From Appendix A it follows that the filtered model (as seen by the investor) is given by
\[
\begin{align*}
    dS_t &= S_t [(r + a + b_x x_t + b_y \hat{y}_t) dt + \sigma_S \hat{z}_{S,t}] \\
    dx_t &= \kappa_x (\mu_x - x_t) dt + \sigma_x dz_{x,t} \\
    d\hat{y}_t &= \kappa_y (\mu_y - \hat{y}_t) dt + K_S \sigma_S \hat{z}_{S,t} + K_x \sigma_x dz_{x,t},
\end{align*}
\]
where $\hat{z}_{S}$ is a standard Brownian motion relative to the reference measure $\mathbb{P}$ and the filtration defined by the observables and
\[
\begin{align*}
    K_S &= \eta b_y + (\rho_{yS} - \rho_{xS}\rho_{xy}) \sigma_S \sigma_y / (1 - \rho^2_{S}) \sigma^2_S, \\
    K_x &= -\eta b_y \rho_{xS} + (\rho_{xy} - \rho_{xS}\rho_{yS}) \sigma_S \sigma_y / (1 - \rho^2_{S}) \sigma_S \sigma_x,
\end{align*}
\]
and $\eta$ is the stationary variance of the estimation error given in (16). The variance of the estimation risk would normally be a deterministic function of time. However, for simplicity, we assume that there have been a sufficiently long period of learning for the investor and hence the variance of the estimation risk have converged to the long-run level of variance $\mathbb{E}$.

The filtered model is the best estimate available to the investor given his current information. However, we assume that he is still skeptical about the degree to which excess returns are predictable as well as the quality of the information he gets from observing the stock price process. Hence, even after learning, the investor is not sure about the true model and thus allow for some ambiguity. In particular, besides being risk-averse, the investor is ambiguity-averse and seeks a robust investment strategy along the lines of Anderson, Hansen, and Sargent (2003). That is, the investor takes the above model as his reference model, but he recognizes that it is only an approximation of reality and also takes some alternative models defined by a set of probability measures $\mathbb{P}^u$ into account. The change from the reference measure $\mathbb{P}$ to an equivalent alternative measure $\mathbb{P}^u$ is defined by the Radon-Nikodym derivative, $\xi^u_t = \mathbb{E}_t [d\mathbb{P}^u / d\mathbb{P}]$ where $\xi^u_t = \exp \{ -0.5 \int_0^t u_s^2 ds - \int_0^t u_s dz_s \}$.

\[\text{Footnote: The same assumption has been made by Scheinkman and Xiong (2003) and Dumas, Kurshev, and Uppal (2009) among others.}\]
By Girsanov’s Theorem it follows that

\[ dz^u_{S,t} = d\tilde{z}_{S,t} + u_t \, dt \]

is a standard Brownian motion under the alternative measure \( \mathbb{P}^u \). Hence, the dynamics of the stock price and the state variables under the alternative measure \( \mathbb{P}^u \) becomes

\[
\begin{align*}
    dS_t &= S_t \left[ (r + a + b_x x_t + b_y \hat{y}_t - \sigma_S u_t) \, dt + \sigma dz^u_{S,t} \right] \\
    dx_t &= \kappa_x (\mu_x - x_t) \, dt + \sigma_x dz_{x,t} \\
    d\hat{y}_t &= \left[ \kappa_y (\mu_y - \hat{y}_t) - K_S \sigma_S u_t \right] \, dt + K_S \sigma_S dz^u_{S,t} + K_x \sigma_x d\tilde{z}_{x,t}.
\end{align*}
\]

In this way, the investor allows for a misspecification of the expected stock return and of the drift of the unobservable predictor. We will refer to \( u \) as a drift distortion. The distance between the reference measure \( \mathbb{P} \) and an alternative measure \( \mathbb{P}^u \) is captured by the relative entropy. The relative entropy is defined as the expectation under the alternative measure of the log Radon-Nikodym derivative, and it can be shown that the increase in the relative entropy from \( t \) to \( t + dt \) equals \( \frac{1}{2} u_t^2 \, dt \). In line with intuition, the distance measure is increasing in the absolute size of \( u \).

The investor is assumed to have a power utility function of terminal wealth \( W_T \) and seeks to maximize his expected utility by choosing an optimal investment strategy \( \alpha = (\alpha_t) \) in the risky asset. In a traditional portfolio choice model with no model uncertainty, the indirect utility of the investor is given by

\[
\tilde{V}(W, x, \hat{y}, t) = \sup_{\alpha} \mathbb{E}_t^\mathbb{P} \left[ \frac{W_t^{1-\gamma}}{1-\gamma} \right],
\]

where \( T \) equals the investor’s investment horizon, \( \gamma \) equals the investor’s relative risk aversion, and the expectation is conditional on the information available to the investor at time \( t \).

As argued above, the investor does not know the true model and is aware and averse about this. In particular, he wants to guard himself against facing some worst-case model. Following Anderson, Hansen, and Sargent (2003), we assume that the investor’s indirect utility function is given by

\[
V(W, x, \hat{y}, t) = \inf_u \sup_{\alpha} \mathbb{E}_t^{\mathbb{P}^u} \left[ \frac{W_t^{1-\gamma}}{1-\gamma} + \int_t^T \frac{u_s^2}{2\Psi(W_s, x, \hat{y}, s)} \, ds \right].
\]
The expected utility is now calculated under the alternative measure $P^u$. The investor chooses the drift distortion $u$ and thus the measure $P^u$ by minimizing the expected utility, that is by considering the worst case. At the same time, he is well aware of the fact that the reference measure is statistically the best representation of the existing data, and he is thus reluctant to deviate arbitrarily much from the reference measure. Therefore, he includes the second term in the indirect utility function which penalizes any deviations. The measure $P^u$ is then found by considering the trade-off between not completely relying on the reference model and, at the same time, not deviating too much from it.

The penalty term is given by the relative entropy scaled by a function $\Psi$. This function captures the investor’s ambiguity aversion and is assumed to be nonnegative. The larger $\Psi$, the less a given deviation from the reference model is penalized, the less faith in the reference model the investor has, and the more the worst case model will deviate from the reference model. Hence the investor’s ambiguity aversion is increasing in $\Psi$. For analytical tractability we assume a suitable form of $\Psi$ proposed by Maenhout (2004), that is

$$\Psi(W, x, \hat{y}, t) = \frac{\theta}{(1-\gamma)} V(W, x, \hat{y}, t).$$

(5)

The investor’s ambiguity aversion is thus increasing in the parameter $\theta$. With this specification the optimal investment strategy is independent of wealth as for an ambiguity-neutral investors with constant relative risk aversion.

3 Optimal robust investment strategy

The investor invests the fraction of wealth $\alpha$ in the stock and the remaining fraction of wealth $1 - \alpha$ in the risk-free asset. For at given investment, $\alpha$, the wealth dynamics under the reference measure is given by

$$dW_t = W_t [r + \alpha_t (a + b_r x_t + b_y \hat{y}_t)] dt + \alpha_t \sigma S W_t d\hat{z}_{S,t}.$$

and under the alternative measure by

$$dW_t = W_t [r + \alpha_t (a + b_r x_t + b_y \hat{y}_t - \sigma S u_t)] dt + \alpha_t \sigma S W_t d\hat{z}_{S,t}.$$

To solve for the optimal investment strategy we use dynamic programming. In particular, we need to solve the robust Hamilton-Jacobi-Bellmann equation developed by Anderson.
Note that if $\Psi = 0$, indicating an ambiguity-neutral investor, we have

$$u^* = \Psi (V_WW_\alpha + V_yK_S) \sigma_S.$$

(7)

Note that if $\Psi = 0$, indicating an ambiguity-neutral investor, we have $u^* = 0$ and we are back in the standard setting with no ambiguity. In Appendix B, we show that the solution is as stated in the following proposition.

