Understanding and Managing Model Risk

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Outline

- Understanding Model Risk: lessons from the past crises
- Theory of Model Risk: uncertainty and Illiquidity
- Regulators and accountants
- Practical management: Model Reserves and Model Limits
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- Example in Equity: Local vs Stochastic Volatility Models
- Example in Rates: One- vs Multi-Factor Models
- Stress-Testing: Detecting the credit correlation mistake
- Stress-Testing: Copula errors in dynamic risk
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- Model Risk in Hedging. SABR and Local Vol
- Model Risk in Algo trading: arbitrage vs uncertainty
- When the payoff is wrong: closeout risk for CVA and DVA
- Model risk in the Funding Debate
What is Model Risk?
What is model risk?

A Model Validation practice requires agreeing on what we mean by Model Risk, from a practical perspective.

There are two main approaches to the problem. One, that we call Value approach, can be described using the words of Derman (1996, 2001). A model risk manager should check the following points:

1. Is the payoff accurately described?
2. Is the software reliable?
3. Has the model been appropriately calibrated to the prices of the simpler, liquid constituents that comprise the derivative?
4. Does the model provide a realistic (or at least plausible) description of the factors that affect the derivative's value?
The Value Approach

“Model risk is the risk that the model is not a realistic (or at least plausible) description of the factors that affect the derivative's value”

By E. Derman

- It is clear that “a model is always an attempted simplification” of reality, and as such there can be no perfectly realistic model. Moreover only what affects the derivatives’s value matters. Yet realism remains a goal, and modellers should avoid the following errors:

  “You may have not taken into account all the factors that affect valuation. You may have incorrectly assumed certain stochastic variables can be approximated as deterministic. You may have assumed incorrect dynamics. You may have made incorrect assumptions about relationships”.

  By E. Derman

- But Rebonato starts from a totally different perspective…
What is Model Risk?

**The Value Approach**

“Model risk is the risk that the model is not a realistic (or at least plausible) description of the factors that affect the derivative's value”

*By E. Derman*

**The Price Approach**

“Model risk is the risk of a significant difference between the mark-to-model value of an instrument, and the price at which the same instrument is revealed to have traded in the market”

*By R. Rebonato*
What is Model Risk?

The **Price** Approach

“Model risk is the risk of a significant difference between the mark-to-model value of an instrument, and the **price** at which the same instrument is revealed to have traded in the market”

*By R. Rebonato*

- Real losses do not appear “because of a discrepancy between the model value and the ‘true’ value of an instrument”, but through the mark-to-market process, because of a discrepancy between the model value and the market price of an instrument.
- As long as the market agrees with our valuation, we do not have large losses due to models, even if “market prices might be unreasonable, counterintuitive, perhaps even arbitrageable”.
- We have no mark-to-market losses, we can sell at the value at which we booked.
Are Value and Price approaches really so different?

The Price approach rightly stresses that model losses usually emerge when a sudden gap opens between market price and model booking.

But this can happen for 3 reasons

1. **We are using a different model from market consensus**
   - **Market Intelligence:** trades, collateral, TOTEM, rumors…

2. **Same model, but implementation or payoff are wrong**
   - **Price Verification:** check software, legal issues…

3. **Same model, but the market consensus suddenly changes**

   “Surmise how today’s accepted model can change in the future…”
Let’s look at the subprime crisis

- **How can the market suddenly change the model consensus?**
  This happened recently for mortgage-backed securities.

- The market consensus modelled the default probabilities in a pool of subprime mortgages based on a Gaussian Copula with default probabilities and correlations that were historically estimated. Both default probabilities and default correlations were low, allowing for low risk estimates and high ratings for senior CDO tranches.

- Historical correlations and default probabilities were low since in the previous decade subprime defaults had led to few and sparse losses. This was due to the increasing trend of house prices, giving cash to mortgagers and and a valuable guarantee to the bank in case mortgager defaulted. Default losses were due to odd, isolated cases.

- This dependence on the trend of house prices, however, created the possibility of big default clusters if trend reversed. Using such low historical probabilities and correlations for the future meant assuming implicitly that house prices were going to grow at same rate also for the subsequent decade.
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How can market consensus on models change suddenly?

Example 1: Mortgage CDOs in the Subprime Crisis

Model consensus was Gaussian Copula with estimated/mapped correlations, very low consistently with the assumption of ever-increasing national house prices.

The reversal of the national house trend reveals that mortgage losses can be very correlated, and the Gaussian Copula market consensus collapses.
How can market consensus on models change suddenly?

Example 2: 1987 Stock Market Crash

Example 3: From One factor to Multifactor in 2000

“How to throw away a billion Dollar”

\[
\begin{align*}
    dr_t &= k (\theta - r_t) \, dt + \sigma \, dW_t \\
    dF_1(t) &= \mu_1(t) \, dt + \sigma_1(t) \, F_1(t) \, dW_1^t \\
    dF_2(t) &= \mu_2(t) \, dt + \sigma_2(t) \, F_2(t) \, dW_2^t \\
    \vdots \\
    dF_n(t) &= \mu_n(t) \, dt + \sigma_n(t) \, F_n(t) \, dW_n^t
    \end{align*}
\]
A synthetic view on Model Risk

- In the subprime crisis and in 1987 crash it is an event in the fundamentals to change the model consensus. Something happens in the reality of the markets that reveals an element of unrealism of the model to be more relevant than previously thought. In the third example it is a piece of research that shows some of the modelling assumptions to be unrealistic in a relevant way.

- Thus realism and reasonableness of the model are relevant not only in Derman’s Value approach, but also in Rebonato’s Price approach to model risk management, since lack of consistency with reality is the main fact that can lead to sudden changes in model consensus.

- On the other hand, consistency with market consensus, the main point in the Price approach, is not necessarily overlooked in the Value approach. Among the factors to be considered in designing a plausible model, Derman’s Value approach includes market sentiment.
A first scheme for model choice, validation and risk management

- **Model Verification**
  - Mathematics
  - Implementation
  - Numerics
  - Correct application to Payoff

- **Model Validation**
  - Calibration
  - Reasonableness
  - Market Intelligence
  - Reality check

The last two points make sense if applied in **Model Comparison**, leading to **model choice** and to the setting of **provisions** (reserves, model limits, monitors..)
Model Comparison for Model Validation

Classic examples of Model Comparison in the literature are:

- **Equity**
  - Stochastic Volatility vs. Local Volatility

- **Rates**
  - Low factor Short Rate vs. Multifactor BGM/HJM

- **Credit**
  - Structural Models vs. Reduced-form Models
Model Uncertainty, Incompleteness and Illiquidity
What is Model Uncertainty?

- The analysis of model realism and the intelligence about the models used in the market does not always eliminate model uncertainty. How should we deal with this uncertainty? What is model uncertainty?
- Gatarek and Brigo and Mercurio formalized a so-called Uncertain Volatility Model where

\[ dS(t) = rS(t)\, dt + \sigma^I S(t)\, dW(t) \]

\[ \sigma^I = \begin{cases} \sigma^1 & \text{with prob } p_1 \\ \sigma^2 & \text{with prob } p_2 \end{cases} \]

- Scenarios depend on random variable \( I \), drawn at \( t=\epsilon \) infinitesimal after 0, independent of \( W \), taking values in \( 1,2 \) with probability

\[ \text{Prob}(I = i) = p_i. \]
What is Model Uncertainty?

- How does one price the option with strike $K$ and maturity $T$? One uses the law of iterated expectation

$$
\Pi (K, 0, T) = P (0, T) \mathbb{E} [(S (T) - K)^+] \\
= P (0, T) \mathbb{E} [\mathbb{E} [(S (T) - K)^+ | \sigma^I]] \\
= P (0, T) \sum_{i=1}^{2} p_i \mathbb{E} [(S (T) - K)^+ | \sigma^I (t) = \sigma^i (t)] .
$$

- The option price is just the average of two Black and Scholes prices

$$
p_1 BS \left( S (0), K, T, \sigma^1 \right) + p_2 BS \left( S (0), K, T, \sigma^2 \right)
$$

- Is this an example of model uncertainty? NO. Here there is only ONE model, with a sketchy random volatility. Volatility is a random variable with a simple distribution which must be drawn before simulating $S$. 

What is Model Uncertainty?

- Model Uncertainty looks like
  \[ dS(t) = rS(t) \, dt + \sigma S(t) \, dW(t) \]
  with us not knowing if \( \sigma = \sigma^1 \) OR \( \sigma = \sigma^2 \)

- What has changed from above?
  - at \( \varepsilon \) we will not draw the right volatility
  - we have no idea of what \( p_1 \) and \( p_2 \) are
  - other market players may know the right value of \( \sigma \) or they may have even less information than us.
Model Uncertainty vs Risk

Cont (2006) treats model uncertainty as multiple probability measures

\[ (\Omega, \mathcal{F}, P_i \mapsto Q_i) , \]

\[ P_i \mapsto Q_i = P_1 \mapsto Q_1, \ldots, P_N \mapsto Q_N \]

as opposed to risk, where we are uncertain about realizations but we know the probability distributions (the roulette for a standard player).

Should we, like in a Bayesian approach, average out expectations under these different measures? This corresponds to treating uncertainty about the model in the same way as uncertainty within the model, in spite of the above differences.
The conservative approach glorified

- Cont (2006) notices that the typical approach of banks is not to average across models but to adopt a worst case approach. Only one choice protects you from any model loss: with $\sigma^2 > \sigma^1$,

$$P_{ask} = BS (S(0), K, T, \sigma^2), \quad P_{bid} = BS (S(0), K, T, \sigma^1).$$

- Gibdoa and Schmeider (1989) show that this old `conservative approach' is rational maximization of utility when we are faced with total ignorance on the probabilities.

- Following this line, Cont (2003) proposes two measures of model uncertainty: one is akin to

$$\max_{Q_i=1, \ldots, N} \text{Price} - \min_{Q_i=1, \ldots, N} \text{Price},$$
Model Risk and Liquid Markets

- The second one weights more or less models depending on the higher or lower capability to price liquid market instruments.

- For Cont there is no model uncertainty when:
  1. the market is liquid (model uncertainty lower than bid-ask)
  2. we can set up a model-independent static hedge

- In principle, this makes sense. But in practice? Should we forget Model Risk management for liquid/hedgeable products?

- We are going to see first that accountancy boards seem to agree, then we will see an example where the collapse of a model consensus shook up a liquid market, disrupted standard practice for discounting, broke up hedging and turned the market illiquid. There will see that and apparent “model independent” static hedge can be broken down by a sudden change of model consensus.
Model Risk and Liquid Markets

- How does this relate with the theory of incomplete markets? The above market would be incomplete if the underlying $S(t)$ was not a traded asset. In this case

$$dS(t) = (\mu_t + \sigma_t \gamma_t) S(t) \, dt + \sigma_t S(t) \, dW(t)$$

- The drift is not necessarily equal to $r$ since the underlying is not traded so there are no `no-arbitrage conditions' that force this equality, and will be determined by each player looking at its own risk aversion $\gamma$. Thus, even if all market players knew the parameters under the real-world probability measure $P$, they would not know the drift under $Q$ used by their counterparties to make prices, creating a kind of model uncertainty.
But there is more. If the underlying is not traded since it is an illiquid financial risk, we will be most often in a situation where we know neither $\gamma$, $\mu$ or $\sigma$.

The uncertainty on $\gamma$ is explained by the standard theory and will be resolved looking at my risk aversion. The uncertainty on $\mu$ and $\sigma$ is not explained by the standard theory but it will also be solved looking at risk aversion, as shown in Cont (2006).

The effect of my arbitrary risk aversion on $\gamma$ and the effect of my uncertainty about the real-world parameters $\mu$ and $\sigma$ will be indistinguishable.

In practice uncertainty on the risk-neutral adjustment $Q|P$ compounds with uncertainty about real world dynamics $P$.

But there is more. With more advanced models such as jumps, market is incomplete by definition, and more parameters are affected by risk aversion, so that in reality risk aversion compounds with model uncertainty for almost all parameters.
Model Risk and Liquid Markets

- This is confirmed by Cherny and Madan (2010), that explain illiquidity (bid-ask spread) based on incompleteness of most markets. For a discounted payoff $X$ to be acceptable in the market, the algebraic sum of price and expected discounted payoff must be positive, irrespectively of the measure used in expectation...

- In incomplete markets each player chooses its own risk aversion and this gives rise to a plurality of possible pricing measures

\[ E^{Q^k} [Ask - X] \geq 0, \ E^{Q^k} [X - Bid] \geq 0 \]

for all $k = a, b, \ldots, z$ so that

\[ Ask = \max_{k=a,b,\ldots,z} E^{Q^k} [X], \ Bid = \min_{k=a,b,\ldots,z} E^{Q^k} [X] \]

- Incompleteness and model uncertainty act jointly in the market to separate bid from ask prices, and they will be most of the time indistinguishable from each other.
Accountants, Regulators and the explosion of credit and liquidity risk
Accountancy for Modellers

There are various aspects of accountancy principles that are linked to the debate about model risk and model uncertainty.

Fair Value
"Fair value is the price that would be received to sell an asset or paid to transfer a liability in an orderly transaction" (FAS 157)

This reminds of the Price approach…market is the real reference.

But in the crisis the "SEC Office of the chief accountants and FASB Staff Clarification on Fair Value Accounting“ says that at times illiquidity, panic or euphoria do not allow for "orderly transactions", so that market prices can be corrected by personal judgment to get

“reasonable risk-adjusted, liquidity-adjusted and credit-adjusted expected cash flows”

This reminds much more of the Value approach…
Level 1, 2, 3… go?

Another analogy are the famous “levels” for prices:

1) **Level 1 (Liquid specific quotes):** the price comes from a liquid market where products identical to the derivative to price are traded.

2) **Level 2 (Comparable quotes or Illiquid quotes):** Prices come from either:
   a) a liquid market where products similar (not identical) to the derivative to evaluate are traded.
   b) a market where products identical to the derivative to evaluate are traded but the market is illiquid.

