Generalized Forecast Error Variance Decomposition for Linear and Nonlinear Multivariate Models

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Generalized Forecast Error Variance Decomposition for Linear and Nonlinear Multivariate Models†

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Abstract

We propose a new generalized forecast error variance decomposition with the property that the proportions of the impact accounted for by innovations in each variable sum to unity. Our decomposition is based on the well-established concept of the generalized impulse response function. The use of the new decomposition is illustrated with an empirical application to U.S. output growth and interest rate spread data.

Keywords: Forecast error variance decomposition, generalized impulse response function, output growth, term spread

JEL codes: C13, C32, C53

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1 Introduction

Impulse response and forecast error variance decomposition analysis are the prominent tools in interpreting estimated linear and nonlinear multivariate time series models. To facilitate such analyses in a reduced-form linear vector autoregressive (VAR) model, identifying restrictions are typically imposed to obtain a structural model with economically meaningful uncorrelated shocks. However, when it is difficult to find credible identification restrictions, so-called generalized impulse response functions (GIRF) and generalized forecast error variance decompositions (GFEVD) are analyzed instead. With nonlinear models, this is virtually always the case.

The main difference between the impulse response function (IRF) and forecast error variance decomposition (FEVD) and their generalized counterparts is the interpretation of the shocks: in the former case, they are uncorrelated and carry an economic meaning, while in the latter case, each of them is just a shock to a given equation of the model. Moreover, because the shocks are not necessarily uncorrelated in the generalized case, the interpretation of the GFEVD as the proportions of the impact accounted for by innovations in each of the variables of the total impact of all innovations after h periods ($h = 0, 1, 2, \ldots$) is somewhat nebulous, as these 'proportions' may not sum to unity.

Our contributions are twofold. First, we propose a simple modification of the GFEVD in linear multivariate models due to Pesaran and Shin (1998) that, by construction, yields the relative contributions to the h-period impacts of the shocks summing to unity, and hence, facilitating convenient interpretation. Second, we generalize this modification to obtain a GFEVD in nonlinear models that, to the best of our knowledge, has not been entertained in the previous literature. Overall, impulse response analysis in nonlinear models has not been frequently considered in the previous literature albeit it has recently awoken increased interest (see, e.g., Karamé, 2012, and Hubrich and Teräsvirta, 2013, 313–315), who discuss GIRFs in Markov-switching and threshold and smooth transition vector autoregressive models, respectively).

The paper is organized as follows. The new GFEVD and its relation to the orthogonalized FEVD and GIRF are introduced in Section 2. In Section 3, we illustrate the GFEVD in an empirical application to U.S. output growth and the spread between long-term and short-term interest rates. Section 4 concludes.

2 A New Generalized FEVD

Let us start out by considering a K-dimensional VAR(p) model

$$\mathbf{y}_t = \sum_{i=1}^{p} \mathbf{\Phi}_i \mathbf{y}_{t-i} + \boldsymbol{\varepsilon}_t \quad (1)$$

where $\boldsymbol{\varepsilon}_t$ is an independent and identically distributed (iid) error term with zero mean and covariance matrix $\mathbf{\Sigma}$. Assuming weak stationarity, $\mathbf{y}_t$ obtains the infinite-order moving-average representation

$$\mathbf{y}_t = \sum_{j=0}^{\infty} \mathbf{A}_j \boldsymbol{\varepsilon}_{t-j}, \quad (2)$$

and if suitable identification restrictions are available such that $\mathbf{\Sigma}$ can be written as $\mathbf{PP}'$, $\boldsymbol{\xi}_t = \mathbf{P}^{-1}\boldsymbol{\varepsilon}_t$ is the orthogonalized error with identity covariance matrix. The orthogonalized impulse response function on $y_{j,t+l}$ of a unit shock to the ith equation is then (see, e.g., Lütkepohl, 2005, Section 2.3)

$$IRF_{ij}(l) = \frac{\partial y_{j,t+l}}{\partial \xi_{it}} = [\mathbf{A}_l \mathbf{P}]_{ji}, \quad l = 0, 1, 2, \ldots, \quad (3)$$

and the corresponding FEVD component for horizon h equals

$$\gamma_{ij}(h) = \frac{\sum_{l=0}^{h} IRF_{ij}^2(l)}{\sum_{i=1}^{K} \sum_{l=0}^{h} IRF_{ij}^2(l)}, \quad i, j = 1, \ldots, K, \quad (4)$$

with $\sum_{i=1}^{K} \gamma_{ij}(h) = 1$ for a given j.