**Proposition 1** The indirect utility function for an ambiguity- and risk-averse investor with $\gamma > 1$ is given by

$$V(W, x, \hat{y}, t) = \frac{W^{1-\gamma}}{1-\gamma} g(t, x, \hat{y}),$$

(8)

with

$$g(x, \hat{y}, t) = \left( e^{A_0(t) + A_1(t)x + A_2(t)\hat{y} + \frac{1}{2} B_1(t)x^2 + \frac{1}{2} B_2(t)\hat{y}^2 + B_3(t)x\hat{y}) \right)^{1-\gamma},$$

(9)

where $\tau = T - t$ and the deterministic functions $A_0$, $A_1$, $A_2$, $B_1$, $B_2$, and $B_3$ solve a system of ordinary differential equations shown in Appendix B. The optimal investment strategy is given by

$$\alpha^* = \frac{a + b_x x + b_y \hat{y}}{\sigma_x} - \frac{(\gamma - 1) \sigma_x \rho_x S}{\sigma_S} \left( A_1(t) + B_1(t)x + B_3(t)\hat{y} \right) - \left( \frac{\theta K_S}{\gamma + \theta} + \frac{\gamma - 1}{\gamma + \theta} \frac{K_S \sigma_S + K_x \sigma_x \rho_x S}{\sigma_S} \right) \left( A_2(t) + B_2(t)\hat{y} + B_3(t)x \right).$$

(10)

The worst-case distortion is given by

$$u^* = \theta \sigma_S \alpha^* + \theta \sigma_S K_S \left( A_2(t) + B_2(t)\hat{y} + B_3(t)x \right).$$

(11)
The optimal strategy has several components. The first term reflects the speculative investment, which corresponds to the investment strategy derived by Maenhout (2004) who assumes no learning and constant investment opportunities. It depends on the estimated expected excess return, and decreases both in the risk aversion and the ambiguity aversion of the investor. The second term hedges changes in the observable state variable $x$. This term disappears if there is no scope for hedging ($\rho_xS = 0$), no need for hedging ($\sigma_x = 0$), or no preference for hedging ($\gamma = 1$, log utility). The remaining third term hedges changes in the unobservable state variable $y$. This term has two parts. The second part involving $\gamma - 1 \frac{K_S\sigma_S + K_x\sigma_x\rho_xS}{\gamma + \theta}$ will again vanish if there is no scope for hedging ($\rho_yS$ is zero when $K_S\sigma_S + K_x\sigma_x\rho_xS = 0$), no need for hedging ($\gamma$ is locally riskfree if $K_S\sigma_S = 0$ and $K_x\sigma_x = 0$), or no preference for hedging ($\gamma = 1$). In contrast, the first part involving $\frac{\theta K_S}{\gamma + \theta}$ is not driven by the hedging needs of a non-myopic investor (with $\gamma \neq 1$), but arises due to the ambiguity aversion of the investor. Obviously the term vanishes for an ambiguity-neutral investor with $\theta = 0$. Note there is no similar hedge term for the observable state variable, $x$. The reason is that there is ambiguity about the drift of the stock return, which via the learning channel then also enters the dynamics of $\hat{y}$. The state variable $x$, on the other hand, is perfectly observable and thus not subject to ambiguity.

The ambiguity about the specification of the return predictability and the investor’s aversion towards ambiguity lowers the speculative investment in the stock as found by Maenhout (2004). Ambiguity aversion is also affecting the hedging demand, but the effect cannot be directly seen from (10) as the $A$- and $B$-functions depend on $\theta$. In addition, ambiguity aversion induces an extra term in the hedge demand related to the unobservable state variable. In the setting of Maenhout (2006) the investor’s ambiguity aversion only enters the optimal portfolio as an addition to the risk aversion of the investor, whereas the ambiguity aversion in our setting also enters as a separate parameter. The reason for this difference is due to our model specification. Maenhout (2006) assumes that the investor worries about model specification about the expected rate of return of the stock as well as the drift of the mean-reverting risk premium. We assume, on the other hand, that the investor only worries about model misspecification about the expected

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4We could introduce model uncertainty about the observable state variable as well, but this is already considered by Maenhout (2006) who analyzes the optimal portfolio problem of an investor worrying about model misspecification and insisting on robust decision rules when facing a mean-reverting risk premium.
rate of return of the stock. The state variable $x$ is perfectly observable and hence not subject to ambiguity. Finally, note that for an ambiguity-neutral investor, the solution in Proposition 1 generalizes the solution derived by Kim and Omberg (1996) in a setting with a single predictor to the case of two predictors.

4 Suboptimal strategies

The expected utility of an ambiguity-averse investor following a given investment strategy $\alpha (t)$ is defined as

$$V^\alpha(W, x, \hat{y}, t) = \inf_u \mathbb{E}^u \left[ 1 - \frac{\gamma}{1 - \gamma} \int_t^T \frac{1}{W^2_s} \Psi(s, W_s) ds \right],$$

and, in line with (5), we assume that

$$\Psi(s, W_s) = \frac{\theta}{(1 - \gamma) V^\alpha(W, x, \hat{y}, t)}.$$

The specific suboptimal strategies we are interested in (see below) are affine in the observed state variable $x$ and the filtered value $\hat{y}$ of the unobservable state variable. We can evaluate such strategies according to the following result. A sketch of the proof can be found in Appendix C.

**Proposition 2** For any investment strategy of the affine form

$$\alpha(t, x, \hat{y}) = F_0(\tau) + F_1(\tau)x + F_2(\tau)\hat{y}, \quad (12)$$

where $\tau = T - t$ and $F_0$, $F_1$, and $F_2$ are deterministic functions, the expected utility is given by

$$V^\alpha(W, x, \hat{y}, t) = \frac{W^{1 - \gamma}}{1 - \gamma} g^\alpha(t, x, \hat{y}),$$

where

$$g^\alpha(t, x, \hat{y}) = \left( e^{A_0^\alpha(\tau)x + A_1^\alpha(\tau)\hat{y} + B_1^\alpha(\tau)x^2 + B_2^\alpha(\tau)\hat{y}^2 + B_3^\alpha(\tau)xy} \right)^{1 - \gamma} \quad (13)$$

and the deterministic functions $A_0^\alpha$, $A_1^\alpha$, $A_2^\alpha$, $B_1^\alpha$, $B_2^\alpha$, and $B_3^\alpha$ solve a system of ordinary differential equations shown in Appendix C.

By definition, with the same initial wealth, a suboptimal investment strategy will generate a lower level of expected utility than the optimal strategy. To evaluate the suboptimal strategies we will determine this loss from following the suboptimal strategy.
We measure the loss as the fraction of initial wealth the investor is willing to give up to know the optimal strategy. That is, the loss $L$ is determined from

$$V(W(1-L), x, \hat{y}, t) = V^\alpha(W, x, \hat{y}, t).$$

It then follows from Proposition 1 and Proposition 2 that for a strategy of the affine form (12), the loss is given as

$$L \equiv L(x, \hat{y}, t) = 1 - \left( \frac{g^\alpha(x, \hat{y}, t)}{g(x, \hat{y}, t)} \right)^{\frac{1}{1-\gamma}}. \quad (14)$$

5 Numerical example

To determine the quantitative effects of learning and ambiguity aversion on the investor’s portfolio planning problem we now look at a numerical example. Besides looking at the optimal investment strategy we also analyze the economic importance of learning and robust control when it comes to portfolio planning. This is done by comparing the expected utility losses and investor suffers either from ignoring the fact can he can learn about the expected excess return or from ignoring model uncertainty.

5.1 Model parameters

For our numerical analysis, we take the benchmark parameters of Brennan and Xia (2010) as a starting point. They study a model in which the expected rate of return follows an Ornstein-Uhlenbeck process. Different from them, we have two predictors instead of one. We assume that the parameter estimates for the dynamics of the unobservable state variable equal the parameter estimates of the observable state variable. This ensures that differences between the observable and the unobservable state variable are driven by differences in the information of the investor and his ambiguity aversion, but not by different parameters.