3) **Level 3 (Model with non-quoted parameters):** Prices come from the a valuation technique that requires inputs which involve a crucial amount of personal judgement of the institution computing the price. This kind of prices are often called 'marked-to-model'.
Level 2 and Level 3

- Standard market wisdom believes that:
  \[
  \text{ModelRisk(Level3)} > \text{ModelRisk(Level2)} > \text{ModelRisk(Level1)} = 0
  \]

- Are we sure?
- Example 1:
  - Realistic model: two factors with a correlation to be estimated: **Level 3**
  - Unrealistic models: one factor (correlation=1) or two factors independent by construction (correlation=0): **Level 2**

- A trader, if penalized by model reserves in case of Level 3 prices, has incentive to choose the most simplified and unrealistic model.

- And what about Level 1 prices, are they really model-risk-free?
Level 1 and model risk

- Level 1 prices are those taken from liquid prices of products *identical* to the derivative under analysis.

- A Level 1 price can involve model risk because it hinges on the subjective decisions that two assets are *identical*, while this may be true only under some hidden assumption that can suddenly become unrealistic.

- For pointing out how model assumptions enter in the evaluation of the simplest vanilla derivatives we show the model assumptions hidden in the valuation of a simple swap. By the way, during the credit crunch – on 9 August 2007 for precision - these assumptions have broken down, opening a radical reform of pricing.
Swaps are the main interest rate derivatives. They are priced ‘without a model’ via replication.

The simplest swap is the Forward Rate Agreement (FRA). We can price it using information in the spot interbank market, where loans are made from bank to bank.

If the prevailing rate is $L(t, T)$, when a lender gives 1 at $t$, he receives

$$1 + L(t, T)(T - t) \text{ at } T$$

If instead he gives

$$\frac{1}{1 + L(t, T)(T - t)} \text{ at } t$$

he receives 1 at $T$. 

When the market model changes. Example from Rates
Replicating a FRA

- If a FRA fixes in $\alpha$ and pays in $2\alpha$, the payoff in $2\alpha$ is:

  $$ (L(\alpha, 2\alpha) - K)\alpha. $$

- To replicate it, lend at 0 to your counterparty

  $$ \frac{1}{1 + L(0, \alpha)\alpha} $$

until $\alpha$, so that you get 1 at $\alpha$. At $\alpha$ lend again to counterparty 1 until $2\alpha$, getting at $2\alpha$:

  $$ 1 + L(\alpha, 2\alpha)\alpha $$

- At 0 you also borrow

  $$ \frac{1 + K\alpha}{1 + L(0, 2\alpha)\alpha} $$

until $2\alpha$, so that you receive at $2\alpha$:

  $$ -1 - K\alpha $$

- Putting together, payoff at $2\alpha$ is:

  $$ (L(\alpha, 2\alpha) - K)\alpha. $$
Replicating a FRA

\[
\begin{align*}
0 & \quad x & \quad 2x \\
\frac{1}{1 + \frac{1}{2}(0, x)x} & \quad +1 & \quad 1 + \frac{1}{2}(x, 2x)x \\
\frac{1 + kx}{1 + \frac{1}{2}(0, 2x)2x} & \quad -1 & \quad 1 - kx \\
\end{align*}
\]

\[
(\lambda(x, 2x) - k)x
\]
The equilibrium FRA rate and the Basis Spreads

The price of a FRA is the cost of its replication. Recalling that in modern markets the discount “bond” comes actually from interbank rates:

\[
\frac{1}{1 + L(0, \alpha)} - \frac{1 + K \alpha}{1 + L(0, 2\alpha)} = P(0, \alpha) - P(0, 2\alpha) - P(0, 2\alpha) K \alpha
\]

and the equilibrium \( K \) is:

\[
F(0; \alpha, 2\alpha) = \frac{1}{\alpha} \left[ \frac{P(0, \alpha)}{P(0, 2\alpha)} - 1 \right]
\]
The equilibrium FRA rate and the Basis Spreads

- With many FRA paying from $T_{a+1}$ to $T_b$ we have swap value:

\[
\sum_{i=a+1}^{b} \left[ P(t, T_{i-1}) - P(t, T_i) - P(t, T_i) K \alpha_i \right] = P(t, T_a) - P(t, T_b) - \sum_{i=a+1}^{b} P(t, T_i) \alpha_i K
\]

- The frequency of payment, and consequently the tenor of the rates paid, does not enter the valuation of floating legs.

- This implies null spreads for Basis swaps, that exchange two floating legs of different frequency and different rate indexation (for example 1y vs 6m).
9 August 2007: the market changes

- Is this pricing Level 1, 2 or 3?

- There are no estimations based on judgements, we are not using comparables but a strategy that yields exactly the same payoff. It should be level 1.

- In spite of this…
9 August 2007: the market changes

- None of these relationships holds anymore. Below see spot replication of 6mX6m FRAs vs real market FRAs.
9 August 2007: the market changes

- Here we see the dynamics of Basis Spreads in the crisis, to be compared with null spread predicted by models.

- Old relations were valid for decades but were rejected in one day.
Introducing Reality

- Which model hypothesis is not valid anymore? The usual suspect is the implicit hypothesis that

   \[ \text{The interbank market is free of default risk} \]

- Default risk can break the above replication. If each bank has its own risk of default, it also has its own fair funding rate, and there is no reason to expect the fair funding rate of my counterparty at \( L_{\text{Counterparty}}(\alpha, 2\alpha) \), to be the same as the payoff rate \( L(\alpha, 2\alpha) \).

- We have used this hypothesis in replication.

- The payoff rate is usually Libor… a very complex basket payoff!

   \[ \text{Libor is a trimming average of reported funding rates for unsecured borrowing deals from a panel of large banks with the highest credit quality} \]
Replicating a FRA after the Credit Crunch

\[
\begin{align*}
0 & \quad \alpha \\
- \frac{1}{1 + L(0, \alpha) \lambda} & \quad + 1 \\
\frac{1 + K \lambda}{1 + L(0, 2\alpha) 2\lambda} & = \left(L(\alpha, 2\alpha) - K\right) \lambda
\end{align*}
\]
An option hidden in Libor…

- However, understanding how to explain the **new market with a new model** is not easy.
- In the Libor (or Euribor) markets there is a **selection mechanism** that creates a bias towards rates with lower risk.

- In fact, banks whose funding rate grows too much, sooner or later…
  - will exit the Libor interquantile average
  - will exit the Libor market and start dummy contributions
  - will exit the Libor panel
  - will exit this world (and Libor) by defaulting…

- Thus the **expected future Libor rate is constantly lower than the expected future borrowing rate of today’s Libor counterparty**
The Libor mechanics: a simple scheme

- A simple scheme to represent the potential refreshment of the counterparty is:

\[
S(\alpha, 2\alpha) = \begin{cases} 
  S^{dX_0}(\alpha, 2\alpha) & \text{if } S^{dX_0}(\alpha, 2\alpha) \leq S^{Exit} \\
  S^{Subst} & \text{if } S^{dX_0}(\alpha, 2\alpha) > S^{Exit}
\end{cases}
\]

- \(S^{Exit}\) is the maximum level of the spread for the initial counterparty to be still a Libor counterparty at \(\alpha\)

- \(S^{Subst}\) is the spread of Libor in case the initial counterparty is no more a Libor bank because

\[
S^{dX_0}(\alpha, 2\alpha) > S^{Exit}
\]
The Libor mechanics: Exit condition

- We have to make a choice on $S^{Exit}$, then on $S^{Subst}$, with $S^{Subst} \leq S^{Exit}$.

- We will propose as an example some extremely simple possible values and we will see if even such simplified assumptions can lead to relevant improvements in explaining the market.

- The level $S^{Exit}$ is the level over which a Libor counterparty is considered an underperformer.

- Thus $S^{Exit}$ can be close to the current expectation of the future spread of the counterparty, so that a counterparty will be excluded from Libor banks if it performs worse than expected:

$$
S^{Exit} = S^{dX_0}(0; \alpha, 2\alpha) = \mathbb{E}^{rf2\alpha}[S^{dX_0}(\alpha, 2\alpha)]
$$
The Libor mechanics: substitution condition

Also for $S_{Subst}$ we make a simple choice, guaranteeing

$$S_{Subst} \leq S_{Exit}$$

$$S^{dX_0} (0; \alpha, 2\alpha) \quad = \quad E^{r_f^2\alpha} \left[ S^{dX_0} (\alpha, 2\alpha) \right]$$

$$S_{Subst} = 2S_{dX_0} (0; \alpha, 2\alpha) - S^{dX_0} (\alpha, 2\alpha).$$
An option hidden in Libor...

If we make some simple assumptions on Libor refreshment

\[ \mathbb{E}^{r f^{2\alpha}} [L(\alpha, 2\alpha)] = F_L (0; \alpha, 2\alpha) - 2\mathbb{E}^{r f^{2\alpha}} \left[ (S^{dX_0}(\alpha; \alpha, 2\alpha) - S^{dX_0}(0; \alpha, 2\alpha))^+ \right] \]

and if we assume that the spread evolves as a geometric brownian motion

\[ dS_{dX_0} (t, \alpha, 2\alpha) = S^{dX_0} (t, \alpha, 2\alpha) \sigma_\alpha dW^P_\alpha (t) , \]

we have the simple option formula

\[ \mathbb{E}^{r f^{2\alpha}} [L(\alpha, 2\alpha)] = F_L (0; \alpha, 2\alpha) - 2\text{BlackCall} \left( S^{dX_0}(0; \alpha, 2\alpha), S^{dX_0}(0; \alpha, 2\alpha), \sigma_\alpha \sqrt{\alpha} \right) \]
An option in Libor…

- We need a volatility. We take it from i-Traxx credit options. We can explain market patterns better than with old models (Morini 2009).
Replicating a FRA after the Credit Crunch

- So we have seen that the replication is at least approximately valid only if
  \[ L(\alpha, 2\alpha) = L^{Counterparty}(\alpha, 2\alpha) \]

- But this is true only if
  - All banks have negligible credit risk
  - Or risk is the same for all banks and stable
  - Or risk is perfectly correlated for all banks

- These conditions were approximately valid for the ten years preceding the credit crunch, but from summer 2007 credit is relevant, heterogeneous and volatile, so Libor selection has clear effect.

- Formally one saw just a market movement for FRAs, but at portfolio level it can only be understood as a change of modelling paradigm, since it broke down the hedging strategies and the construction of the discounting curve for liquid and illiquid derivatives, all hinging on the above model assumptions.

- We can have model risk about liquid prices and provisions like reserves or position limits can be useful also there.
What Regulators (Basel Committee) say after the crisis:

- Validation includes evaluations of the model's theoretical soundness and mathematical integrity and the appropriateness of model assumptions, including consistency with market.

- Bank processes should emphasise the importance of assessing fair value using a diversity of approaches and having in place a range of mechanisms to cross-check valuations.

- A bank is expected to test and review the performance of its valuation models under possible stress conditions, so that it understands the limitations of the models.

- Assess the impact of variations in model parameters on fair value, including under stress conditions.
A first scheme for model choice, validation and risk management

- **Model Verification**
  - Mathematics
  - Implementation
  - Numerics
  - Correct application to Payoff

- **Model Validation**
  - Calibration
  - Reasonableness
  - Market Intelligence
  - Reality check

The last two points make sense if applied in **Model Comparison**, leading to **model choice** and to the setting of **provisions** (reserves, model limits, monitors...).

After first decision is taken…
Stress Testing

- **Stress-testing**
  - **Stressability:** Verify if the model can be used to express stress-conditions
  - **Stress-testing implementation:** detect what can impair the precision of approximations and numerics
    - for market conditions
    - for payoff features
  - **Stress testing assumptions:** subject the model to stress conditions to limit and monitor its application
    - for market conditions
    - for payoff features

- **Model Evolution:** analyze how the current market consensus (or lack of it) could change in the future and how this affects the model use.

- **Review Model Reserve/Limits,** set restrictions - monitoring (triggers)
Model Comparison for Model Validation

Classic examples of Model Comparison in the literature are:

**Equity**

- Stochastic Volatility vs. Local Volatility

**Rates**

- Low factor Short Rate vs. Multifactor BGM/HJM

**Credit**

- Structural Models vs. Reduced-form Models
Comparing models: an example from Credit Modelling
Structural First-Passage models vs Reduced-form Intensity Models

Structural Models

\[ dV_t = (r - q) V_t dt + \sigma V_t dW \]

\[ \tau = \inf \{ t | V_t \leq H_t \} \]

\( V \) is the firm value, \( H \) is the default barrier.

Intensity Models

\[ \tau = \inf \left\{ t : \int_0^t \lambda(s) ds \geq \varepsilon \right\} \]

\[ \Pr(\varepsilon \leq z) = 1 - e^{-z}. \]

\( \lambda(t) \) is the intensity or instantaneous default probability, giving credit spreads, \( \varepsilon \) is the impredicable default trigger.
THE MODELS: First-Passage Structural Models

- If $V$ goes below the barrier $H$ there is default. $V$ is a standard Geometric Brownian Motion
  \[ dV_t = (r - q) V_t dt + \sigma V_t dW \]

- Default happens at
  \[ \tau = \inf \{ t | V_t \leq H_t \} \]

- Survival probability is given by
  \[ Q\{\tau > T\} = N\left( \frac{\ln \frac{V_0}{L} + (r-q-\frac{\sigma^2}{2})T}{\sigma \sqrt{T}} \right) + \]
  \[ - (\frac{V_0}{L}) - \frac{2(r-q-\frac{1}{2}\sigma^2)}{\sigma^2} \]
  \[ N\left( \frac{-\ln \frac{V_0}{L} + (r-q-\frac{\sigma^2}{2})T}{\sigma \sqrt{T}} \right) \]

- Brigo and Morini (2006) give analogous closed-form formulas also for time-dependent parameters, fitting credit spreads.
We take $\tau$ to be the first jump of a Poisson Process of intensity $\lambda$, so that it has a negative exponential distribution with parameter $\lambda$:

$$Pr(\tau > T) = e^{-\lambda T},$$

$$F_\tau(T) = Pr(\tau \leq T) = 1 - e^{-\lambda T},$$

If we set $\tau \lambda = \varepsilon$, we see $\varepsilon$ has an exponential distribution with unit parameter, since

$$F_\varepsilon(\bar{z}) = Pr(\varepsilon \leq \bar{z}) = F_\tau\left(\frac{\bar{z}}{\lambda}\right) = 1 - e^{-\bar{z}}.$$ 

so that we can write

$$\tau = \frac{\varepsilon}{\lambda}$$
Reduced-Form Intensity Models

- When intensity is time-dependent, we have

\[ P(r > T) = e^{- \int_0^T \lambda(s) ds}. \]

- We define

\[ \Lambda(T) = \int_0^T \lambda(s) ds \]

so that we can write

\[ \tau = \Lambda^{-1}(\varepsilon) = \Lambda^{-1}(- \ln(1 - U)) \]

- In practice, e.g. montecarlo simulation, \( \tau \) is computed as

\[ \tau = \inf \left\{ t : \int_0^t \lambda(s) ds \geq \varepsilon \right\}. \]
Consider a simple note sold by a bank to a client. The client pays $1$, with this notional the bank sells leveraged protection on a reference entity. The notional of this sale is $Lev$. If a default happens, the bank may have a loss given by

$$Lev \times Lgd \quad \text{paid by noteholder}$$

A typical way to mitigate this gap risk is to set a trigger to stop the note if spreads are growing too much:

The trigger can be set at such a level $trigger^*$ that when touched, if unwinding is timely, we expect to have a loss around $1$, minimizing gap risk.
Preliminary test

1. Calibration
Both models must be able to calibrate the CDS term structure of the reference entity. This is natural for standard (deterministic) intensity models, while structural models must be upgraded to time-dependent parameters and barrier (i.e. Brigo and Morini (2006))

2. Reasonableness
Both models must admit stochastic credit spreads, since spread dynamics is crucial for gap risk. This is natural for structural models, while intensity must be upgraded to stochastic intensity (i.e. Lando (1999)).

\[ d\lambda (t) = k[\theta - \lambda (t)]dt + \sigma \sqrt{\lambda (t)}dW (t) \]
Are now the two families of models equivalent for Gap Risk?