For the case where sufficient restrictions cannot be found to identify the structural error $\boldsymbol{\xi}_t$, Pesaran and Shin (1998) have proposed an approach originally put forth by Koop et al. (1996). For generality, consider a K-dimensional nonlinear multivariate model (with the linear VAR as a special case),

$$\mathbf{y}_t = G(\mathbf{y}_{t-1}, \ldots, \mathbf{y}_{t-p}\,; \boldsymbol{\theta}) + \boldsymbol{\varepsilon}_t, \tag{5}$$

where $G(\cdot)$ is a nonlinear function depending on the parameter vector $\boldsymbol{\theta}$ and $\boldsymbol{\varepsilon}_t$ is an iid error term. Following Pesaran and Shin (1998), we concentrate on shocks hitting only one equation at a time, and define the GIRF of $\mathbf{y}_t$ to the shock δ_{it} at horizon l as

$$GI(l, \delta_{it}, \boldsymbol{\omega}_{t-1}) = E(\mathbf{y}_{t+l}|\varepsilon_{it} = \delta_{it}, \boldsymbol{\omega}_{t-1}) - E(\mathbf{y}_{t+l}|\boldsymbol{\omega}_{t-1}), \quad l = 0, 1, 2, \ldots, \tag{6}$$

where $\boldsymbol{\omega}_{t-1}$ and δ_{it} are the history and the shock to the ith equation that the expectations are conditioned on, respectively. The GIRF (6) can be interpreted as the time profile at time $t+h$ of the effect of the shock δ_{it} hitting at time t, obtained as the difference between the expectations conditional on the shock and the history $\boldsymbol{\omega}_{t-1}$, and the expectations conditioned only on the history $\boldsymbol{\omega}_{t-1}$. Each history $\boldsymbol{\omega}_{t-1}$ consists of the matrix of initial values needed to compute the conditional expectations (forecasts) in (6) which are typically obtained by simulation. In the linear VAR model, the GIRF is history and shock invariant and is obtained by the formulas of Pesaran and Shin (1998, 19).

Based on (6) for $i = 1, \ldots, K$, Pesaran and Shin (1998) suggested a GFEVD for linear models that has the shortcoming that the contributions of the shocks to the forecast error variance of a given variable at horizon l do not sum to unity if the covariance matrix of the error $\boldsymbol{\varepsilon}_t$ is not a diagonal matrix. This makes their interpretation problematic. In contrast, we define the GFEVD of shock i, variable j, horizon h and history $\boldsymbol{\omega}_{t-1}$ by replacing the IRF in (4) by the GIRF:

$$\lambda_{ij,\boldsymbol{\omega}_{t-1}}(h) = \frac{\sum_{l=0}^{h} GI(l, \delta_{it}, \boldsymbol{\omega}_{t-1})_j^2}{\sum_{i=1}^{K} \sum_{l=0}^{h} GI(l, \delta_{it}, \boldsymbol{\omega}_{t-1})_j^2}, \quad i, j = 1, \ldots, K. \tag{7}$$

The denominator measures the aggregate cumulative effect of all the shocks, while the numerator is the cumulative effect of the ith shock. By construction, as in (4),

$\lambda_{ij,\boldsymbol{\omega}_{t-1}}(h)$ lies between 0 and 1, measuring the relative contribution of a shock to the ith equation to the total impact of all K shocks after h periods on the jth variable in $\mathbf{y}_t$, and these contributions sum to unity. Our GFEVD is thus easily interpretable and applicable in any nonlinear model for which the conditional expectations in (6) can be computed.

In the linear VAR model with history and shock invariant GIRFs, (7) can be computed by just plugging in the GIRFs computed by the formulas of Pesaran and Shin (1998). In contrast, in a nonlinear model, the effects of a shock typically depend on its size and sign as well as the history, and, in the same way as shown by Koop et al. (1996) for the GIRF, (7) is readily generalized by averaging over the relevant shocks and histories.

In practice, we recommend computing the GFEVD as the average of $\lambda_{ij,\boldsymbol{\omega}_{t-1}}(h)$ over shocks obtained by bootstrapping from the residuals of the estimated model, and over all the histories. This should yield the GFEVD characteristic of the data at hand, and naturally it solves the problem of selecting the size of shocks to each equation in a multivariate model. If the interest concentrates on only a subset of the histories, averaging can be restricted to the relevant histories, with shocks bootstrapped from among the residuals related to these histories only. For instance, we might be interested in finding the GFEVDs of positive and negative shocks to the ith equation separately.