Following Brennan and Xia (2010) we set the interest rate equal to $r = 3\%$. The average equity risk premium is $6\%$, and the volatility of the stock return is put equal to $\sigma_s = 14\%$. We assume that the expected return of the stock is a linear function of $x$ and $y$. This implies that its constant part $r + a$ is zero, so that $a = -0.03$. The expected return is then given by $b_x x + b_y y$. We normalize $b_x + b_y = 1$. With an expected stock return of
9%, this implies \( \mu_x = \mu_y = 9\% \). In our benchmark case we will put an equal weight on the predictability from the observable and unobservable state variable, i.e. \( b_x = b_y = 0.5 \).

The speed of mean reversion for the two state variables is set equal to \( \kappa_x = \kappa_y = 0.10 \), whereas the standard deviation of the state variables is assumed to equal \( \sigma_x = \sigma_y = 0.018 \). The correlation between the innovations in the observable state variable and the expected rate of return equals \( \rho_{xS} = -0.5 \). Brennan and Xia (2010) consider nine scenarios, combining three values of the mean reversion parameter (0.02, 0.10, and 0.50) and three values of the correlation between the state variable and the expected rate of return (-0.90, -0.50, and 0.0). In our benchmark case we use the middle of the three values, but will also consider the other scenarios in robustness checks. In particular, the correlation of -0.5 between the observable state variable and the return of the stock is a bit low in magnitude. For example, Xia (2001) reports a correlation of -0.93 between monthly innovations in the dividend yield and stock returns, which is similar to the one reported by Barberis (2000). For the unobservable state variable \( y \), we deviate from the benchmark values and set \( \rho_{yS} = 0 \). The investor has no prior about what \( y \) actually is, so the best choice is to set both its correlation with the stock price and the observable predictor \( x \) equal to zero, i.e. \( \rho_{yS} = \rho_{xy} = 0 \).

Our benchmark investor has an investment horizon of \( T - t = 10 \) years. Furthermore, we assume the investor’s risk aversion is \( \gamma = 4 \), a standard assumption in the portfolio choice literature. We will vary the level of his risk aversion to see how this affects our results. The literature on investor’s preferences involving ambiguity aversion is relatively new. Maenhout (2004) finds the preference parameters required to match the risk-free rate and equity premium in the data from Campbell (1999) are \( \gamma = 7 \) and \( \theta = 14 \). However, as noted by Maenhout (2006), values for \( \theta \) that seem reasonable when investment opportunities are constant, as in Maenhout (2004), are not plausible for investors facing stochastic investment opportunities. In particular, Maenhout (2006) argues that the level of the investor’s ambiguity aversion, \( \theta \), should be less than or equal to the size of the investor’s risk aversion. Anderson, Ghysels, and Juergens (2009) investigate the relation between risk, uncertainty, and expected returns. They find evidence of a \( \theta \) about 1500. However, the level of the investor’s risk aversion is only about 0.08. As noted, the literature on model uncertainty and parameter risk is still relatively new and only a few papers have tried to estimate the ambiguity parameter. With the above references in mind there is no
general consensus regarding the magnitude of these aversions, i.e. a realistic level of $\theta$ is still hard to give. We choose $\theta = 3$ as our benchmark, in line with the paper by Maenhout (2006). Again, we will also study the impact a variation in $\theta$ has on the results.

5.2 Optimal robust investment strategy

Figure 1 illustrates the optimal investment strategy as well as the worst-case drift distortion as a function of the investor’s investment horizon. The three rows of panels correspond to different combinations of $b_x$ and $b_y$ and thus different weights of the observable and the unobservable predictor of stock returns. Panels A and B correspond to our benchmark case, i.e. $b_x = b_y = 0.5$. Panels C and D display the case where the expected excess return only depends on the unobservable variable, that is $b_x = 0$, and $b_y = 1$. Finally, Panels E and F illustrate the case where the expected excess return only depends on the observable state variable, i.e. $b_x = 1$ while $b_y = 0$. Panels A, C, and E show the optimal portfolio weights in terms of fractions of wealth invested in the stock and the bank account. The solid black line displays the total investment in stock, while the dashed black line displays the investment in the bank account. Finally, the four colored lines show the four components of the optimal stock investment: the red line shows the speculative demand, the green line the hedge against changes in the observable state variable, the blue line the hedge against changes in the unobservable state variable due to learning, and finally the purple line displays the ambiguity-induced hedge demand.

[Figure 1 about here.]

The speculative demand is constant over time and the same in all three cases. Differences in the optimal weight of the stock stem from the two hedge terms for $x$ and $y$ and from the hedge term for ambiguity. These hedge terms are zero for an investment horizon of zero. For our benchmark case in Panel A, the weight in the hedge term for $x$ (green line), which is negatively correlated with the stock, is positive and slightly increasing over time.$^5$ The other two hedge terms for $\hat{y}$ (blue line and purple line), which is positively correlated with the return on the stock, are negative and decreasing in the investment horizon. In total, the hedge terms for $\hat{y}$ dominate those for $x$. An investor with a long

---

$^5$This is consistent with the findings in Kim and Omberg (1996) and Wachter (2002).
investment horizon should thus put a lower fraction of his total wealth into the stock compared to an investor with a shorter investment horizon.

Panel C gives the portfolio in the special case where the expected excess return only depends on the unobservable state variable, that is $b_x = 0$ and $b_y = 1$. It then holds that $A_1(\tau) = B_1(\tau) = B_3(\tau) = 0$, and in line with intuition, the intertemporal hedge due to stochastic changes in the observable state variable (green line) is equal to zero. The hedging demand for $y$ (blue line) and the term arising from ambiguity aversion (purple line) are negative and decrease in the investment horizon, as in the benchmark case. Put together, the optimal investment in the stock decreases in the length of the planning horizon. With the positive impact of the hedge term for $x$ missing, the decrease is more pronounced than in the benchmark case.

Finally, Panel E gives the result for the opposite case where the expected excess return only depends on the observable state variable, that is $b_x = 1$ and $b_y = 0$. This first implies $A_2(\tau) = B_2(\tau) = B_3(\tau) = 0$, so that the investor obviously does not hedge changes in $y$. Furthermore, he does also not invest anything in the hedge fund associated with model uncertainty. The hedging demand for the observable state variable (green line) is positive, again as in the benchmark case. Putting the results together, the total demand for stocks is now increasing in the investment horizon.

Panels B, D, and E show the optimal drift distortion $u^*$ (red line), which describes the worst case scenario, and the expected excess return on the stock in the worst case (green line) when $x$ and $y$ are at their long-term levels. Irrespective of the values of $b_x$ and $b_y$, $u^*$ is increasing in the investment horizon. The longer the horizon, the more the investor thus cares about ambiguity. This in turn implies that the expected excess rate of return is (slightly) decreasing in the investment horizon. The investment horizon has the biggest effect on $u^*$ in the case with an observable predictor only, i.e. in Panel F.

Figure 2 illustrates the dependence of the optimal strategy and its components as well as of the worst case drift distortion on the ambiguity aversion $\theta$. Again the three rows of figures corresponds to different combinations of $b_x$ and $b_y$. Panels A and B display our benchmark case, i.e. $b_x = b_y = 0.5$. In Panels C and D, we assume that $b_x = 0$ and $b_y = 1$, while Panels E and F illustrate the opposite case with $b_x = 1$ and $b_y = 0$. 