- Big gap losses are associated to entity defaulting without touching the trigger first. Thus gap risk depends crucially on the behaviour of spreads in the time preceding default: will default be preceded by a significant rally in the spread, or we will have an abrupt leap to default?

A) Default anticipated by spread rally: if the rally is not too fast, the deal is stopped with no gap losses.

\[
\text{GapRisk} \approx 0.
\]

B) Sudden leap to default, not sufficiently anticipated by spread rally: the bank has the maximum possible gap risk, given by

\[
\text{GapRisk} = \text{Discount} \times \Pr(\tau < T_M) \times (\text{Lev} \times (1 - \text{Rec}) - 1)^+
\]
Comparing Structural Models and Intensity Models

**Structural Models**

Credit spreads grow with default probability. In standard structural models default probabilities are higher the closer Firm Value gets to the Barrier, so **credit spreads rally to infinity when default approaches.** In a trigger note this means **LOW GAP RISK.**

**Intensity Models**

In intensity models, if the intensity is a stochastic diffusion - CIR for example – in most cases default is a sudden leap not anticipated by spreads. In a trigger note it is **HIGH GAP RISK.**
Why Gap Risk is so high (leap to default) in intensity models?

The blue line is simulated diffusive (CIR) intensity.

The green line is its own integral.

Default is when integral reaches simulated $\varepsilon$. With these patterns, it is $\varepsilon$ that decides default, and $\varepsilon$ is independent of spreads.
Adding Market Information: CDS Options?

- Do we have liquid quotes to make a choice? CDS do not help, because the above models are both perfectly calibrated to CDS.
- One possibility could be CDS options, when available. But a CDS option is knock-out, so if maturity is $T$ and strike is $K$ we have in the payoff

$$1_{\{\tau > T\}} (S(T) - K)^+$$

thus it depends crucially on

$$\Pr (S(T) > K | \tau > T)$$

while for a note with trigger= $K$ and same maturity $T$ we are interested in

$$\Pr (S(\tau) > K | \tau < T)$$

- The two derivatives speak of two opposite states of the world! One is conditional to survival, the other one on default. Options increase our information but are not crucial for model choice when we have to evaluate a trigger note.
Realism: in search of historical evidence

- Do we have a clear historical evidence of what is the most likely possibility, a leap to default or a default preceded by a spread increase?
- The first caveat is that history of past defaults has little to do with future defaults.
- Moreover the history of past defaults is mixed.

In general, the likelihood of a sudden default is inversely proportional to the transparency of the balancesheet of the reference entity, but...

**Parmalat** and **Enron** were little predictable. **Argentina**, a sovereign default, was fully predictable, with spreads growing steadily. **Lehman** was mixed.
Market Intelligence and the practical side of Incompleteness

- The leveraged note depends crucially on a risk factor, driving the relation between spreads and default time, which is not observable in any liquid market. This has two related consequences:
- We have no quotes to be used to calibrate the model to market consensus about the behaviour of this risk factor. One may think of market intelligence…
- The market on this risk factor is incomplete. In an incomplete market, even if all players agreed on the actual probability to have a sudden leap-to-default rather than a spread rally anticipating default, they may disagree on the compensation for the risk of this event. A conservative bank selling this note, for example, will increase the risk-adjusted probability of such a leap to default.
Dealing with Model Uncertainty

- How to deal with a situation where model uncertainty is so high, and the two most popular modelling solution give two opposite values to the derivative?

- First, find models with behaviour in-between the “null” Gap Risk of standard Structural Models and the “maximum” Gap Risk of the Intensity Models seen. There are at least 2 possibilities.

1. More realistic structural models, with a default barrier that can jump between different levels. An unpredictable barrier creates in these models a controllable probability of leaps to default

2. Intensity models where intensity stops being a diffusion:

\[ d\lambda (t) = k[\theta - \lambda (t)]dt + \sigma \sqrt{\lambda (t)}dW(t) + dJ_t^{\alpha,\gamma}. \]

where \( dJ_t^{\alpha,\gamma} \) is a Poisson jump with intensity \( \alpha \) and jumps which are exponentially distributed with average size \( \gamma \).
How can intensity jumps create predictable defaults?

The blue line is simulated intensity CIR with jumps.

The green line is its own integral.

Default is when integral reaches $\varepsilon$.

Here it is not $\varepsilon$ that decides default, but the jump in the intensity, which first creates a spread rally anticipating default.
Default times and spreads when intensity is jump-diffusion
A parametric family of models covering the possible values

In this family of models, moving the expected jumps size we move gap risk from maximum to almost zero.

How to use parametric family for model risk management?
A parametric family of models covering the possible values
Model Risk Management: Reserves, Limits, Revisions
Suppose for simplicity a class of models parameterized by the parameter $\gamma$,

$$\Pi_t^T (\gamma)$$

of the derivative increases with $\gamma$.

The model validation process validates the model with $\gamma=0.5$ but recognizes that there is a residual model uncertainty.

If counterparty is eager to buy the derivative at $\gamma=0.75$, the difference

$$R = \Pi_t^T (0.75) - \Pi_t^T (0.5)$$

appears a day-0 profit for the salespeople and the traders that closed the deal.

The model risk managers can instead use $R$ to create a model capital reserve, that will be released to the trader only along the life of the derivative, for example from time $t_1$ to $t_2$ he may linearly receive only

$$\frac{R(t_2 - t_1)}{T}$$
Model risk management: Model Lines

- Model *position limits* or *lines* are analogous to the credit lines used to manage credit risk. If a model is considered subject to high model risk, the bank can set a limit to the exposure built through the model.

1. First decide first the **total model line** - the maximum exposure to model uncertainty allowed.

2. Then compute the ‘**add-on**’, i.e. how much a single deal contributes to filling the total line. It is the potential loss due to model uncertainty.

- For a product sold at $\prod (\widetilde{\gamma}, T)$ an estimate is

\[
\text{Notional} \times \left[ \prod_t^T (1) - \prod_t^T (\widetilde{\gamma}) \right]
\]

- The line does not regard deals sold at the conservative price $\prod_t^T (1)$

- If the error we expect on $\gamma$ is no more than $\Delta \gamma$, one can use

\[
\text{Notional} \times \left. \frac{\partial \prod_t^T (\gamma)}{\partial \gamma} \right|_{\gamma=\widetilde{\gamma}} \Delta \gamma.
\]

But we have high uncertainty on the point where sensy is computed.
Model risk management: Model Revisions

- For credit lines the exposure is often a quantile of the future mark-to-market. But with uncertainty on $y$ the quantile is less reliable. Additionally, model risk is not associated to uncertainty about future values – market risk - but about present prices.

- **Model Revisions**: specify dates at which the validation must be revised.

  1. Periodic Revisions: scheduled regularly, for example yearly. After one year experience and evidence have increased.

  2. Triggered Revisions: stress-test may have revealed that under particular market conditions a model may become unreliable. If quantitative triggers signal such conditions are reached, validation must be revised.
Classic examples of Model Comparison in the literature are:

<table>
<thead>
<tr>
<th>Equity</th>
<th>Rates</th>
<th>Credit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stochastic Volatility</td>
<td>Low factor Short Rate</td>
<td>Structural Models</td>
</tr>
<tr>
<td>VS</td>
<td>VS</td>
<td>VS</td>
</tr>
<tr>
<td>Local Volatility</td>
<td>Multifactor BGM/HJM</td>
<td>Reduced-form Models</td>
</tr>
</tbody>
</table>
Comparing models: an example for Rates
Low factor Short Rate vs Multifactor BGM/HJM

- Popular debate around 2000. The paper “How to throw away a billion dollar” claims bermudan swaptions are undervalued by 1-factor interest rate models, that have have instantaneous correlations among rates set to 1.

\[ dr_t = k (\theta - r_t) \, dt + \sigma dW_t \]

- This can be an element of model risk. Think of a montecarlo simulation. If correlation among the different payoffs you can get exercising is forced to be high, even as high as one, then having Bermudan rather than European is not big deal. Bermudans will have low values.

- Compared to real world correlations, mipic strategies.
Low factor Short Rate vs Multifactor BGM/HJM

- When correlations are allowed to be lower, in any scenario there can be low exercise values at some times and higher exercise values at other times. With a Bermudan you can find convenient exercise time that you could not find with one European.

- It seems that for getting this one needs multifactor models with lower correlations.

\[
\begin{align*}
\frac{dS_{a,b}(t)}{dt} &= \mu_a(t) \, dt + \sigma_a(t) \, dW_a(t), \\
\frac{dS_{a+1,b}(t)}{dt} &= \mu_{a+1}(t) \, dt + \sigma_{a+1}(t) \, dW_{a+1}(t), \\
& \vdots \\
\frac{dS_{b-1,b}(t)}{dt} &= \mu_{b-1}(t) \, dt + \sigma_{b-1}(t) \, dW_{b-1}(t).
\end{align*}
\]
However, when pricing Bermudans…

It is **SERIAL** correlations that matter, not **INSTANTANEOUS** correlations.
Low factor vs Multifactor

- **Serial correlations** depend not only on instantaneous correlations but also on **expiry time** and **volatility**…

\[ R_{i,j} \approx \frac{\int_0^{T_i} \sigma_i(t) \sigma_j(t) \rho_{i,j} \, dt}{\sqrt{\int_0^{T_i} \sigma_i^2(t) \, dt} \sqrt{\int_0^{T_i} \sigma_j^2(t) \, dt}} \]

- These can be controlled also in models with a lower number of factors.

- They need not be 1 even in a one-factor model, particularly if we consider one-factor models more advanced than Vasicek.
Throwing away a billion dollar

A more general one factor model may be just a collapse of the previous multifactor model.

\[
\begin{align*}
    dS_{a,b}(t) &= \mu_a dt(t) + \sigma_a dW(t), \\
    dS_{a+1,b}(t) &= \mu_{a+1} dt + \sigma_{a+1} dW(t), \\
    &\vdots \\
    dS_{b-1,b}(t) &= \mu_{b-1} dt + \sigma_{b-1} dW(t).
\end{align*}
\]

where we are still in a one-factor model. Thanks to choosing one single \(W(t)\) and flat volatilities: \(\sigma_i(t) = \sigma_i\) the entire term structure can still be seen as a function of the one factor \(W(t)\).
Throwing away a billion dollar

In this model we have

\[ R_{i,j} = \frac{\int_{0}^{T_i} \sigma_i \sigma_j 1 \, dt}{\sqrt{\int_{0}^{T_i} \sigma_i^2 \, dt} \sqrt{\int_{0}^{T_j} \sigma_j^2 \, dt}} \]

\[ = \frac{T_i \sigma_i \sigma_j}{\sigma_i \sqrt{T_i \sigma_j} \sqrt{T_j}} = \frac{\sqrt{T_i}}{\sqrt{T_j}}, \]

which is lower than perfect correlation 1.

There is a further element of realism that we can introduce, gaining control of serial correlations while remaining in a one factor model. It is the introduction of time-dependent volatility in the following way:
Throwing away a billion dollar

\[ dS'_{a,b}(t) = \mu_a dt(t) + \sigma_a v(t) \, dW(t), \]
\[ dS'_{a+1,b}(t) = \mu_{a+1} dt + \sigma_{a+1} v(t) \, dW(t), \]
\[ \vdots \]
\[ dS'_{b-1,b}(t) = \mu_{b-1} dt + \sigma_{b-1} v(t) \, dW(t). \]

where \( v(t) \) is a time-dependent deterministic function common to all rates. Thus the term structure can remain driven by 1 factor,

\[ \int_0^T v(t) \, dW(t) \]

but now the serial correlation is

\[ R_{i,j} = \frac{\int_0^{T_i} \sigma_i \sigma_j v^2(t) \, dt}{\sqrt{\int_0^{T_i} v^2(t) \, dt}} \frac{\sqrt{\int_0^{T_j} v^2(t) \, dt}}{\sqrt{\int_0^{T_j} v^2(t) \, dt}} = \frac{\sqrt{\int_0^{T_i} v^2(t) \, dt}}{\sqrt{\int_0^{T_j} v^2(t) \, dt}} \]
Throwing away a billion dollar

If one observes the historical behaviour of interest rates, one will see that the time-dependent part of interest rate volatility is dominated by an increasing behaviour: the more rates get close to their fixing, the more they are influenced by fast-changing short term expectations, becoming more volatile.

In a calibration to swaptions, an important part of correlations, also terminal and serial, of the model are fixed with limited indetermination. Thus, tests that alter correlation or volatility without recalibrating european swaptions and show big bermudan swaption discrepancies may be of limited practical relevance.