3 Empirical Illustration

We illustrate the different generalized FEVDs in the bivariate linear and nonlinear autoregressive leading indicator models of Anderson et al. (2007) for the U.S. GDP growth rate and term spread between the long-term (10-year) and short-term (3-month) interest rates. The term spread reflects the stance (direction) of monetary policy, and it typically decreases (increases) prior to recessions (during recessions), suggesting that it might be a leading indicator of output growth. The estimates of the restricted fifth-order VAR and logistic smooth-transition vector autoregressive (LSTVAR) models on quarterly data from 1961Q1 to 1999Q4 are reported in Anderson et al. (2007, Appendix B).

The GFEVDs of the VAR and LSTVAR models are presented in Tables 1 and 2, respectively. In accordance with the discussion in Section 2, the GFEVDs of Pesaran and Shin (1998) do not sum to unity for all h (see the left panel in Table 1 and, in particular, the decomposition for the term spread), whereas this problem does not arise with our GFEVDs, facilitating interpretation.

The GFEVDs based on the VAR and LSTVAR models appear somewhat different. Especially at short forecast horizons, the shock to the term spread has a larger relative contribution to the forecast error variance of output growth in the LSTVAR model compared with the VAR model. This is in line with the importance of nonlinearity found by Galbraith and Tkacz (2000) and Anderson et al. (2007), among others, suggesting that as a leading indicator of output, the predictive information of the term spread is not fully exploited in a linear model. As to the term spread itself, it is dominated by its own shock in the VAR model while the contribution of the shock to output growth is far more important in the LSTVAR model. Finally, the results for the low growth regime (consisting of histories with lagged output growth rate less than 0.32%, see Anderson et al. (2007)) reported in the right panel of Table 2, suggest that the term spread shock plays a slightly more important role for output growth than implied by the results based on all histories.

4 Conclusions

We propose a new generalized forecast error variance decomposition for multivariate linear and, in particular, nonlinear models. In the linear VAR model, the proposed GFEVD encompasses the usual orthogonalized case, and it has a convenient interpretation also when the shocks are non-orthogonal. An empirical application to U.S. output growth and term spread highlights the advantages of the new GFEVD in interpreting estimated linear and nonlinear multivariate models.

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Table 1: GFEVDs of the linear VAR model.

	Pesaran and Shin (1998)				GFEVD (7)			
Variable:	Growth		Spread		Growth		Spread	
Shock to:	Growth	Spread	Growth	Spread	Growth	Spread	Growth	Spread
h								
1	1.00	0.01	0.04	1.00	1.00	0.01	0.00	1.00
2	0.98	0.01	0.10	0.99	0.96	0.04	0.01	0.99
3	0.96	0.02	0.16	0.97	0.92	0.08	0.01	0.99
4	0.94	0.03	0.20	0.96	0.88	0.12	0.02	0.98
8	0.91	0.05	0.59	0.80	0.82	0.18	0.11	0.89
16	0.91	0.05	0.66	0.77	0.82	0.18	0.13	0.87
20	0.91	0.05	0.67	0.77	0.82	0.18	0.13	0.87

Notes: The GFEVDs for different forecast horizons (quarters) h are based on the Pesaran and Shin (1998) approach and the new formulation (7) in the left and right panels, respectively. The latter are given by expression (3) assuming $\mathbf{P} = \mathbf{I}_K$.

Table 2: GFEVDs (7) of the LSTVAR model.

	LSTVAR				LSTVAR, Low Growth Regime			
Variable:	Growth		Spread		Growth		Spread	
Shock to:	Growth	Spread	Growth	Spread	Growth	Spread	Growth	Spread
h								
1	1.00	0.00	0.27	0.73	1.00	0.00	0.21	0.79
2	0.82	0.18	0.35	0.65	0.80	0.20	0.31	0.69
3	0.81	0.19	0.39	0.61	0.78	0.22	0.36	0.64
4	0.81	0.19	0.42	0.58	0.78	0.22	0.40	0.60
8	0.80	0.20	0.53	0.47	0.77	0.23	0.52	0.48
16	0.79	0.21	0.54	0.46	0.77	0.23	0.53	0.47
20	0.79	0.21	0.54	0.46	0.77	0.23	0.53	0.47

Notes: The GFEVDs are based on 1000 shocks bootstrapped from among the residuals of the estimated LSTVAR model. For each pair of shocks the GIRF is computed for each of the 154 histories (consisting of five consecutive observations), yielding, in total, 154000 GIRFs, over which (7) is averaged. The conditional expectations in (6) are computed based on 1000 simulated realizations of the model. In the right panel, the low growth regime applies when the GDP growth rate is less than 0.32%

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