[Figure 2 about here.]
In all three cases we see that the speculative demand (the red line) is decreasing in $\theta$, which also follows directly from (10). In Panels A and E, we have $b_x > 0$, which implies that there is a hedging demand for $x$. This hedging demand (green line) is positive and decreases when ambiguity aversion increases. In Panel C, $x$ has no impact on the expected excess return, $b_x = 0$, and the hedging demand is then of course identically equal to zero. The hedging demand for $y$ is negative and decreases in absolute terms in the investor’s ambiguity aversion as illustrated in Panels A and C. In Panel E, we have $b_y = 0$, and the hedging demand for $y$ obviously equals zero. The decrease in absolute terms in the hedging demand for both state variables reflects the fact that the lower speculative demand leads to a lower exposure to changes in the expected return, and thus also the need to hedge becomes smaller.

The purple line illustrates the investor’s hedging demand due to his ambiguity aversion. Obviously the hedging demand equals zero for $\theta = 0$. For $\theta > 0$, the hedging demand is negative and increases in $\theta$ in absolute terms in the two cases where $y$ has an impact on the expected excess return, i.e. in Panels A and C. The hedging demand is zero in Panel E, where the unobservable state variable has no impact on the expected excess return. To get the intuition, note that there is ambiguity about the true drift of the stock in all three cases. If the investor uses the stock return to learn about the unobservable state variable (i.e. if $b_y \neq 0$), this ambiguity also enters the dynamics of $\hat{y}$. The state variable $x$, on the other hand, is perfectly observable and thus not subject to ambiguity. For $b_y = 0$, ambiguity thus only affects the drift of the stock return, and we are back in the setting by [Maenhout (2004)]. In that setting the investor’s ambiguity aversion has an impact on the speculative demand, but not on the hedging demand.

The right-hand side panels of Figure 2 depict the optimal $u^*$ (red line), which describes the worst case drift distortion, and the expected excess stock return in the worst case (green line). Irrespective of the values of $b_x$ and $b_y$, the worst case drift distortion, not surprisingly, increases in the level of ambiguity aversion, and the equity risk premium decreases in $\theta$. The higher the ambiguity aversion, the less the investor trusts his reference model, and the more he allows the alternative models to deviate from this reference model. In all three cases a change in $\theta$ has the biggest effect for small values of $\theta$. 

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5.3 Suboptimal strategies: no learning

We now turn to the analysis of suboptimal strategies. First, we consider an investor who ignores the fact that he can learn about the expected excess return by observing the stock price process. That is, he replaces the random unobservable predictor by its average and assumes that the expected excess return on the stock is given by

$$\mu_t = \bar{a} + b_x x_t,$$

where $\bar{a} = a + b_y \mu_y$. The resulting investment strategy is a special case of Proposition 1 with $A_2(\tau) = B_2(\tau) = B_3(\tau) = 0$ for all $\tau$, i.e.

$$\alpha = \frac{\bar{a} + b_x x}{(\gamma + \theta) \sigma_S^2} - \frac{(\gamma - 1) \sigma_x \rho_{xS}}{(\gamma + \theta) \sigma_S} (A_1(\tau) + B_1(\tau)x).$$

(15)

The strategy is affine in the state variables, and we can write it in the form given in (12) with

$$F_0(\tau) = \frac{\bar{a}}{(\gamma + \theta) \sigma_S^2} - \frac{(\gamma - 1) \sigma_x \rho_{xS}}{(\gamma + \theta) \sigma_S} A_1(\tau)$$

$$F_1(\tau) = \frac{b_x}{(\gamma + \theta) \sigma_S^2} - \frac{(\gamma - 1) \sigma_x \rho_{xS}}{(\gamma + \theta) \sigma_S} B_1(\tau)$$

$$F_2(\tau) = 0.$$

The loss can then be determined by Equation (14).

Figure 3 gives the loss from ignoring the possibility to learn as a function of the investment horizon. As expected, the (non-annualized) loss increases exponentially in the investment horizon. Furthermore, the losses are the smaller the larger the ambiguity aversion of the investor. To get the intuition, note that an increase in the ambiguity aversion makes the investor more conservative. With a lower portfolio weight of the stock, however, the failure to use the best estimate available for the expected return matters less than it does for $\theta = 0$.

[Figure 3 about here.]

Figure 4 shows the dependence of the loss from refusing to learn on the value of $b_y$, i.e. the average percentage of the stochastic part of the equity risk premium explained by $y$. If $b_y$ is equal to zero, the loss from ignoring learning about $y$ is of course zero, too. It then increases exponentially in $b_y$, and reaches its largest value when the stochastic component of the expected return is due to variation in $y$ only.
Overall, the utility losses due to ignoring learning are rather small. For our setup, they are well below 4% of the initial portfolio value even for an investment horizon of 20 years and a value of $b_y = 1$. While the investor thus profits from learning, the consequences of a suboptimal strategy are by no means devastating. Furthermore, ambiguity aversion dampens the losses and thus attenuates the consequences of refusing to learn.

5.4 Suboptimal strategies: ignoring ambiguity

Now consider an investor who ignores model uncertainty. He invests as if he were ambiguity-neutral and follows the investment strategy given in Proposition 1 with $\theta = 0$. The strategy is affine in the state variables and can be written in the form given in (12) with

\[
F_0(\tau) = \frac{a}{\gamma \sigma_S^2} - \frac{(\gamma - 1) \sigma_x \rho x S}{\gamma \sigma_S} A_1(\tau) - \frac{\gamma - 1}{\gamma \sigma_S} (K_S \sigma_S + K_x \sigma_x S) A_2(\tau)
\]
\[
F_1(\tau) = \frac{b_x}{\gamma \sigma_S^2} - \frac{(\gamma - 1) \sigma_x \rho x S}{\gamma \sigma_S} B_1(\tau) - \frac{\gamma - 1}{\gamma \sigma_S} (K_S \sigma_S + K_x \sigma_x S) B_3(\tau)
\]
\[
F_2(\tau) = \frac{b_y}{\gamma \sigma_S^2} - \frac{(\gamma - 1) \sigma_x \rho x S}{\gamma \sigma_S} B_3(\tau) - \frac{\gamma - 1}{\gamma \sigma_S} (K_S \sigma_S + K_x \sigma_x S) B_2(\tau).
\]

The loss from ignoring model uncertainty can again be determined by Equation (14).

Figure 5 displays the loss from ignoring ambiguity. The loss is again increasing in the investment horizon, but now at a rate that is less than linear. Obviously, the loss is larger the greater level of model uncertainty which is ignored by the investor. If the investor took the model uncertainty into account he would follow a much more conservative strategy. It follows from the figure that incorrectly relying on a strategy with $\theta = 0$ leads to huge utility losses. For example with a level of model uncertainty of $\theta = 3$, the investor would be willing to give up close to 20% of his initial wealth to know the robust investment strategy. The loss the investor suffers from ignoring model uncertainty is thus much larger than the losses from refusing to learn.

In Figure 6 we display the expected loss as a function of $\theta$ for four different levels of $\gamma$. In line with Figure 5 we see that the loss intuitively increases in the level of model uncertainty.
uncertainty. Furthermore, the loss is larger the smaller the risk aversion. An investor with a higher risk aversion invests less in the stock and thus suffers less from ignoring the model uncertainty about the expected return on the stock. For $\gamma = 2$ and $\theta > 5$, the expected loss exceeds 50% of the initial portfolio value.

The losses from ignoring model uncertainty are thus highly economically significant. They far exceed the utility losses from refusing to learn. It is thus much more important to take ambiguity into account than to learn about the true value of the unobservable predictor. Furthermore, ambiguity aversion lowers the utility losses from not learning even further. In our setup, ambiguity aversion is thus of first order importance. An investor who does not adjust his strategy to ambiguity aversion suffers much larger losses than an investor who ignores to learn about the true predictor.

5.5 Robustness

As mentioned in Section 5.1 we base our parameter estimates on Brennan and Xia (2010). They consider nine scenarios, combining three different values of mean reversion and the correlation between the state variable and the expected rate of return, respectively. We have used one of these scenarios as our benchmark case, but will now consider other scenarios to check for robustness of our results.