<table>
<thead>
<tr>
<th>maturity-dependent vol</th>
<th>mc-low</th>
<th>mc-mid</th>
<th>mc-up</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1,300,000.00</td>
<td>1,340,000.00</td>
<td>1,380,000.00</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>time-to-mat-dependent vol</th>
<th>mc-low</th>
<th>mc-mid</th>
<th>mc-up</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1,000,000.00</td>
<td>1,030,000.00</td>
<td>1,060,000.00</td>
</tr>
</tbody>
</table>
**Low factor vs Multifactor**

With some more modern tools we can compare models with different numbers of factors both perfectly calibrated to swaptions. We see little price differences, confirming that in this case calibration to swaptions reduces model indetermination on Bermudans.

<table>
<thead>
<tr>
<th>high-rank $\rho$, calibrated vol</th>
</tr>
</thead>
<tbody>
<tr>
<td>mc-low</td>
</tr>
<tr>
<td>1,180,000.00</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>low-rank $\rho$, calibrated vol</th>
</tr>
</thead>
<tbody>
<tr>
<td>mc-low</td>
</tr>
<tr>
<td>1,210,000.00</td>
</tr>
</tbody>
</table>
Evolution of the Term Structure of Smiles

- However, nowadays the price of a Bermudan depends also on the implied evolution of the smile, that we can calculate.
Comparing models: an example for Equity
Local vs Stochastic Volatility Models

Stochastic Volatility
\[
dS = rSdt + \sigma \sqrt{V} SdW \\
dV = k(V - \theta) dt + \nu \sqrt{V} dZ
\]

Local Volatility
\[
dS(t) = r(t) S(t) dt \\
+ \Sigma(S(t),t) S(t) dW^s(t)
\]

We can force them to have the same Marginal distributions
\[
Pr(S(T) \in [x, x + dx] | S(0))
\]

But transition distributions still differ strongly
\[
Pr(S(T_2) \in [x_2, x_2 + dx] | S(T_1) = x_1)
\]
The simplest example on marginal and transition densities

**Sort of Stoch Vol?**
\[
\begin{align*}
X^A(0) &= X(0) \\
X^A(t_1) &= \mathcal{N}(0, 1) \\
X^A(t_2) &= \mathcal{N}(0, 1) \perp X^A(t_1) \\
X^A(t_3) &= \mathcal{N}(0, 1) \perp X^A(t_2) \perp X^A(t_1)
\end{align*}
\]

**Sort of Local Vol?**
\[
\begin{align*}
X^B(0) &= X(0) \\
X^B(t_1) &= \mathcal{N}(0, 1) \\
X^B(t_2) &= X^B(t_1) \\
X^B(t_3) &= X^B(t_2) = X^B(t_1)
\end{align*}
\]

**Same Marginals**

**Totally different transition densities**
How relevant it is? What the market believes, and what in reality?

- Less than 2% difference on compound option (1 bivariate transition density)
- More than 50% difference on barrier options (many multivariate transition densities)

- Local Volatility used to be market standard, shaken by some results (Hull and Suo on asset-volatility correlation, Hagan et al. on hedging)
- Stochastic Volatility is not opposite assumption. Models admitting jumps have a lot to say on this model risk.
Stress Testing

- **Stress-testing**
  - **Stressability**: Verify if the model can be used to express stress-conditions
  - **Stress-testing implementation**: detect what can impair the precision of approximations and numerics
    - for market conditions
    - for payoff features
  - **Stress testing assumptions**: subject the model to stress conditions to limit and monitor its application
    - for market conditions
    - for payoff features

- **Model Evolution**: analyze how the current market consensus (or lack of it) could change in the future and how this affects the model use.

- Review Model Reserve/Limits, set restrictions - monitoring (triggers)
Modelling default dependence. The correlation mistake
The market model for Credit Correlation

- We have seen that for the standard credit (CDS) model we have

\[ \tau = \inf \left\{ t : \int_0^t \lambda(s) \, ds \geq \varepsilon \right\} \]

that for flat intensity simplifies to

\[ \tau = \frac{\varepsilon}{\lambda} \]

- How to apply this model to multiname credit products like First/Last-to-default and CDO? The market choice was to use a copula to link the \( \varepsilon \)'s of different obligors

\[ \tau_1 = \frac{\varepsilon_1}{\lambda_1}, \quad \tau_2 = \frac{\varepsilon_2}{\lambda_2} \]

\[ F(\tau_1, \tau_2) = GaussCopula(\varepsilon_1, \varepsilon_2 | \rho) \]

- Copulas were chosen since they provide "a way to analyze the dependence structure of multivariate distributions without studying marginal distributions“ (Bouyè et al., 2000). They were used by banks with a common, flat correlation.
The Correlation Skew

Gaussian Copula in the market is used with flat correlation: equal for all couples of names.

This "model" fit the CDO tranche market only by changing the correlation number for different tranches. This is no real model, and such one-tranche/one-parameter approach makes it difficult to design a consistent stress-scenario for the market.
There exists more complex models that fit the correlation skew with one consistent set of parameters (first one is Hull and White's Perfect Copula). These models show that the implied probability density of the Loss of the i-Traxx Index has some distinctive bumps.
The Loss density implied by market data is multimodal, with bumps in the far tail. It is considered incompatible with gaussian copula.

The problem may be the way copula is used. Can we relate the existence of the correlation skew to the obvious fact that $\rho_{ij} \neq \rho$?
A realistic correlation structure

- A model with a flat correlation around an average level tends to miss:

  A. Some pairs of names are characterized by a very high correlation ($\rho_{ij}=1$ in the most extreme case)
  B. Some pairs of names are characterized by a very low correlation ($\rho_{ij}=0$ in the most extreme case)

- Notice that clusters of high correlations make **big losses and small losses** (far from average) **more likely**, while clusters of low correlations make **losses around the expected loss more likely**
A realistic correlation structure

Take a simple pool with two classes of names:

A. a smaller group (40%) of riskier names, $\lambda = 0.015$, quite loosely related to any other (their risk is mostly idiosyncratic),
B. a larger group of more senior names, $\lambda = 0.005$, that we expect to default only in case of a systemic (or sector-wide) crisis.

We assign correlation 20% to riskier names and 100% for senior names. The distribution is:
A realistic correlation structure

- Can we go beyond this simple trick and use heterogenous correlation to fit the market? In the simple example we made correlation a function of default risk:
  - the risk of subinvestment grade borrowers usually comes from idiosyncratic, or firm-specific, risk factors
  - the risk of senior borrowers usually comes from the risk of more systemic crisis.

  “The loans with the highest predictable losses have the lowest sensitivity to macroeconomic conditions. Conversely the lowest risk loans have the highest sensitivity to macroeconomic conditions (Breeden 2006).”

- This works well also for companies, including financials that are very sensitive to global economic. In spite of many exceptions (fallen angels, business links) this is a reasonable starting point.
A realistic correlation structure

- We start from the simplest gaussian framework: One Factor Gaussian Copula:
  \[ X_i = \rho_i M + \sqrt{1 - \rho_i^2} Y_i \quad i = 1, \ldots, n \]
  where M and the Y_i are independent standard gaussian, so that
  \[ \rho_{ij} = \text{Corr} (X_i X_j) = \rho_i \rho_j. \]

- If \( \rho_i, \rho_j \) are high, then \( \rho_{ij} \) will be high too, while if either is low the resulting correlation \( \rho_{ij} \) will be even lower, and very low when \( \rho_i \approx 0, \rho_j \approx 0 \)

- We now take
  \[ \rho_i = f (\text{Spread}_i). \]
  with
  \[ \text{Spread}_i < \text{Spread}_j \implies \rho_i = f (\text{Spread}_i) > \rho_j = f (\text{Spread}_j) \]
Parameterizing Correlations

- We want the parameterization to be sufficiently general to allow for correlation factors as high as $\rho_i = 1$ and as low as $\rho_i = 0$. This can be obtained with a 3-parameter constrained polynomial (see Duminuco and Morini (2008)).

On February 15, 2008, this results in the following dependence of the correlation factors $\rho_i$ on the spreads of the i-Traxx names:

![Parametric correlation factors as a function of spreads - i-Traxx](image)
Parametric Correlation: the resulting Correlation Matrix

- This parameterization corresponds to the heterogenous correlation:
Parametric Correlation: Implied Correlation Skew

- This correlation implies a rather realistic correlation skew, consistent with market correlation skew of February 15.

One can take more factors (e.g. sectors) or scenarios to increase flexibility.
The implied Loss distribution can be easily computed, and it has the desired multimodal structure, although we are still in a standard Gaussian Copula framework.
Pitfalls in stress testing: default correlation and loss concentration
Stress testing payoff: loss concentration in copulas

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- Copulas were chosen since they provide "a way to analyze the dependence structure of multivariate distributions without studying marginal distributions" (Bouyè et al., 2000).
Portfolio credit derivatives and correlation

- What does this imply for a credit derivative, for example for a senior CDO tranche or for the similar last-to-default, that pays only when all names in a portfolio default?
- First, it implies that the value of protection (riskiness for protection seller) grows with gaussian copula correlation $\rho$.
- The value is always $\leq$ than the value of one CDS, and it grows with $\rho$ until at $\rho = 1$ it equals the least risky CDS. With two names, at $\rho = 1$: $\varepsilon_1 = \varepsilon_2 = \varepsilon$.
- With two obligors and unit correlation, the last default is always the default of name 2, so it has the value of a CDS on 2, its maximum possible value.
The meaning of correlation in a gaussian copula

- Two observations are now in order:

  1) Obligors with very different default risk can have default correlation 1
  2) Even when correlation is 1, the default times can be very different

  This **contradicts** the general wisdom that

  "**default correlation is the probability that the default of a loan occurs simultaneously with the default of another**"

  **Oyama (2010)**

  Can this be misleading is some relevant financial computation?

  ➢ A company is exposed to losses from default of two names, with

  \[ \lambda_1 = 0.03, \quad \lambda_2 = 0.2 \]

  and it is worried about the probability that they **both** default in a particular **quite short period** of time, say in \([3y,5y]\), when company will be short of liquidity. Not such bizarre worry now… It can buy protection from this loss concentration with a forward-start last-to-default.
A paradox in market copulas: missing concentration risk

- The value of this protection depends on the probability of joint default:

\[ P\left(3y \leq \tau_1 \leq 5y, 3y \leq \tau_2 \leq 5y\right) \]

- We may expect this probability to go up with correlation, as it always happen in spot-starting last to default…

For a general interval in time and general default risks, the probability of having joint losses can go down with default correlation!

Very little link between copula parameter and default timing…
A mathematical explanation of the paradox

- We can understand mathematically this behaviour. In the previous chart we have
the probability of joint default in [3y,5y] for two investments with default probability

\[ \lambda_1 = 0.03, \quad \lambda_2 = 0.2 \]

- When correlation is 0, the probability that the two names default together is

\[ \Pr(3 \leq \tau_1 \leq 5) \times \Pr(3 \leq \tau_2 \leq 5) = \]

\[ = 0.053 \times 0.181 = 0.01 \]

- When correlation is 1,

\[ \tau_1 = \frac{\varepsilon}{\lambda_1} = \frac{\varepsilon}{0.03} \quad \text{and} \quad \tau_2 = \frac{\varepsilon}{\lambda_2} = \frac{\varepsilon}{0.2}, \]

so

\[ 3 \leq \tau_1 \leq 5 \quad \text{means} \quad 0.09 \leq \varepsilon \leq 0.15 \]

while

\[ 3 \leq \tau_2 \leq 5 \quad \text{means} \quad 0.6 \leq \varepsilon \leq 1 \]

so it is impossible that the two names default in the same period. Then correlation
decreases monotonically between these two extremes.
A paradox in market copulas

- When the deal is spot starting, or when the obligors have the same risk, the probability of having joint losses goes up with default correlation, as one would expect.

Copula does not really control dependence irrespective on individual default probabilities. They are crucial for probability of joint losses.
Dynamic VAR and CDS counterparty risk. What does $\rho$ mean?

- In the previous chart the worst case typical of spot-starting intervals or homogenous risk ($\rho=1$) is actually the best case. But the situation can be even more confuse. For example the probability of just having defaults within three months of each other, that can arise in dynamic credit VAR or in CDS counterparty risk (see Brigo and Chourdakis 2008), has the following dependence on correlation:

How to guess that worst case correlation is 0.7?
Alternative and benchmark for model risk: Marshall-Olkin

- The Gaussian Copula looks like a black box in these credit applications. You put in intensities and correlations, and you get a model where it is difficult to understand and anticipate how the timing of the defaults is regulated.

- In the Marshall-Olkin model from 1967, there is a physical representation of the happening of default events: for two names there are 3 poisson processes and one of them regulates default dependency, making them default together.

  - The relation between the jump intensity of this process and the probability of joint losses is always an increasing one, no matter the interval considered and the risks of the obligors.
A remark on Correlation risk
Today’ Correlation errors

- Here the error was in replacing correlations $\approx 1$ and $\approx 0$ with an intermediate correlation.

- The most typical correlation errors are opposite, as we mentioned when speaking of level 3 prices:

  - **Zero-correlation errors** (*underestimation of a high correlation*): different examples can be seen in wrong-way risks which are missed in computing CVA

  - **One-correlation errors** (*introducing strong links that may not exist*): this is the mistake we make today when we model double-curves with Libor as a deterministic spread over OIS. We are implying perfect correlation between discounting and forwarding curves, a hypothesis easily falsified when looking at historical behaviour. It leads in particular to hedges that leave basis risk unhedged.
Technical errors when dealing with correlations

- These are errors in our correlation assumptions. Other errors come from lack of technical tools.
- For example with stochastic volatility models for equity baskets we have in the model a complex correlation structure between assets and the volatility factor ($S$):

\[
\rho = \begin{bmatrix}
\rho_{\text{Asset}} \\
\left(\rho_{\text{Asset}/Vol}\right)'
\end{bmatrix}
\begin{bmatrix}
\rho_{\text{Asset}/Vol} \\
1
\end{bmatrix}.
\]

- If the different correlation bits are not modelled consistently, total correlation will be meaningless (and not positive semidefinite)
- This can be addressed using parameterizations that put the minimal constraint to make different bits consistent (e.g. Mercurio e Morini (2006)).
Technical errors when dealing with correlations

\[
\begin{bmatrix}
\theta_{ij} + \rho_i^v \rho_j^v \\
\frac{(\rho^v)'}{1}
\end{bmatrix}
\]

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<th>0.046</th>
<th>0.013</th>
<th>0.006</th>
<th>0.003</th>
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## Model Validation and Model Risk Management

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- Payoff features
- Market conditions
Stress-Testing Numerics. An example on the classic CMS approximation
Stress-testing approximations: CMS classic convexity adjustment

- Consider a constant maturity swap where one leg pays Libor + Spread while the other Leg pays every year a 5y annual swap rate fixing at the time it is paid.