In particular, the correlation of $-0.5$ between the observable state variable and the rate of return of the stock seems a bit low in magnitude. As mentioned above Xia (2001) and Barberis (2000) report a correlation of approximately -0.90 between monthly innovations in the dividend yield and stock returns. In Figure 7 we display the expected loss from ignoring either learning or model uncertainty as a function of the correlation between the stock return and the observable state variable, $\rho_{xS}$. The loss is displayed for four different values of the investor’s ambiguity aversion, $\theta$. We only consider negative values of the correlation since we use the parameter estimates from Brennan and Xia (2010). They rely on the dividend yield as the predictive state variable, and hence a negative correlation is expected. In Panel A the loss from ignoring learning is illustrated, whereas the loss from ignoring model uncertainty is illustrated in Panel B.
Panel A shows that the loss from ignoring learning increases in the absolute size of the correlation between the observable state variable and expected rate of return. As the correlation between the observable variable $x$ and the stock returns increases so does the correlation between the filtered state variable $\hat{y}$ and the stock returns, and hence the investor gains more from taking learning into account. The increase in the gains from learning is most pronounced for ambiguity-averse investors. In the benchmark case with $\rho_{xS} = -0.5$, an ambiguity-neutral investor suffers a loss of approximately 0.55%, while ambiguity-averse investors suffer smaller losses. For example an ambiguity-averse investor with $\theta = 3$ suffers a loss of 0.40%. In contrast, for $\rho_{xS} = -0.9$, the loss is approximately 1.01% for an ambiguity-neutral investor, but higher for ambiguity-averse investors. For example for the ambiguity-averse investor with $\theta = 3$ the loss increases to 2.18%. With a correlation of $-0.9$, the loss is still modest, but when the correlation gets very close to $-1$, the loss increases considerably for ambiguity-averse investors. For example, for a correlation of $\rho_{xS} = -0.99$, an investor with an ambiguity aversion of $\theta = 3$ suffers a loss of approximately 30% when ignoring learning.

Panel B of Figure 7 displays the loss the investor suffers from ignoring model uncertainty as a function of the correlation between the stock returns and the observable state variable. The loss is displayed for four different levels of ambiguity aversion, $\theta \in \{0, 3, 5, 8\}$. Of course, for $\theta = 0$, the investor does not care about model uncertainty, so the loss is zero for all levels of $\rho_{xS}$. For $\theta > 0$, the loss increases slightly in the absolute value of the correlation up to some point close to a correlation of $-0.8$ after which the loss decreases. To get the intuition, note that there are two effects. First, as the correlation goes to minus one, both the optimal and the suboptimal portfolio weight of the stock increases, since the investor can learn better about the unknown drift. With the investor being more aggressive, the difference between the two portfolio weights increases, too. This effect in turn implies that the loss from following the suboptimal strategy should also increase. Second, recall that the predictive power of the observable and unobservable state variable is given by the constants $b_x$ and $b_y$, and that in the benchmark case we have $b_x = b_y = 0.5$. Therefore, the investor also learns about the unobservable predictor $y$ from observing the stock return and the known predictor $x$. With a larger absolute correlation between stock returns and $x$, the quality of his information increases. Hence, the decrease in the loss from ignoring ambiguity as $\rho_{xS}$ approaches $-1$ occurs because the gain from learning (which
the investor does take into account) increases in the absolute value of the correlation. In our benchmark case with \( \rho_{xS} = -0.5 \) the investor suffers a loss of 8.45% if \( \theta = 3 \). Hence, the loss from ignoring ambiguity is significantly bigger than the loss the investor suffers from refusing to learn about the true value of the unobservable predictor. This is still the case if we assume a correlation of \( \rho_{xS} = -0.9 \), where the loss from refusing to learn equals 2.18%, whereas the loss from ignoring ambiguity equals 11.43%. However, as \( \rho_{xS} \) approaches \(-1\), the gain from learning about the unobservable predictor will eventually exceed the gain from taking model uncertainty into account. For example, for a correlation of \( \rho_{xS} = -0.99 \) the investor suffers a loss of approximately 30% from refusing to learn while he suffers a loss of approximately 7% from ignoring model uncertainty in the case where \( \theta = 3 \). Intuitively, the loss increases in the level of model uncertainty, and we find the same overall correlation dependence for all levels of \( \theta > 0 \).

Figure 8 illustrates the importance of the size of the speed of mean reversion in the two predictors. Panel A shows the loss from refusing to learn about the value of the unobservable state variable, and Panel B depicts the loss from ignoring model uncertainty. The loss is again displayed for four different levels of ambiguity aversion, \( \theta \in \{0, 3, 5, 8\} \). We increase the volatility of the predictors together with the speed of mean reversion so that the variance of the stationary distribution of the expected rate of return on the stock remains unchanged. As the speed of mean reversion, \( \kappa = \kappa_x = \kappa_y \), increases, the predictors become more stable over time and, hence, we would expect to see a decrease in the gain an investor can earn from learning about the unobservable predictor. However, the simultaneous increase in the volatility of the predictors leads to more volatile predictors and one would expect an increase in the gain one can earn from learning about the predictors. The two effects more or less offset each other so that the loss from ignoring learning is relatively insensitive to the simultaneous changes in the speed of mean reversion and the volatility.

6In the special case with \( b_x = 1 \) and \( b_y = 0 \) the loss increases in the absolute size of the correlation for all values of \( \rho_{xS} \).

7Following Breman and Xia (2010) we assume a volatility of 4% of the stationary distribution of \( \mu \), i.e. \( 0.04 = \sigma_i / \sqrt{2\kappa_i} \) for \( i \in \{x, y\} \).
Panel B of Figure 8 shows that the loss from ignoring model uncertainty increases slightly in the speed of mean reversion. An increase in $\kappa$ makes the process more stable and the difference between the optimal and suboptimal investment strategy therefore increases, which suggests bigger losses. The simultaneous increase in the predictor volatility will in itself induce a smaller difference between the optimal and suboptimal strategy smaller and thus a decrease in the loss, but this effect turns out to be quantitatively smaller than the effect of the increase in the speed of mean reversion.

6 Conclusion

In this paper, we have studied the optimal portfolio choice of an investor who faces uncertainty about the predictable expected rate of return on the stock. There are two predictors for stock returns one of which is unobservable and thus has to be inferred from observed stock returns and the other predictor. Furthermore, the investor recognizes that even the filtered model, which is the best description of the data-generating process given his current information, might not be the correct one. He thus takes ambiguity into account.

First, we find that both learning and ambiguity aversion have an impact on the level and structure of the optimal demand for the stock. Learning about the unobservable predictor induces some additional hedging demand, and the same is true for ambiguity.

Second, we find that suboptimal strategies resulting from either not learning or from not considering ambiguity can lead to economically significant losses. The losses from refusing to learn are in our setup rather modest. Ambiguity aversion reduces them even further since it makes the investor more conservative. The losses from not taking ambiguity aversion into account, however, can exceed more than 50% of the initial wealth over an investment horizon of 20 years. It is thus of first order importance to take ambiguity aversion into account, while learning is of second order importance only.
A  Filtering

To get the same notation as Liptser and Shiryaev (2001) section 12.3 we rewrite the dynamics of the stock price and the state variables. The dynamics of the observable variables, that is the stock price and the state variable $x$, are written as

$$
\begin{align*}
\left( \frac{dS_t}{S_t} \right) &= \left[ \left( r + a + b_x x_t \right) + \left( b_y \right) y_t \right] dt + \left( 0 \right) dy_t,

&+ \left( \frac{\sigma_S}{\sigma_x \rho_{xS}} \frac{0}{\sigma_x \sqrt{1 - \rho_{xS}^2}} \right) \left( dz_{S,t} \right).
\end{align*}
$$