- The payment of a swap rate as if it was a Libor rate is a non-natural payment. This led to the practice of pricing with a Convexity Adjustment. We can understand it mathematically as follows.

\[
CMS = \mathbb{E} \left[ \sum_{i=1}^{n} D(0, T_i) (S_{i\times c}(T_i) - F_i(T_{i-1}) - X) \delta \right]
\]

\[
= \sum_{i=1}^{n} P(0, T_i) \left( \mathbb{E}^{T_i} \left[ S_{i\times c}(T_i) \right] - \mathbb{E}^{T_i} \left[ F_i(T_{i-1}) \right] - \mathbb{E}^{T_i} \left[ X \right] \right) \delta
\]

\[
= \sum_{i=1}^{n} P(0, T_i) \left( \mathbb{E}^{T_i} \left[ S_{i\times c}(T_i) \right] - F_i(0) - X \right) \delta.
\]

Discount factors for n payment dates, 1y-spaced

Swap rates, c=5y

Libor rates

Spreads

Bonds

Not martingale!

martingale

fixed

T_i forward measure. Gets rid of stochastic discount D replaced by bond P

12/01/2014
Massimo Morini, Understanding and Managing Model Risk
The classic convexity adjustment approximation

\[ \mathbb{E}^{T_i} [S_{i \times c}(T_i)] = S_{i \times c}(0) + CA \]

Would be enough if \( S \) was martingale

Required correction, since \( S \) is no martingale

The market approach, formalized by Hagan in 2002, exploits the fact that, using change of measure to the swap measure, we can write

\[ CA = \mathbb{E}^{i \times c} \left[ S_{i \times c}(T_i) \left( \frac{\sum_{j=i+1}^{i+c} P(0, T_j) \tau_j}{P(0, T_i) \sum_{j=i+1}^{i+c} P(T_i, T_j) \tau_j} - 1 \right) \right] \]

Under this measure
It is a martingale

The difficulty is to compute the distribution of this stochastic quantity, while it is not difficult to compute the expectation of the swap rate \( S \), now a martingale. Things would be simpler if this stochastic quantity could be written as a function of \( S \)…
A flat curve at the level of the swap rate

The trick that allows to write

\[ \sum_{j=i+1}^{i+c} P(T_i, T_j) \tau_j, \]

which depends on the various points of the term structure in the interval spanned by the swap, as a function of the swap rate \( S_{i \times c}(T_i) \), is:

**to approximate the real term structure, for the interval spanned by the swap, with a flat term structure at a level equal to the swap rate \( S_{i \times c}(T_i) \) itself.**

This allows to write

\[ P(T_i, T_j) = \frac{1}{(1 + S_{i \times c}(T_i))^{j-i}} \]

and using the formula for the sum of terms in geometric progression,

\[ \sum_{j=i+1}^{i+c} P(T_i, T_j) = \frac{1}{S_{i \times c}(T_i)} \left( 1 - \frac{1}{(1 + S_{i \times c}(T_i))^c} \right) \]
A flat curve at the level of the swap rate

This leads to an easy and popular closed-form formula.

\[ CA \approx S_{a,b}(0) \times \theta(S_{a,b}(0)) \times \left( \frac{\mathbb{E}^{a,b} \left[ S_{a,b}(T_a)^2 \right]}{S_{a,b}(0)^2} - 1 \right) \]

This comes from the approximation

\[ \theta(S_{a,b}) := 1 - \frac{\tau S_{a,b}}{1 + \tau S_{a,b}} \left( \Delta + \frac{b - a}{(1 + \tau S_{a,b})^{b-a} - 1} \right) \]

“Variance” of the swap rate
Comparing with numerical but non-approximated methods

Can you guess why this formula worked wonderfully well for computing the convexity adjustments used in CMS pricing during all the CMS boom of 2005-2007, but was not equally efficient starting in 2008?

The error made with this approximation in the two periods can be seen below, where the approximation is compared with non-approximated montecarlo simulation.

In the next slide you can see the reason for these results.
What to stress-test: the shape of the term structure

Convexity adjustments start being of a relevant size for long maturity and tenor, i.e. 10X20. Here the curve was really almost flat in 2005, and the flat swap rate approximation was almost exact. The slope of the term structure, however, is much more rapid in 2008.
A different use of models, different model risks: Hedging
Hagan et al. introduce SABR model as a solution to the unrealistic behaviour of local volatility models in hedging, proven analytically.

\[
\sigma_F(K) = \text{LocVol} \left( \frac{F + K}{2} \right) \left\{ 1 + \frac{1}{24} \frac{\text{LocVol}''(F+K)}{\text{LocVol}(F+K)} (F-K)^2 + \ldots \right\}
\]
Model Risk in Hedging

- So they introduce SABR, a stochastic volatility model

\[ \begin{align*}
    dF(t) &= V(t) F(t)^\beta dW_F(t), \\
    dV(t) &= \epsilon V(t) dW_V(t), \\
    V(0) &= \alpha,
\end{align*} \]

- Hence, \( \mathbb{E}[dW_V dW_F] = \rho \, dt \)

- But how does this model behave in the above hedging test?
When you take correlation into account, behaviour is analogous to local volatility.
Model Risk in Hedging

- It seems that either one gets the same sort of behaviour of local vol, or, to avoid it, one must abandon the replication of the skew. But...

- It seems that when skew is captured via correlation, then we observe the desired move. But...
Model Risk in Hedging

- But this is now a correlated stochastic volatility model….

\[ dW_V = \sqrt{1 - \rho^2} dZ + \rho dW_F, \]

\[ \mathbb{E}[dW_V \mid dW_F] = \rho dW_F. \]

- We want to assess the effect of a shift of the underlying from 0:05 to 0:051.

\[ \alpha = 0.1, \nu = 0.3, \beta = 1, \rho = -0.7 \]

- The shift in the underlying corresponds to: \[ \Delta W_F = \frac{\Delta F}{\alpha F} = 0.2, \]

\[ \mathbb{E}[\Delta W_V \mid \Delta W_F = 0.2] = \rho \Delta W_F = -0.14 \]
Model Risk in Hedging

- When you forget correlation, SABR seems to behave as desired.

- When you take correlation into account, behaviour is analogous to local volatility.
Model Risk in Hedging

But they are also used inconsistently in the market, with implicit recalibration to ATM options in hedging, and in this case behaviour is the desired one...

SABR works badly when used consistently, but it is easier to use inconsistently. Traders like ‘inconsistent’ use because it corresponds to implicit recalibration in hedging.

So what about local volatility? The theoretical behaviour is not the one desired......
Model Risk in Hedging

- Local and stochastic volatility behaviour do not differ qualitatively when used in hedging.
- They both work well when we introduce and implicit recalibration.
- This is inconsistent with model assumption, but consistent to the market reality of model frequent recalibration.
- What we conclude about the appropriateness of risks of a model in pricing cannot be automatically translated in hedging.
- There models are used differently from their theoretical dynamics, and a different assessment and management of model risk must be performed.
Uncertainty in payout: the default closeout
The default payment: computing the closeout amount

- When a default event happens to one party in a deal, the deal is stopped and marked-to-market: the net present value of the residual part of the deal is computed. This net present value, called closeout amount, is then used to determine the default payments.

- Up to the Lehman default, there was no ambiguity in the above computation: the net present value of the residual deal was usually computed as the expectation of the future payments discounted back by a Libor-based curve of discount factors, consider good both for risk-free collateralized deals and interbank uncollateralized deals.

- Today we are aware that discounting a deal which is default-free should be performed using a curve based on riks-free overnight quotes, whereas a deal which is not collateralized and is thus subject to default risk should be discounted taking liquidity costs into account and should include a counterparty risk adjustment. In case of a default, which net present value should we compute?
Unilateral Counterparty Risk

- Before the credit crunch, the answer was obvious. Default risk regarded mainly deals between a bank and a corporate. The bank had usually a credit quality so higher than the corporate that the bank could safely be assumed to be default-free, and only the risk of default of the corporate was considered. This setting is called ‘unilateral counterparty risk’.

- In such a case the net present value after a default will be computed as if the residual deal was default-free: in fact the only party that can default has just defaulted, while the surviving party has been assumed to be risk-free from the start. We call this assumption a risk-free closeout.
Unilateral Counterparty Risk

- The risk-free net present value of the derivative at \( t \) is

\[
NPV^0_t(t) := \mathbb{E}_t [Cash(t, T)]
\]

- If only the default of counterparty \( C \) is considered, while the investor \( I \) is treated as default free, the price at time 0 is

\[
NPV^C_t(t) = \mathbb{E}_t \left[ 1_{\{\tau^C > T\}} Cash(t, T) \right] + \\
\mathbb{E}_t \left[ 1_{\{\tau^C \leq T\}} Cash(t, \tau^C) \right] + \\
\mathbb{E}_t \left[ 1_{\{\tau^C \leq T\}} D(t, \tau^C) \left\{ Rec \ast (NPV^0(\tau^C))^+ - (NPV^0(\tau^C))^\ominus \right\} \right]
\]

- At default the closeout amount is the risk-free one,

\[
NPV^0(\tau^C)
\]

obviously.
Bilateral Counterparty Risk

- When default of parties is considered, we have in Picoult (2005), Gregory (2009), Brigo and Capponi (2008)

\[
NPV^{C,I}(t) = E_t \{ 1_0 Cash(t, T) \} + \\
+ E_t \left\{ 1_I \left[ Cash(t, \tau^I) + D(t, \tau^I) \left( (NPV^0(\tau^I))^+ - Rec(NPV^0(\tau^I))^\ast \right) \right] \right\} \\
+ E_t \left\{ 1_C \left[ Cash(t, \tau^C) + D(t, \tau^C) \left( Rec(NPV^0(\tau^C))^+ - (NPV^0(\tau^C))^\ast \right) \right] \right\}
\]

- Also with bilateral risk of default the closeout amount is computed treating the residual deal as default-free

\[
NPV^0(\tau^{first})
\]

- Is this assumption as obvious as it was in the unilateral case? Not quite.
Unilateral Counterparty Risk

- Not quite. The second party has not defaulted, and there is a probability that it defaults before the maturity $T$ of the residual deal. Not clear why the mark-to-market of a deal where one party has not yet defaulted and can default in the future should be treated as default free.

- Here the risk-free closeout is not a replacement closeout. In fact if the survived party wanted to substitute the defaulted deal with another one where the market counterparty is default free, the counterparty would ask it to pay not

$$V^{0}(\tau_{first})$$

but

$$V^{Survived}(\tau_{first})$$

because the market counterparty cannot ignore the default risk of the survived party from first default to the maturity.

- This amount will be called in the following replacement closeout.
Replacement closeout

"Upon default close-out, valuations will in many circumstances reflect the replacement cost of transactions calculated at the terminating party's bid or offer side of the market, and will often take into account the credit-worthiness of the terminating party" 


Analogously the ISDA Close-out Amount Protocol, published in 2009, says that in determining a closeout amount the information used includes "quotations (either firm or indicative) for replacement transactions supplied by one or more third parties that may take into account the creditworthiness of the Determining Party at the time the quotation is provided"
Replacement closeout

- If we take into account that the survived party has still a default probability at the default time of the counterparty, the expression for credit adjustment changes:

\[
\overline{NPV}_{C,I} (t) = \mathbb{E}_t \{ 1_0 Cash (t, T) \} + \\
+ \mathbb{E}_t \left\{ 1_I \left[ Cash (t, \tau^I) + D (t, \tau^I) \left( (NPV^C (\tau^I))^+ - Rec (NPV^C (\tau^I))^- \right) \right] \right\} \\
+ \mathbb{E}_t \left\{ 1_C \left[ Cash (t, \tau^C) + D (t, \tau^C) \left( Rec (NPV^I (\tau^C))^+ - (NPV^I (\tau^C))^- \right) \right] \right\}
\]

and this change has a relevant impact.

Here we are taking into account the residual default probability of the survived party. The defaulted party is instead assumed to be replaced by a default-free one. See later on this.
Consider the simplest payoff possible: a party B (borrower) promises to pay a unit payoff \((1)\) at \(T\) to a counterparty L (lender), in return for the net present value of 1 to be lent to him at time 0.

Even such a simple payoff has a different price depending on the closeout conventions…
Risk-free vs Replacement closeout

With few passages we get eventually with a replacement closeout:

\[ V_A^{\text{Replacement}}(t) = e^{-\int_t^T r(s) ds} \left[ \Pr_t(\tau^B > T) + R^B \Pr_t(\tau^B \leq T) \right]. \]

- A risk-free bond
- When borrower survives
- Discounted Recovery when borrower defaults

With similar techniques, we get with a risk-free closeout:

\[ V_A^{\text{RiskFree}}(t) = e^{-\int_t^T r(s) ds} \left( \Pr_t[\tau^B > T] + \Pr_t[\tau^A < \tau^B < T] + R^B \Pr_t[\tau^B < \min(\tau^A, T)] \right). \]

- A risk-free bond
- When borrower survives
- Here we have a risk-free bond also when lender defaults first
- Discounted Recovery when borrower defaults first
Value of a 'bond' with substitution closeout
The case of a bond payoff

Value of a 'bond' with risk-free closeout

Borrower spread

Lender Spread
Risk-free vs Replacement closeout

With a risk-free closeout, we have the paradox that the price is affected by the default probability of a party who has nothing to pay in the deal.