The dynamics of the unobservable state variable, $y$, is

$$
dy_t = (\kappa_y \mu_y - \kappa_y \hat{y}_t) dt + \sigma_y \rho_2 dz_{y,t} + \left( \begin{array}{c} \sigma_y \rho_{yS} \\ \sigma_y \rho_1 \end{array} \right) \left( \begin{array}{c} dz_{S,t} \\ dz_{x,t} \end{array} \right),
$$

where $\rho_1$ and $\rho_2$ are chosen such to get the prespecified correlations between the state variable $y$ and the observable variables. It now follows from Theorem 12.7 that

$$
d\hat{y}_t = \kappa_y (\mu_y - y_t) dt + K_S \sigma_{yS} dz_{S,t} + K_x \sigma_x dz_{x,t},
$$

where $K_S$ and $K_x$ are given in (1) and (2). The variance of the estimation error in the steady state is given by

$$
\eta = \sqrt{\kappa_y^2 + 2 \kappa_y b_y \frac{(\rho_{yS} - \rho_{xS} \rho_{xy}) \sigma_y}{(1 - \rho_{xS}^2) \sigma_S} + b_y^2 \frac{(1 - \rho_{xS}^2) \sigma_y^2}{(1 - \rho_{xS}^2) \sigma_S}} - \kappa_y - b_y \frac{(\rho_{yS} - \rho_{xS} \rho_{xy}) \sigma_y}{(1 - \rho_{xS}^2) \sigma_S}.
$$

B  Optimal strategies

Substituting $u^*$ from (7) into the HJB-equation (6) implies that

$$
0 = \sup_{\alpha} \left\{ V_t + V_t W \left( r + \alpha \left( a + b_x x + b_y y - \Psi \left( V_t W \alpha + V_y K_S \right) \sigma_{S}^2 \right) \right) \right. $$

$$
+ \frac{1}{2} V_{yy} W^2 \sigma_S^2 \sigma_{xy}^2 + \frac{1}{2} V_{yy} \left( K_S^2 \sigma_S^2 + K_x^2 \sigma_x^2 + 2 K_S K_x \sigma_S \sigma_x \rho_{xy} \right) \right. $$

$$
+ \frac{1}{2} V_{yy} W \sigma_S \sigma_{xy} + V_{yy} W \alpha \left( K_S \sigma_S^2 + K_x \sigma_x \sigma_{xy} \right) $$

$$
+ V_{yy} \left( K_S \sigma_S \sigma_{xy} \rho_{xy} + K_x \sigma_x^2 \right) + \frac{\Psi}{2} \left( V_t W \alpha + V_y K_S \right)^2 \sigma_S^2 \right\}.
$$

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The first order condition with respect to $\alpha$ implies the following candidate for the optimal robust investment strategy

$$
\alpha^* = -\frac{V_W(a + b_x x + b_y \hat{y})}{(V_W - V^2 W) W \sigma_S^2} + \frac{\Psi V_W V_y K_S}{(V_W - V^2 W) W} - \frac{V_{Wx} \sigma_x \rho_x S}{(V_W - V^2 W) W \sigma_S} - \frac{V_{Wy} (K_S \sigma_S + K_x \sigma_x \rho_x S)}{(V_W - V^2 W) W \sigma_S}.
$$

(18)

Substituting the candidate for the optimal investment strategy into the HJB-equation yields a partial differential equation (PDE). If this PDE has a solution, $V(W, x, \hat{y}, t)$, such that the strategy $\alpha^*$ is well-defined, it follows from a verification theorem that the strategy is optimal and that the function $V(W, x, \hat{y}, t)$ equals the indirect utility function. As in a standard setting with power utility and no ambiguity, we get with the specification (5) that the optimal strategy becomes independent of the current level of wealth. Now make the following conjecture about the solution to the HJB-equation

$$
V(W, x, \hat{y}, t) = \frac{W^{1-\gamma}}{1-\gamma} g(t, x, \hat{y}).
$$

Substituting the relevant derivatives of $V$ into the candidate for the optimal investment strategy as well as our candidate for the worst-case measure yields

$$
\alpha^* = \frac{a + b_x x + b_y \hat{y}}{(\gamma + \theta) \sigma_S^2} - \frac{1}{\gamma + \theta} \left( \frac{\theta K_S}{1-\gamma} - \frac{K_S \sigma_S + K_x \sigma_x \rho_x S}{\sigma_S} \right) \frac{g_y}{g} + \frac{\sigma_x \rho_x S}{(\gamma + \theta) \sigma_S} \frac{g_x}{g}.
$$

(19)

$$
u^* = \theta \sigma_S \alpha + \frac{\theta \sigma_S}{1-\gamma} \frac{g_y K_S}{g}.
$$

(20)

Plugging these as well as the relevant derivatives of our guess of $V$ into (17) it turns out that our conjecture does solve the HJB-equation if the function $g$ solves the partial differential equation

$$
0 = \left(1 - \gamma \right) r + \frac{1-\gamma}{2(\gamma + \theta) \sigma_S^2} (a + b_x x + b_y \hat{y})^2 g + g_t
$$

$$
+ \left[ \kappa_x (\mu_x - x) + \frac{1-\gamma}{(\gamma + \theta) \sigma_S} (a + b_x x + b_y \hat{y}) \sigma_x \rho_x S \right] g_x
$$

$$
+ \left[ \kappa_y (\mu_y - \hat{y}) + c_1 (a + b_x x + b_y \hat{y}) \right] g_y
$$

$$
+ \frac{1}{2} g_{xx} \sigma_x^2 + \frac{1}{2} g_{yy} c_2 + g_{xy} c_3 + \frac{1-\gamma}{2(\gamma + \theta) \sigma_x^2 \rho_x S} \frac{g_x^2}{g} + \frac{\sigma_S^2}{2} c_4 \frac{g_y^2}{g} + c_5 g_x g_y,
$$

\footnote{See for example Theorem 11.2.2 in Øksendal (2000).}
where we have introduced the auxiliary constants

\[ c_1 = \left(\frac{1}{\gamma + \theta} - 1\right) K_S + \frac{1 - \gamma}{\gamma + \theta} \frac{\rho_0}{\sigma_S} K_x \]  
(21)

\[ = -\frac{1 - \gamma}{\gamma + \theta} \left(\frac{\theta K_S}{1 - \gamma} - \frac{K_S \sigma_S + K_x \sigma_x \rho_0}{\sigma_S}\right) \]  
(22)

\[ c_2 = K_S^2 \sigma_S^2 + K_x^2 \sigma_x^2 + 2K_S K_x \sigma_x \sigma_0 \rho_x S \]  
(23)

\[ c_3 = K_S \sigma_x \sigma_0 \rho_0 \rho_x S + K_x^2 \sigma_x^2 \]  
(24)

\[ c_4 = \frac{1 - \gamma}{\gamma + \theta} \left(\frac{\theta K_S}{1 - \gamma} - \frac{K_S \sigma_S + K_x \sigma_x \rho_0}{\sigma_S}\right)^2 - \frac{\theta}{1 - \gamma} K_S^2 \]  
(25)

\[ c_5 = \left(\frac{1}{\gamma + \theta} - 1\right) \sigma_S \sigma_x \sigma_0 \rho_0 \rho_x S K_x + \frac{1 - \gamma}{\gamma + \theta} \frac{\sigma_x^2 \rho_0^2 S}{\sigma_S} K_x \]  
(26)