With a simple credit model we can get a numerical example:

\[
\Pr (\tau_X > T) = e^{-\lambda_X T},
\]

Assuming independence of the two default times, we get

\[
\Pr \left( \min \left( \tau^A, \tau^B \right) > T \right) = \Pr (\tau^A > T) \Pr (\tau^B > T) = e^{-(\lambda_A+\lambda_B)T}
\]

and

\[
\Pr (\tau^A < \min (\tau^B, T)) = \frac{\lambda_A}{\lambda_A + \lambda_B} \left( 1 - e^{-(\lambda_A+\lambda_B)T} \right),
\]

as one can easily show by solving the integral

\[
\Pr \left( \{\tau^B > \tau^A\} \cap \{T > \tau^A\} \right) = \int_0^T \Pr (\tau^B > t) \Pr (\tau^A \in dt) = \int_0^T e^{-\lambda_B t} \lambda_A e^{-\lambda_A t} dt
\]

We have also the special case

\[
\Pr (\tau^A < \tau^B) = \frac{\lambda_A}{\lambda_A + \lambda_B}
\]

which is obtained as a limit case of the earlier expression when $T \uparrow \infty$. We have all bits for the above formulas.
Risk-free vs Replacement closeout

\[ \Pr (\tau^{Borrower} \leq 5y) = 63.2\%. \]
\[ \Pr (\tau^{Lender} \leq 5y) = 18.1\%. \]

A risk free bond issuance with the same maturity and notional would cost

\[ P_T = 860.7mn \]

Using the formula with risk-free closeout, we get that a risky bond has price

\[ V_{Lender} = 359.5mn \]

to be compared with the price coming from the formula with replacement closeout

\[ \hat{V}_{Lender} = 316.6mn. \]

- The difference of the two valuations for the above 1bn funding deal is not negligible but not dramatic. More relevant is the difference of what happens in case of a default under the two assumptions on closeout.
Risk-free vs Replacement closeout

\[ \Pr \left[ \tau^{Borrower} < \min \left( 5y, \tau^{Lender} \right) \right] = 58\%, \]
\[ \Pr \left[ 5y < \min \left( \tau^{Lender}, \tau^{Borrower} \right) \right] = 30\%, \]
\[ \Pr \left[ \tau^{Lender} < \min \left( 5y, \tau^{Borrower} \right) \right] = 12\%. \]

- Let’s see what happens in the last case, when lender defaults first
Risk-free vs Replacement closeout

Suppose the exact day when default happens is in 2.5 years from now. Just before default we have

\[ V_{\text{Borrower}}^{\text{Risk-free}} (T_{\text{Lender}} - 1d) = -578.9 \text{mln} \]

under a risk-free closeout, or

\[ V_{\text{Borrower}}^{\text{Replacement}} (\tau_{\text{Lender}} - 1d) = -562.7 \text{mln} \]

under a replacement closeout.

If there is default, in case of a replacement closeout the liability remains \(-562.7 \text{mln}\).

In case of a risk-free closeout, the value of the bond becomes the value of a risk free bond,

\[ V_{\text{Borrower}}^{\text{Risk-free}} (\tau^A + 1d) = -927.7 \text{mln} \]

The borrower must pay this amount entirely to the lender’s liquidators. He has a sudden loss of 348.8mln.
Payoff Uncertainty: Closeout and Contagion

\[ -\Pr (\tau^{Bor} > T) e^{-rT} \]
replacement closeout

\[ -\Pr (\tau^{Len} < \tau^{Bor} < T) e^{-rT} \]
risk-free closeout

\[ -\Pr (\tau^{Bor} > T) e^{-r(T-\tau^{Len})} \]
replacement closeout

\[ -e^{-r(T-\tau^{Len})} \]
risk-free closeout
Replacement closeout

- The increase will be larger the larger the credit spreads of the debtor. This is a dramatic surprise for the debtor that will soon have to pay this increased amount of money to the liquidators of the defaulted party. There is a true contagion of a default event towards the debtors of a defaulted entity, that does not exist in the bond or loan market. Net debtors at default will not like a risk-free closeout. They will prefer a replacement closeout, which does not imply a necessary increase of the liabilities since it continues taking into account the credit-worthiness of the debtor also after the default of the creditor, consistently with what happens in the bond or loan markets.

- Additionally the replacement closeout has a crucial property: if one of the two parties in the deal has no future obligations, like a bond or option buyer, his default probability does not influence the value of the deal at inception. Instead, the risk-free closeout implies a dependence also on the credit risk of the lender, in spite of the fact that the lender has no obligation in the deals.
Risk-free closeout

- On the other hand, the replacement closeout has shortcomings opposite to those of the risk-free closeout. While the replacement closeout is preferred by debtors of a defaulted company, symmetrically a risk-free closeout will be preferred by the creditors. The more money debtors pay, the higher the recovery will be.

- The replacement closeout, while protecting debtors, can in some situations worryingly penalize the creditors by abating the recovery. Consider the case when the defaulted entity is a company with high systemic impact, so that the credit spreads of the counterparties are expected to increase at default. Under a replacement closeout, the market value of their liabilities will be strongly reduced, since it will take into account the reduced credit-worthiness of its debtors themselves.

- All the claims of the liquidators towards the debtors of the defaulted company will be deflated, and the low level of the recovery may be again a dramatic surprise, but this time for the creditors of the defaulted company.
Payoff Uncertainty: Closeout and Contagion

- \(- \Pr (\tau^{Bor} > T) \ e^{-rT}\) for a risk-free closeout.
- \(- \Pr (\tau^{Bor} > T) \ e^{-r(T-t)}\) for a strong default dependency with replacement closeout.
- \(- \Pr_{\tau^{Len}} (\tau^{Bor} > T) \ e^{-r(T-T^{Len})}\) for replacement closeout.
- \(-e^{-r(T-T^{Len})}\) for risk-free closeout.
After these results were published, various papers are appearing on the issue. Gregory (2012) says that “risk-free closeout is an approximation that makes quantification more straightforward, but the actual payoff is more complex” “Including DVA in the closeout amount seems to make more sense” “risky closeout recently discussed in Brigo and Morini [2011] leads to more intuitive results than risk-free” “risky closeout does seem to be the most common method for terminating transactions at default of one party”.

Gregory (2012)
Even more interesting is the analysis by Dehapiot and Patry (2002). They totally agree with the idea of a replacement closeout, but they go beyond and tackle the issue of what to assume for the party that, in the replacement transaction, replaces the defaulted counterparty.

\[
\overline{NPV}^{C,I}(t) = E_t \{1_0 Cash(t, T)\} + \\
+ E_t \left\{1_I \left[ Cash(t, \tau^I) + D(t, \tau^I) \left((NPV^C(\tau^I))^+ - Rec(NPV^C(\tau^I))^+\right)\right]\right\} \\
+ E_t \left\{1_C \left[ Cash(t, \tau^C) + D(t, \tau^C) \left(Rec(NPV^I(\tau^C))^+ - (NPV^I(\tau^C))^+\right)\right]\right\}
\]

Here we are taking into account the residual default probability of the survived party. This is ok, but why to assume that the defaulted party replaced by a default-free one?

**A solution could be to take the replacement as having the same credit quality as C, as C would surely accept to invest in an asset that has at least the same credit quality as himself.**
After these results were published, various papers are appearing on the issue. Gregory (2012) says that “risk-free closeout is an approximation that makes quantification more straightforward, but the actual payoff is more complex” “Including DVA in the closeout amount seems to make more sense” “risky closeout recently discussed in Brigo and Morini [2011] leads to more intuitive results than risk-free” “risky closeout does seem to be the most common method for terminating transactions at default of one party”. 
Dehapiot and Patry (2002)

Even more interesting is the analysis by Dehapiot and Patry (2002). They totally agree with the idea of a replacement closeout, but they go beyond and tackle the issue of what to assume for the party that, in the replacement transaction, replaces the defaulted counterparty. A solution could be to take the replacement as having the same credit quality as C, as C would surely accept to invest in an asset that has at least the same credit quality as himself.
Model Risk in Arbitrage Trading
Capital Structure Arbitrage with Structural Model

- Here we want to explore the relation between model uncertainty and algorithmic trading the uses models and tries to profit from market arbitrages.
- We focus on the most classic arbitrage trade: capital structure trade, where one tries to profit from allegedly short/medium-term discrepancies between pricing of the value and the health of a company in different but related markets, typically bonds and stocks.
- We consider a modern version where the markets involved are CDS and Equity options, using equity cash data.
- As a tool we use a structural model, implicitly capable to price consistently different parts of a company’s capital structure, specifically we use the Brigo and Morini (2009) tractable model.
Credit model: tractable fit to CDS term structure

In this model the value of the form has standard geometric brownian motion dynamics, but with time-varying parameters,

\[ dV_t = V_t (r_t - q_t) \, dt + V_t \sigma_t \, dW_t \]

while the barrier takes into account that when a company grows its debt will also change, what can remain unchanged is just expected leverage:

\[ H_t = H \exp \left( \int_0^t (r_s - q_s) \, ds \right) \]

\[ = \frac{H}{V_0} E^Q [V_t] \]

With this choices, the model allows for tractable expression for default probability and thanks to time-dependent parameters it allows for perfect fit to a term structure of CDS.
From credit to equity: simple equity dynamics

Moving from credit to equity requires the same funny assumption of Merton: that a company has an expiry date $T$, where

$$
E(T) = 1_{\{V(s) > H(s), 0 \leq s < T\}} (V(T) - D(T))^+.
$$

Different from Merton is just that we took into account early default. Notice that barrier at final day of company can only be its debt so

$$
E(T) = 1_{\{V(s) > H(s), 0 \leq s < T\}} (V(T) - H(T))^+.
$$

$$
E(t) = 1_{\{V(s) > H(s), 0 \leq s \leq t\}} \cdot 
\mathbb{E}_t \left[ 1_{\{V(s) > H(s), t < s < T\}} e^{-\int_t^T r(s) - q(s)ds} (V(T) - H(T))^+ \right].
$$

This is a down and out barrier option! Solving the rather complex pricing problem (Lo et al (2003)) we get a surprisingly simple result for equity

$$
E(t) = 1_{\{V(s) > H(s), 0 \leq s < t\}} (V(t) - H(t))
$$

Where equity does not depend anymore on the funny expiry date!
Equity smile embed in credit model

- With this solution for $V(t)$, there is a tractable expression also for an equity option with Maturity $\hat{T}$ and Strike $\hat{K}$

$$
E_t \left[ e^{-\int_t^{\hat{T}} r(u) du} \left( E \left( \hat{T} \right) - K \right)^+ \right] = 
$$

$$
= 1 \{ V(s) > H(s), 0 \leq s \leq t \} e^{\int_t^{\hat{T}} -q(u) du} \left\{ V(t) N \left( \left( \ln \left( \frac{V(t)}{\hat{K}} \right) + \int_t^{\hat{T}} \frac{\sigma_x(s)^2}{2} ds \right) / \Sigma \right) 
- \hat{K} N \left( \left( \ln \left( \frac{V(t)}{\hat{K}} \right) - \int_t^{\hat{T}} \frac{\sigma_x(s)^2}{2} ds \right) / \Sigma \right) 
- H(t) N \left( \left( \ln \left( \frac{H(t)^2}{V(t)\hat{K}} \right) + \int_t^{\hat{T}} \frac{\sigma_x(s)^2}{2} ds \right) / \Sigma \right) 
+ \hat{K} \left( \frac{V(t)}{H(t)} \right) N \left( \left( \ln \left( \frac{H(t)^2}{V(t)\hat{K}} \right) - \int_t^{\hat{T}} \frac{\sigma_x(s)^2}{2} ds \right) / \Sigma \right) \right\}.
$$

- Notice that $V(t)$ is simple but not lognormal, it has a shifted dynamics leading to an asymmetric smile. It turns out this smile is rather consistent with shapes visible in liquid (short term) equity option markets, allowing model calibration.
Testing the model on real companies: Credit/Equity calibration

Calibrating the Model to Credit and Equity: Market data

The most liquid equity options usually have a short maturity, so we will test the model behaviour when fit to credit and equity using a set of European options with maturity around 1y, and the 1y CDS.

We have 2 parameters

\[ \sigma, H \]

and we fit 6 market data

\[ CDS^{1y}, EqOpt_{K1}^{1y}, \ldots, EqOpt_{K5}^{1y} \]

We have selected two companies with different characteristics: British Petroleum (BP), an example of a company that, in spite of the crisis, is still deemed by the market to be financially solid and with good growth perspectives, and Fiat Spa, that on the other hand experienced a recent run from speculators. We show results for April 6, 2009. Very similar results were obtained for March 19, 2009.
Testing the model on real companies: Credit/Equity calibration

Market Data: BP

<table>
<thead>
<tr>
<th>BP, April 6, 2009</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Equity:</strong></td>
</tr>
<tr>
<td>Equity price: 458.25</td>
</tr>
<tr>
<td>Dividend yield = 6%</td>
</tr>
<tr>
<td>9m expiry option:</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td><strong>Credit:</strong></td>
</tr>
<tr>
<td>Recovery: 40%</td>
</tr>
<tr>
<td>1y CDS spread: 64.7 bps</td>
</tr>
</tbody>
</table>

We calibrate our model using 2 parameters (the 1y volatility of the assets and the initial value of the barrier H) to the above set of 6 market data (CDS spread and equity options, with the equity price as a constraint to be fit exactly by $V(0)$) obtaining
Testing the model on real companies: Credit/Equity calibration

CDS Spread Error: 0 bp

<table>
<thead>
<tr>
<th>Strike</th>
<th>420</th>
<th>440</th>
<th>460</th>
<th>480</th>
<th>500</th>
</tr>
</thead>
<tbody>
<tr>
<td>Implied Vol</td>
<td>1%</td>
<td>0.1%</td>
<td>-0.4%</td>
<td>0.2%</td>
<td>-0.2%</td>
</tr>
</tbody>
</table>

Calibration: BP

BP - Equity Options implied volatility (Dot: Market - Continuous: Model)

Errors are within bid-ask spread.
Testing the model on real companies: Credit/Equity calibration

Market Data: FIAT

For Fiat we have the following market data:

<table>
<thead>
<tr>
<th>Equity:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equity price: 7.215</td>
</tr>
<tr>
<td>Dividend yield=0%</td>
</tr>
<tr>
<td>9m expiry option:</td>
</tr>
<tr>
<td>Strike</td>
</tr>
<tr>
<td>Implied Vol</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Credit:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Recovery: 40%</td>
</tr>
<tr>
<td>1y CDS spread: 1211 bps</td>
</tr>
</tbody>
</table>

Both markets clearly reveal a more stressed situation, particularly according to the credit market. One can see when this dramatic worsening of Fiat credit spread started by looking at the next figure.
Testing the model on real companies: Credit/Equity calibration

FIAT: recent history in the credit market

Fiat CDS

Fiat Historical Data
Testing the model on real companies: Credit/Equity calibration

**Calibration: FIAT**

CDS Spread Error (Market-Model): 126bp

<table>
<thead>
<tr>
<th>Strike</th>
<th>6.8</th>
<th>7</th>
<th>7.2</th>
<th>7.4</th>
<th>7.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Implied Vol</td>
<td>-3.2%</td>
<td>-3.2%</td>
<td>-3%</td>
<td>-2.8%</td>
<td>-2.5%</td>
</tr>
</tbody>
</table>

Fiat - Calibration to Equity and Credit - Equity Options implied volatility
(Dot: Market - Continuous: Model)
Testing the model on real companies: Credit/Equity calibration

FIAT: calibration focused on credit

Calibration is bad. In particular, the model underestimates credit spreads and overestimates implied volatility. It is then interesting to see which level of implied volatility the model predicts if we force good fit to the CDS spread leaving equity volatilities to be determined by credit calibration:

Calibration Results

CDS Spread Error (Market-Model): 2.6bp

<table>
<thead>
<tr>
<th>Strike</th>
<th>6.8</th>
<th>7</th>
<th>7.2</th>
<th>7.4</th>
<th>7.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Implied Vol</td>
<td>-6.7%</td>
<td>-6.7%</td>
<td>-6.4%</td>
<td>-6%</td>
<td>-5.7%</td>
</tr>
</tbody>
</table>
Testing the model on real companies: Credit/Equity calibration

**FIAT: calibration focused on equity**

Now we see what happens if we force good fit to equity options leaving the credit spread to be determined by equity calibration.

**Calibration Results**

CDS Spread Error (Market-Model): 225bp

<table>
<thead>
<tr>
<th>Errors (Market-Model) on 9m expiry option:</th>
<th>6.8</th>
<th>7</th>
<th>7.2</th>
<th>7.4</th>
<th>7.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strike</td>
<td>-0.4%</td>
<td>-0.5%</td>
<td>-0.3%</td>
<td>-0.1%</td>
<td>-0.1%</td>
</tr>
<tr>
<td>Implied Vol</td>
<td>-0.4%</td>
<td>-0.5%</td>
<td>-0.3%</td>
<td>-0.1%</td>
<td>-0.1%</td>
</tr>
</tbody>
</table>
Testing the model on real companies: Credit/Equity calibration

FIAT: calibration focused on equity

Fiat - Calibration focused on Equity - Equity Options implied volatility
(Dot: Market - Continuous: Model)

A good fit to equity options implies a CDS which is 225 bp lower than the prevailing market CDS spread. This inconsistency is revealed by the very same model that for BP explained both markets with small errors. Is this a classic case of capital structure arbitrage?
Capital structure arbitrage or wrong model?

- We are making the fundamental assumption that the default threshold is a deterministic, known function of time.
- Very often this assumption does not hold (Parmalat is a dramatic example - fraud - but this happened for product opacity and illiquidity also with Lehman).
- Public investors have only a partial and coarse information about the true value of the firm assets or the related liability-dependent firm condition that would trigger default.
- $H$ in our models is the ratio between the initial level of the default barrier and the initial value of company assets.
- To take market uncertainty into account in a realistic and albeit simple manner, $H$ is replaced by a random variable assuming different values in different scenarios.
Testing the model on real companies: Credit/Equity calibration

Scenario Barrier Time-Varying Volatility Model

The assets value $V_t$ risk neutral dynamics is

$$dV_t = V_t (r_t - q_t) dt + V_t \sigma_t dW_t$$

The default time $\tau$ is the first time where $V_t$ hits above the scenario barrier

$$H^I_t = \frac{H^I}{V_0} \mathbb{E}^Q [V_t] \exp \left( -B \int_0^t \sigma_s^2 ds \right),$$

where $\mathbb{Q} (I = i) = p_i$, $p_i \in [0, 1]$, $i = 1, 2, \ldots, N$, and $\sum_{i=1}^N p_i = 1$, so

$$H^I = \begin{cases} H^1 & \text{with probability } p_1 \\ H^2 & \text{with probability } p_2 \\ \vdots & \vdots \\ H^N & \text{with probability } p_N \end{cases}$$
Testing the model on real companies: Credit/Equity calibration

Calibration: BP. Introducing uncertainty in the barrier

We first apply this model to BP. We consider one Optimistic scenario \((H_2, p_2)\) and a Pessimistic one \((H_1, 1 - p_2)\). The results are

**Calibration Results**

CDS Spread Error: 0 bp

<table>
<thead>
<tr>
<th>Strike</th>
<th>420</th>
<th>440</th>
<th>460</th>
<th>480</th>
<th>500</th>
</tr>
</thead>
<tbody>
<tr>
<td>Implied Vol</td>
<td>0.9%</td>
<td>0.0%</td>
<td>-0.5%</td>
<td>0.2%</td>
<td>-0.0%</td>
</tr>
</tbody>
</table>

The addition of two parameters (one level of the barrier \(H_2\) and its probability \(p_2\)) do not improve the situation considerably. In fact only one scenario appears relevant: \(p_2 \approx 0\). Uncertainty in the company liabilities does not seem crucial to fit jointly BP equity and credit data. This is done easily in the more parsimonious model where the default barrier is deterministic.
Testing the model on real companies: Credit/Equity calibration

FIAT Calibration: introducing uncertainty in the barrier

When the Equity-Credit Calibration of FIAT is performed the results are very different. If we assume two-scenario uncertainty, we obtain now that both scenarios are relevant:

\[
\begin{align*}
H_1/V(0) &= 0.89 & p_1 &= 0.7 \\
H_2/V(0) &= 0.2 & p_2 &= 0.3
\end{align*}
\]

which means that we have a pessimistic scenario (1), more likely, and an optimistic scenario (2), less likely.
Testing the model on real companies: Credit/Equity calibration

**FIAT Calibration: introducing uncertainty in the barrier**

CDS Spread Error (Market-Model): 0bp

<table>
<thead>
<tr>
<th>Strike</th>
<th>6.8</th>
<th>7</th>
<th>7.2</th>
<th>7.4</th>
<th>7.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Implied Vol</td>
<td>-0.11%</td>
<td>-0.14%</td>
<td>-0.1%</td>
<td>-0.0%</td>
<td>0.1%</td>
</tr>
</tbody>
</table>

**Fiat - Calibration to Equity and Credit - Equity Options implied volatility Model with barrier uncertainty (Dot: Market - Continuous: Model)**
A better model for Fiat in the crisis…and before?

- Although, due to uncertainty on the default barrier, this model has 4 rather than 2 parameters, the model is still parsimonious since we are fitting 6 market quotes (CDS spread and equity options, with equity price as a constraint to be fit exactly).

- Now we have perfect fit, and since perfect fit has been obtained by introducing a relevant element of uncertainty in the model, which is a reasonable assumption that does not make the model overly complex, we consider this second proposal a better representation of Fiat in the crisis.

- Yet, let us see if in March 11, 2008, before the beginning of the run on FIAT, uncertainty is low as the model seems to suggest. Market data are:
**Testing the model on real companies: Credit/Equity calibration**

**FIAT market data before its recent crisis**

<table>
<thead>
<tr>
<th>Fiat, March 11, 2008</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Equity:</strong></td>
</tr>
<tr>
<td>Equity price: 12.588</td>
</tr>
<tr>
<td>Dividend yield=0%</td>
</tr>
<tr>
<td><strong>6m expiry option:</strong></td>
</tr>
<tr>
<td>Strike</td>
</tr>
<tr>
<td>11.33</td>
</tr>
<tr>
<td>12.27</td>
</tr>
<tr>
<td>12.9</td>
</tr>
<tr>
<td>13.85</td>
</tr>
<tr>
<td>Implied Vol</td>
</tr>
<tr>
<td>44.6%</td>
</tr>
<tr>
<td>43.2%</td>
</tr>
<tr>
<td>42.8%</td>
</tr>
<tr>
<td>41.8%</td>
</tr>
<tr>
<td><strong>Credit:</strong></td>
</tr>
<tr>
<td>Recovery: 40%</td>
</tr>
<tr>
<td>1y CDS spread: 173 bps</td>
</tr>
</tbody>
</table>
Testing the model on real companies: Credit/Equity calibration

FIAT Calibration before its recent crisis

The market views on the credit quality of the company appear much better. Calibrating the model with deterministic barrier we obtain:

Calibration Results

CDS Spread Error (Market-Model): 0bp

<table>
<thead>
<tr>
<th>Errors (Market-Model) on 6m expiry options:</th>
</tr>
</thead>
<tbody>
<tr>
<td>6m expiry option:</td>
</tr>
<tr>
<td>Strike</td>
</tr>
<tr>
<td>Implied Vol</td>
</tr>
</tbody>
</table>
Testing the model on real companies: Credit/Equity calibration

Fiat - Calibration to Equity and Credit - Equity Options implied volatility
Model without barrier uncertainty on 11-03-08 (Dot: Market - Continuous: Model)

Differently from what happened after the beginning of the crisis, on these earlier data the simpler model with deterministic barrier can fit exactly the credit spread while implying errors around the bid-ask spread on equity options, similarly to the results obtained for the healthy company BP.
BP in an unexpected crisis

There is an additional test of our conclusion that we can perform. The other company we considered, BP, entered into an unexpected crisis when on April 20, 2010, the BP's Deepwater Horizon drilling rig exploded, killing 11 employees and starting a dramatic oil spill.

This created a real disarrange in the market views about BP's perspectives. BP could be considered responsible for dozens of billions of damages to the environment; but it may also prove to have followed best practice and due diligence, reducing its responsibilities. From being one of the most solid international companies in the world it turned into one with a troubled future.

The value of its assets was certainly high, but the value of its liabilities had become very difficult to assess, making BP a typical example of a company affected by high uncertainty.
Testing the model on real companies: Credit/Equity calibration

Figure 19: BP Historical Data
Testing the model on real companies: Credit/Equity calibration

**BP in its recent crisis**

See below data for the day 17 June 2010, just after BP agreed to pay and initial 20 bn dollar to the US government to repay the damages created by the oil spill.

<table>
<thead>
<tr>
<th>BP, June 17, 2010</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Equity:</strong></td>
</tr>
<tr>
<td>Equity price: 337</td>
</tr>
<tr>
<td>Dividend yield=0%</td>
</tr>
<tr>
<td>9m expiry option:</td>
</tr>
<tr>
<td>Strike</td>
</tr>
<tr>
<td>Implied Vol</td>
</tr>
<tr>
<td><strong>Credit:</strong></td>
</tr>
<tr>
<td>Recovery: 40%</td>
</tr>
<tr>
<td>1y CDS spread: 635 bps</td>
</tr>
</tbody>
</table>
Testing the model on real companies: Credit/Equity calibration

BP in its recent crisis

If the interpretation we gave above is reasonable, after this event it should be very difficult to calibrate jointly to BP CDS and Equity Option with the basic model that assumes a deterministic default barrier. In fact, the results we obtain are as follows

Calibration Results

CDS Spread Error (Market spread minus Model spread): 244 bp

<table>
<thead>
<tr>
<th>Strike</th>
<th>280</th>
<th>320</th>
<th>360</th>
<th>400</th>
<th>440</th>
</tr>
</thead>
<tbody>
<tr>
<td>Implied Vol</td>
<td>-7.2%</td>
<td>-6.6%</td>
<td>-6.3%</td>
<td>-5.2%</td>
<td>-3.7%</td>
</tr>
</tbody>
</table>

In such a situation, a model with explicit uncertainty on the level of liabilities (default barrier) should work better. We have tried this, obtaining the following results.
Testing the model on real companies: Credit/Equity calibration

**BP in its recent crisis**

**Calibration Results**

CDS Spread Error (Market spread minus Model spread): 3 bp

<table>
<thead>
<tr>
<th>Strike</th>
<th>280</th>
<th>320</th>
<th>360</th>
<th>400</th>
<th>440</th>
</tr>
</thead>
<tbody>
<tr>
<td>Implied Vol</td>
<td>-0.4%</td>
<td>-0.9%</td>
<td>-1.3%</td>
<td>-0.6%</td>
<td>0.8%</td>
</tr>
</tbody>
</table>

This has been obtained with the following parameters for the uncertain barrier

\[
\frac{H^1}{V_1} = 0.92 \quad \quad \quad p_1 = 0.52
\]

\[
\frac{H^2}{V_2} = 0.6 \quad \quad \quad p_2 = 0.48
\]

which show uncertainty between a higher level of BP’s liabilities and a lower one, with almost the same probability. In confirmation of the above interpretation, only with these two scenarios we are able to calibrate market data for BP at the peak of the Deepwater Horizon’s oil spill crisis.
Model Arbitrage vs More Uncertainty in the Model

- It is hard to know if two markets have a long-standing relation with temporary variations from it that we can exploit as trading opportunities.
- Certainly, no-arbitrage models do not allow to make such claims. We have seen that an apparently obvious arbitrage disappears immediately if we move to a model which is actually more realistic than the previous one. This model explains the different patterns of BP and Fiat quotes not by saying that there is an arbitrage in the pricing of the different parts of the Fiat's capital structure, but just by saying that there is more uncertainty. One can bet on how and when this uncertainty is going to diminish or grow.
- This is still a trading opportunity, but it is not an arbitrage. Which implies, by the way, that you cannot expect the patterns of Fiat quotations to become closer to those of BP soon, but, in the best case, just sooner or later. It is difficult to figure out market configurations that cannot be explained by some model. A no-arbitrage model cannot, by definition, decide if an inconsistency it detects is a problem in the market (arbitrage) or in the model itself. This is a trader's job, and the model offers no alibi.
Model Risk in the Funding Debate
Collateral

What happens when a deal is collateralized, with no Minimum Transfer Amount, cash collateral, and continuous collateral regulation (approximation of standard collateral)? Market simplification is

Collateral = Exposure

At default the obligations of the parties cancel out

Default risk is eliminated

Coll. Interest = Overnight Rate

We finance the deal with an OIS based account

We discount with OIS
The first approach, suggested also by Piterbarg 2010, was to extend this to non-collateralized deals...

No collateral

Interest=Funding rate

We finance the deal with a funding-based account

We discount with Funding Rate

Default risk is full

However, as revealed by Morini and Prampolini 2011, things are not so simple...
The players

- We consider companies with the following features: company $X$ has instantaneous credit spread $\pi_X$ and funding spread

$$s_X = \pi_X + \gamma_X$$

usually measured in the bond market, and $\gamma_X$ is the liquidity basis.