A qualified guess of a solution \( g(x, \hat{y}, t) \) to the above PDE is given by [0]. It turns out that our guess does solve the PDE if the functions \( A_0(\cdot), A_1(\cdot), A_2(\cdot), B_1(\cdot), B_2(\cdot), \) and \( B_3(\cdot) \) solve the following system of ordinary differential equations:

\[
A_0'(\tau) = \left(\kappa_x \mu_x + \frac{1 - \gamma}{(\gamma + \theta) \sigma_S} a \sigma_x \rho_x S \right) A_1(\tau) + (\kappa_y \mu_y + c_1 a) A_2(\tau) \\
+ \frac{1 - \gamma}{2 \sigma_S^2} B_1(\tau) + \frac{1}{2} c_2 B_2(\tau) + c_3 B_3(\tau) \\
+ \left(\frac{1 - \gamma}{(\gamma + \theta) \sigma_S} a \sigma_x \rho_x S + \kappa_x \mu_x \right) B_1(\tau) + (\kappa_y \mu_y + a c_1) B_3(\tau) \\
+ \left(1 - \gamma \right) \left(\frac{1 - \gamma}{\gamma + \theta} \right) \sigma_x^2 A_1(\tau) B_1(\tau) + (1 - \gamma) \left(c_2 + c_4 \sigma_S^2\right) A_2(\tau) B_3(\tau) \\
+ (1 - \gamma) \left(c_3 + c_5\right) (A_1(\tau) B_3(\tau) + A_2(\tau) B_1(\tau)) + \frac{a b_y}{(\gamma + \theta) \sigma_S} \]  
(27)

\[
A_1'(\tau) = \left(\frac{1 - \gamma}{(\gamma + \theta) \sigma_S} b_x \sigma_x \rho_x S - \kappa_x \right) A_1(\tau) + c_1 b_x A_2(\tau) \\
+ \left(\frac{1 - \gamma}{(\gamma + \theta) \sigma_S} b_x \sigma_x \rho_x S + \kappa_x \mu_x \right) B_1(\tau) + (\kappa_y \mu_y + a c_1) B_3(\tau) \\
+ \left(1 - \gamma \right) \left(\frac{1 - \gamma}{\gamma + \theta} \right) \sigma_x^2 A_1(\tau) B_1(\tau) + (1 - \gamma) \left(c_2 + c_4 \sigma_S^2\right) A_2(\tau) B_3(\tau) \\
+ (1 - \gamma) \left(c_3 + c_5\right) (A_1(\tau) B_3(\tau) + A_2(\tau) B_1(\tau)) + \frac{a b_y}{(\gamma + \theta) \sigma_S} \]  
(28)

\[
A_2'(\tau) = \frac{1 - \gamma}{(\gamma + \theta) \sigma_S} b_y \sigma_x \rho_x S A_1(\tau) - (\kappa_y - b_y c_1) A_2(\tau) \\
+ (\kappa_y \mu_y + c_1 a) B_2(\tau) + \left(\frac{1 - \gamma}{(\gamma + \theta) \sigma_S} a \sigma_x \rho_x S + \kappa_x \mu_x \right) B_3(\tau) \\
+ \left(1 - \gamma \right) \left(\frac{1 - \gamma}{\gamma + \theta} \right) \sigma_x^2 A_1(\tau) B_3(\tau) + (1 - \gamma) \left(c_2 + c_4 \sigma_S^2\right) A_2(\tau) B_2(\tau) \\
+ (1 - \gamma) \left(c_3 + c_5\right) (A_1(\tau) B_2(\tau) + A_2(\tau) B_3(\tau)) + \frac{a b_y}{(\gamma + \theta) \sigma_S} \]  
(29)
Conjecture that the PDE (6) without the supremum over $\alpha$

After substitution into the PDE for $C$ Suboptimal strategies by the investor follows from (20).

The optimal investment strategy follows from (19), and the worst-case measure accepted by the investor follows from (20).

C Suboptimal strategies

Here is a sketch of the proof of Proposition 2. For a general $\Psi$, $V^\alpha(W, x, \hat{y}, t)$ satisfies the PDE (6) without the supremum over $\alpha$. Analogous to (7), the optimal $u$ is then

$$u^* = \Psi \left( V_W^\alpha W \alpha + V_y^\alpha K_S \right) \sigma_S = \frac{\theta}{1 - \gamma} \sigma_S \left( \alpha \hat{\Psi} + \frac{\alpha}{V^\alpha} \frac{V_W^\alpha}{V^\alpha} W \alpha + \frac{V_y^\alpha}{V^\alpha} K_S \right).$$

Conjecture that

$$V^\alpha(W, x, \hat{y}, t) = \frac{W^{1-\gamma}}{1 - \gamma} g^\alpha(t, x, \hat{y}).$$

After substitution into the PDE for $V^\alpha$, we see that $g^\alpha$ has to satisfy the PDE

$$0 = g_x^\alpha + (1 - \gamma)g^\alpha \left( r + [a + b_x x + b_y \hat{y}]\alpha - \frac{\gamma + \theta}{2} \sigma_x^2 \right)$$

$$+ g_x^\alpha (\kappa_x \mu_x - x) - (\gamma - 1)\sigma_S \sigma_x \rho_{x \alpha}$$

$$+ g_y^\alpha (\kappa_y \mu_y - \hat{y}) - K_S \alpha S^2 (\theta + \gamma - 1)\alpha - (\gamma - 1)K_x \sigma_x \sigma_S \rho_{x \alpha}$$

$$+ \frac{(g_y^\alpha)^2}{g^\alpha} \frac{\theta}{2(\gamma - 1)} \sigma_y^2 K_S^2 + c_3 g_{xy}^\alpha + \frac{1}{2} \sigma_x^2 g_y^\alpha + \frac{1}{2} c_2 g_{yy}^\alpha.$$
where \( c_2 \) and \( c_3 \) are given in (23) and (24). Now substitute in both the affine form of \( \alpha \) from (12) and the conjectured form of \( g^\alpha \) from (13) and divide by \( g^\alpha \). Then the right-hand side of the above equation involves terms with \( x \), \( x^2 \), \( y \), \( y^2 \), \( x \dot{y} \), as well as terms without \( x \) and \( \dot{y} \). As the equation has to be satisfied for all values of \( x \) and \( \dot{y} \), each of the six groups of terms has to equal zero. This leads to the following system of ODEs:

\[
\begin{align*}
(A_0^0)'(\tau) &= (\kappa_x \mu_x - (\gamma - 1)\sigma_S \sigma_x \rho_x F_0(\tau)) A_1^0(\tau) + (\kappa_y \mu_y - c_6 F_0(\tau)) A_2^0(\tau) \\
&\quad + \frac{1}{2} \sigma_x^2 B_1^0(\tau) + \frac{1}{2} c_2 B_2^0(\tau) + c_3 B_3^0(\tau) \\
&\quad - \frac{\gamma - 1}{2} \sigma_x^2 A_1^0(\tau)^2 - \frac{1}{2} ((\gamma - 1)c_2 + \theta \sigma_S^2 K_3^0) A_2^0(\tau)^2 \\
&\quad - (\gamma - 1) c_3 A_2^0(\tau) A_3^0(\tau) + r + a F_0(\tau) - \frac{1}{2} \sigma_S^2 (\gamma + \theta) F_0(\tau)^2 \\
(A_1^0)'(\tau) &= - (\kappa_x + (\gamma - 1)\sigma_S \sigma_x \rho_x F_1(\tau)) A_1^0(\tau) - c_6 F_1(\tau) A_2^0(\tau) \\
&\quad + (\kappa_x \mu_x - (\gamma - 1)\sigma_S \sigma_x \rho_x F_0(\tau)) B_1^0(\tau) + (\kappa_y \mu_y - c_6 F_0(\tau)) B_2^0(\tau) \\
&\quad - (\gamma - 1) \sigma_x^2 A_1^0(\tau) B_1^0(\tau) - ((\gamma - 1)c_2 + \theta K_3^0 \sigma_S^2) A_2^0(\tau) B_2^0(\tau) \\
&\quad - (\gamma - 1) c_3 (A_1^0(\tau) B_2^0(\tau) + A_2^0(\tau) B_1^0(\tau)) \\
&\quad + b_x F_0(\tau) + a F_1(\tau) - \sigma_S^2 (\theta + \gamma) F_0(\tau) F_1(\tau) \\
(A_2^0)'(\tau) &= -(\gamma - 1)\sigma_S \sigma_x \rho_x F_2(\tau) A_1^0(\tau) - (\kappa_y + c_6 F_2(\tau)) A_2^0(\tau) \\
&\quad + (\kappa_y \mu_y - c_6 F_0(\tau)) B_2^0(\tau) + (\kappa_x \mu_x - (\gamma - 1)\sigma_S \sigma_x \rho_x F_0(\tau)) B_3^0(\tau) \\
&\quad - (\gamma - 1) \sigma_x^2 A_1^0(\tau) B_3^0(\tau) - ((\gamma - 1)c_2 + \theta K_3^0 \sigma_S^2) A_2^0(\tau) B_3^0(\tau) \\
&\quad - (\gamma - 1) c_3 (A_1^0(\tau) B_3^0(\tau) + A_2^0(\tau) B_2^0(\tau)) \\
&\quad + b_y F_0(\tau) + a F_2(\tau) - \sigma_S^2 (\theta + \gamma) F_0(\tau) F_2(\tau) \\
(B_1^0)'(\tau) &= -2 (\kappa_x + (\gamma - 1)\sigma_S \sigma_x \rho_x F_1(\tau)) B_1^0(\tau) - 2 c_6 F_1(\tau) B_2^0(\tau) \\
&\quad - (\gamma - 1) \sigma_x^2 B_1^0(\tau)^2 - ((\gamma - 1)c_2 + \theta \sigma_S^2 K_3^0) B_2^0(\tau)^2 \\
&\quad - 2 (\gamma - 1) c_3 B_1(\tau) B_3(\tau) + 2 b_x F_1(\tau) - (\gamma + \theta) \sigma_S^2 F_1(\tau)^2 \\
(B_2^0)'(\tau) &= -2 (\kappa_y + c_6 F_2(\tau)) B_2^0(\tau) - 2 (\gamma - 1) \sigma_S \sigma_x \rho_x F_2(\tau) B_3^0(\tau) \\
&\quad - ((\gamma - 1)c_2 + \theta \sigma_S^2 K_3^0) B_3^0(\tau)^2 - (\gamma - 1) \sigma_x^2 B_2^0(\tau)^2 \\
&\quad - 2 (\gamma - 1) c_3 B_2^0(\tau) B_3^0(\tau) + 2 b_y F_2(\tau) - (\gamma + \theta) \sigma_S^2 F_2(\tau)^2
\end{align*}
\]
\begin{equation}
\begin{aligned}
(B_3^\alpha)'(\tau) &= - (\gamma - 1) \sigma_S \sigma_x \rho_{xS} (F_1(\tau)B_3^\alpha(\tau) + F_2(\tau)B_1^\alpha(\tau)) - c_6 \left( F_1(\tau)B_2^\alpha(\tau) + F_2(\tau)B_3^\alpha(\tau) \right) \\
&\quad - (\kappa_x + \kappa_y) B_3^\alpha(\tau) - (\gamma - 1) \sigma_x^2 B_1^\alpha(\tau)B_3^\alpha(\tau) \\
&\quad - \left( (\gamma - 1)c_2 + \theta K_S^2 \sigma_S^2 \right) B_2^\alpha(\tau)B_3^\alpha(\tau) - (\gamma - 1) c_3 \left( B_1^\alpha(\tau)B_2^\alpha(\tau) + B_3^\alpha(\tau)^2 \right) \\
&\quad + b_y F_1(\tau) + b_x F_2(\tau) - (\gamma + \theta) \sigma_S^2 F_1(\tau)F_2(\tau),
\end{aligned}
\end{equation}

and we have introduced the additional auxiliary constant

\begin{equation}
c_6 = (\theta + \gamma - 1) K_S \sigma_S^2 + (\gamma - 1) K_x \sigma_x \sigma_S \rho_{xS}.
\end{equation}

The terminal condition \( V^\alpha(W, x, \hat{y}, T) = W^{1-\gamma}/(1 - \gamma) \) implies that \( A_0^\alpha(0) = A_1^\alpha(0) = A_2^\alpha(0) = B_1^\alpha(0) = B_2^\alpha(0) = B_3^\alpha(0) = 0. \)
References


Figure 1: Optimal investment strategy and the optimal worst-case drift distortion as a function of the investor’s investment horizon. In Panels A, C, and E the solid black line displays the total investment in the stock, whereas the dashed black line displays the investment in the bank account, i.e. $1 - \alpha$. The colored lines display the allocations in the four funds. The red line displays $\alpha_{spec}$, the green line displays $\alpha_{hdg,x}$, the blue line displays $\alpha_{hdg,y}$, and the purple line displays $\alpha_{amb}$. Panels B, D, and F display the worst case drift distortion, $u^*$, by the red line, whereas the corrected excess return is displayed by the green line. Panels A and B display our benchmark case. In Panels C and D we have assumed that $b_x = 0$ and $b_y = 1$, and in Panels E and F we have assumed that $b_x = 1$ and $b_y = 0$. The investor is assumed to have a risk aversion of $\gamma = 4$ and an ambiguity aversion of $\theta = 3$. 
Figure 2: Optimal investment strategy and the optimal worst-case drift distortion as a function of the investor’s ambiguity aversion. In Panels A, C, and E the solid black line displays the total investment in the stock, while the dashed black line displays the investment in the bank account, i.e. \(1 - \alpha\). The colored lines display the allocations in the four funds. The red line displays \(\alpha_{\text{spec}}\), the green line displays \(\alpha_{\text{hdg},x}\), the blue line displays \(\alpha_{\text{hdg},y}\), and the purple line displays \(\alpha_{\text{amb}}\). Panels B, D, and F show the worst case drift distortion, \(u^*\), by the red line, whereas the corrected excess return is shown by the green line. Panels A and B correspond to our benchmark case. In Panels C and D we have assumed that \(b_x = 0\) and \(b_y = 1\), and in Panels E and F we have assumed that \(b_x = 1\) and \(b_y = 0\). The investor is assumed to have a risk aversion of \(\gamma = 4\) and an investment horizon \(T - t = 10\).
Figure 3: The expected loss from ignoring learning as a function of the investor’s investment horizon, $T - t$. The loss is displayed for four different levels of model uncertainty, $\theta$. The investor is assumed to have a risk aversion of $\gamma = 4$. 
Figure 4: The expected loss from ignoring learning as a function of the predictive constant $b_y$. The loss is displayed for four different levels of model uncertainty, $\theta$. The investor is assumed to have a risk aversion of $\gamma = 4$ and an investment horizon of $T - t = 10$. 
Figure 5: The expected loss from ignoring model uncertainty as a function of the investor’s investment horizon, $T - t$. The loss is displayed for four different levels of model uncertainty, $\theta$. The investor is assumed to have a risk aversion of $\gamma = 4$. 
Figure 6: The expected loss from ignoring model uncertainty as a function of the level of model uncertainty, $\theta$. The loss is displayed for four different values of the investor’s risk aversion, $\gamma$. The investor is assumed to have an investment horizon of $T - t = 10$ years.
Figure 7: The expected loss from ignoring either learning or model uncertainty as a function of the correlation, $\rho_{xS}$. The loss is displayed for four different values of the investor’s ambiguity aversion, $\theta$. Panel A displays the loss from ignoring learning, while Panel B displays the loss from ignoring model uncertainty. The investor is assumed to have an investment horizon of $T - t = 10$ years and a risk aversion of $\gamma = 4$. 
Figure 8: The expected loss from ignoring either learning or model uncertainty as a function of the speed of mean reversion, $\kappa = \kappa_x = \kappa_y$. The loss is displayed for four different values of the investor’s ambiguity aversion, $\theta$. Panel A displays the loss from ignoring learning, while Panel B displays the loss from ignoring model uncertainty. The investor is assumed to have an investment horizon of $T - t = 10$ years and a risk aversion of $\gamma = 4$. 