- With an instantaneous credit spread $\pi_X$, if we assume zero recovery, the survival probability until maturity $T$ can be expressed as

$$\Pr (\tau_X > T) = e^{-\pi_X T}$$
The deal

- We want to price a very simple deal. One entity B (borrower), commits to pay a fixed amount $K$ at time $T$ to a party L (lender). This is the derivative equivalent of a zero-coupon bond issued by B or a loan from L to B. So we will be able to compare the results with well-established market practice.

- In standard credit modelling (Gregory (2009), Brigo (2010)), pricing this deal is simple

\[
P = \mathbb{E} \left[ e^{-rT} 1_{\{\tau_B > T\}} K \right] = e^{-rT} \Pr(\tau_B > T) K = e^{-rT} e^{-\pi_B T} K
\]

CVA/DVA term
Credit and Funding

- In this setting the parties agree on the premium of the deal, but we are not considering explicitly the value of liquidity. In fact at time 0 party B receives a cash flow from party L equal to P, so while party L has to finance the amount until the maturity of the deal at its funding spread, party B can reduce its funding by P.

- So party B should see a funding benefit, and party L a financing costs. How come that these funding components do not appear in the above valuation? The absence of the funding term for L can indeed be justified by assuming \( s_L = 0 \), that implies \( \pi_L = 0 \)

- However the same assumption cannot be made for B without changing completely the nature of the deal. In fact assuming \( s_B = 0 \) would imply \( \pi_B = 0 \) which would cancel the DVA and CVA term. When B has non-negligible risk of default he must have a funding cost given at least by

\[
 s_B = \pi_B > 0
\]

- The effect of this funding costs seems to be missing in the above formula...
To introduce liquidity costs, we follow Piterbarg (2010), that corrects the discounting rate by the funding spread. According to Piterbarg (2010), the funding spread includes the CDS spread (default intensity) of an institution; but he does not consider explicitly default of the two parties. We now instead introduce also defaultability of the payoff, getting for the lender:

$$PL = \mathbb{E} \left[ e^{-(r+s_L)T} K 1_{\{\tau_B > T\}} \right] = \mathbb{E} \left[ e^{-rT} e^{-\gamma LT} e^{-\pi LT} K 1_{\{\tau_B > T\}} \right] = e^{-(r+\gamma_L+\pi_L+\pi_B)T} K$$

and for the borrower

$$PB = -\mathbb{E} \left[ e^{-(r+s_B)T} K 1_{\{\tau_B > T\}} \right] = -\mathbb{E} \left[ e^{-rT} e^{-\pi_B T} e^{-\gamma_B T} K 1_{\{\tau_B > T\}} \right] = -e^{-(r+\gamma_B+2\pi_B)T} K$$
Liquidity by discounting plus Credit

- To compare this result, including CVA, DVA and liquidity from discounting, with results on obtained previously, it is convenient to reduce ourselves to the simplest situation where $L$ is default free and with no liquidity spread, while $B$ is defaultable and has the minimum funding cost allowed in this case:

$$s_L = 0, \quad s_B = \pi_B > 0$$

- Thus for the lender nothing changes, while for the borrower we must multiply by a discount correction factor

$$P_L = e^{-rT}e^{-\pi_B T}K$$
$$P_B = e^{-rT}e^{-\pi_B T}e^{-\pi_B T}K$$

- There are two bizarre aspects in this representation.
  1. Even with no liquidity basis, two counterparties do not agree on the simplest transaction with default risk. A day-one profit should be accounted by all borrowers.
  2. We price a zero-coupon bond by multiplying their risk-free present value by the survival probability twice...
Considering explicitly the funding strategy

- In order to solve the puzzle, we model explicitly the funding strategy. Here companies capitalize and discount money with the risk-free rate $r$, and then add or subtract credit and funding costs.

- The above deal has two legs. For the lender $L$, one is the deal leg, with net present value

$$\mathbb{E} [-P + e^{-rT}G]$$

where $G$ is the payoff at $T$, including a potential default indicator; the other leg is the funding leg with net present value

$$\mathbb{E} [+P - e^{-rT}F]$$

where $F$ is the funding payback at $T$, including a potential default indicator. In the general case the total net present value is

$$V_L = \mathbb{E} [-P + e^{-rT}G + P - e^{-rT}F] = \mathbb{E} [e^{-rT}G - e^{-rT}F]$$
The borrower. Default on the funding strategy

- The borrower B has a liquidity advantage from receiving P, as it allows to reduce its funding requirement by P. This amount of funding would have generated a negative cashflow at T, when funding must be paid back, equal to

\[ -P e^{rT} e^{sBT} 1\{\tau_B > T\} \]

- The outflow equals P capitalized at the cost of funding, times a default indicator \( 1\{\tau_L > T\} \). Why do we need a default indicator? Because in case of default the borrower does not pay back the borrowed funding. Thus reducing the funding by P corresponds to receiving at T a positive amount equal to

\[ P e^{rT} e^{sBT} 1\{\tau_B > T\} = P e^{rT} e^{\pi BT} e^{\gamma BT} 1\{\tau_B > T\} \]

to be added to what B has to pay in the deal. Thus the total payoff at T is

\[ 1\{\tau_B > T\} P e^{rT} e^{\pi BT} e^{\gamma BT} - 1\{\tau_B > T\} K \]

Default on the funding strategy

- Taking discounted expectation,

\[
V_B = e^{-\pi_B T} P e^{\pi_B T} e^{\gamma_B T} - K e^{-\pi_B T} e^{-r T}
\]

- Notice that

\[
V_B = 0 \quad \Rightarrow \quad P_B = K e^{-r + \gamma_B + \pi_B T}
\]

- Assume, as above, that \( \gamma_B = 0 \). In this case the breakeven premium is

\[
P_B = K e^{-\pi_B T} e^{-r T}.
\]

- Taking into account the probability of default in the valuation of the funding benefit shows that there is no pure liquidity charge, and no double counting of survival probability.
The accounting view

- DVA is disturbing since it evaluates as an asset our own default. But see what happens if the borrower pretends to be default-free. In this case the premium P paid by the lender gives B a reduction of the funding payback at T corresponding to a cashflow at T

\[ P e^{rT} e^{sBT} \]

where there is no default indicator because B is treating itself as default-free. This cashflow must be added to the payout of the deal at T, again without indicator. Thus the total payoff at T is

\[ P e^{rT} e^{sBT} - K \]

- This yields an accounting breakeven premium for the borrower equal to the previous breakeven, irrespectively of considering our default or not. If there is no basis:

\[ P_B = K e^{-rT} e^{-\pi_B T} \]

- The borrower recognizes on its liability a funding benefit that takes into account its own market risk of default plus additional liquidity basis.
Avoiding double counting

- Putting credit and liquidity together since deal is not collateralized,

\[ P = e^{-rT} e^{-\pi_B T} e^{-\pi_B T} K \]

- Is this correct? No. We have to take into account explicitly our funding strategy, and the possibility of a default there (Morini and Prampolini (2011)), getting

\[ P = e^{-rT} e^{-\pi_B T} K \]

- And when we introduce the basis, there are surprising consequences also for the funding charge of the lender...
The lender case: foundations of the Funding Debate

- The lender pays \( P \). He needs to finance (borrow) \( P \) until \( T \). At \( T \), \( L \) will give back the borrowed money with interest, but only if he has not defaulted, so the outflow is

\[
P e^{rT} e^{sLT} 1\{\tau_L > T\} = P e^{rT} e^{\pi_LT} 1\{\tau_L > T\}.
\]

The total payoff at \( T \) is

\[
-P e^{rT} e^{\pi_LT} 1\{\tau_L > T\} + K 1\{\tau_B > T\}.
\]

Taking discounted expectation

\[
V_L = -P e^{\pi_LT} e^{-\pi_LT} + K e^{-rT} e^{-\pi_BT} = -P + K e^{-rT} e^{-\pi_BT}
\]

The condition that makes the deal fair for the lender is

\[
V_L = 0 \quad \Rightarrow \quad P_L = K e^{-(r + \pi_B)T}
\]

The lender, when valuing all future cashflows as seen from the counterparties, does not include a charge for the credit component \( \pi_L \) of its own cost of funding, compensated by the fact that funding is not given back in case of default.
DVA or not DVA (of funding strategy)

- If the lender does not take into account its probability of default in the funding strategy, there is no simplification and he gets a different breakeven premium,

\[ P_L = K e^{-rT} e^{-\pi_B T} e^{-\pi_L T} \]

Now the funding spread of the Lender is charged

- What is fair for an external observer, and for the borrower? What is fair is not charging the lender’s funding costs: in fact they are compensated by the probability that the lender’s defaults on funding, and the borrower has nothing to do with the credit risk of the lender that leads to funding costs.

- What is logic for the lender? Certainly charging the funding costs. If he charges only \( \pi_B \) (the credit risk of the borrower) and not \( \pi_L \) (its own credit risk, that leads to its funding costs), in case of no defaults his carry

\[ \pi_B - \pi_L \]

can even be negative. This is not possible when he charges funding costs since its carry becomes

\[ \pi_B + \pi_L - \pi_L = \pi_B \]
The debate on FVA

- So, can we say with Hull & White that
  1. If a dealer takes into account the DVA of the funding strategy, then the FVA disappears, since FVA = DVA of the funding strategy?

  Yes

- So, can we say with Hull & White that
  2. The DVA of the funding strategy is a benefit to shareholders, so it should be taken into account?

  This is really much more controversial...
Is DVA shareholder’s value or company tragedy (or both)

2. The DVA of the funding strategy is a benefit to shareholders, so it should be taken into account

- This “benefit” emerges only in case of a default of a company. It is not clear why a company should consider benefits coming after its death. If there is no death, such an approach would lead to consider beneficial a deal with negative cashflows.

- Yet, even if it seems logic that a company reasons in terms of “going concern” not considering benefits after default, Hull & White actually talk of a benefit to shareholders, not to the company.

- Is the DVA of the funding strategy a benefit to shareholders? Shareholders of a company with Limited Liability, compared to those of a company with Unlimited Liability, hold a sort of call option, that allows them to take all the equity of a company when it is positive, but are not taken responsible when equity is negative (instead, shareholders with Unlimited Liability hold a forward contract). In this sense, the DVA of the funding strategy is a crucial component of the value of this option. A reasoning in line with the decisions of accountancy boards of including DVA in fair value accounting.
The debate on FVA

- So we have two approaches to FVA:

- From inside a company, FVA seems a real cost and must be charged to counterparties

- From outside a company, which includes certainly counterparties and possibly shareholders, FVA cancels out with the DVA of the funding strategy and there is no justification for charging to counterparties

- This misalignment of interest seems really disturbing. Hull & White, however, propose a point of view that justifies reconciles the two views, as we will see.
Market’s Feeback according to Hull and White

- They say that even from an internal company’s perspective there is no FVA cost. In fact, when for example a company that is worth 1bn and has a credit/funding spread of 100bps,

\[
\text{BANK (1bn)}
\]

\[
\begin{array}{c|c}
100bps & \\
credit spread & \\
\end{array}
\]

invests 1 additional bn into a new project or derivative which is risk-free (0bps of credit spread), then the market recognizes that the bank is now

\[
\text{BANK (2bn)}
\]

\[
\begin{array}{c|c}
100bps & 0bps \\
credit spread & credit spread \\
\end{array}
\]

so in terms of funding costs the market will treat it as

\[
\text{BANK (2 bn)}
\]

\[
\begin{array}{c}
50bps \\
credit spread \\
\end{array}
\]

There is no FVA
The debate on FVA

This reasoning leads to say that there is no FVA based on three crucial assumptions

1. The market has *instantaneous* efficiency

2. Funding of a deal happens *after* the market knows about the deal

3. The effect of a new deal on the funding costs of a bank is *linear*
The debate on FVA

This reasoning leads to say that there is no FVA based on three crucial assumptions

1. The market has instantaneous efficiency: this is not the case in the reality of funding markets, although we always use indirectly this assumption in pricing.

2. Funding of a deal happens after the market knows about the deal: this can be true when a project is funded rolling short-term funding, but prudential management includes often part of funding at maturity.

3. The effect of a new deal on the funding costs of a bank is linear: let’s see if this must always be the case.
Linear Funding Feedback?

- Under some assumptions, the effect is actually approximately linear. Consider for example the simplest Merton Model.
Linear Funding Feedback?

- What does it mean here to add a risk-free project (worth in this case around 20% of the firm)? It can be a project whose value never changes (no vol).
- In this case, under 0 rates, default probability is unchanged but recovery increases proportionally. Spreads are reduced linearly by around 20%.
Linear Funding Feedback? First Passage Models

- Let's add some realism. First, we move to default barrier models where there is not one single default date but covenants can lead to earlier default.

- Due to its current asset composition, the bank is very exposed towards its sovereign, which is represented by the two basic scenarios below: the blue one if the sovereign goes well (no default), the red one when sovereign enters a crisis (default when loss of value triggers covenants).

Covenants make bank default when value is 50% of debt.

Default in 5y
Excluding our own default. Internal symmetry

- Now the bank adds a project worth 20% of the company and risk-free, namely without volatility, having maturity of 10y, so that the bank looks for 10y funding.

- As we can see, the only effect of the project is to shift the default time in the bad scenario (sovereign crisis) by 3m. This has no effect on the 10y cost of funding. For decoupling the bank from the sovereign risk on such an horizon, probably the project should be worth more than 100% of the company itself.

![Graph showing default scenarios](image-url)
The debate on FVA

- Thus, even if one believes in instantaneous market efficiency, only under some assumptions a new project has a linear effect on credit spread. Under rather realistic assumptions the effect is highly non-linear.

- Hull and White have the merit of pointing out that FVA is a distortion compared to an efficient market. As seen in Morini 2011, FVA makes deals fair for lenders but can make them unfair for borrowers in a negative spiral of growing funding costs.

- Yet, in the current market situation a dealer following a going concern must take some FVA into account
A way out

- A market where the main lenders are the most risky payer is unavoidably distorted. FVA cannot be neglected by banks and yet is an unjustified burden on the economy.

- Banks can recognize the other side of the distortion, that is funding through central bank facilities and guaranteed deposits.

- This way, FVA can be kept within reasonable size, and theory and practice can agree on this.
Thank you!
The main references are the books:

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