On the calibration of distortion risk measures to bid-ask prices

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Abstract

We investigate the calibration of a non-linear pricing model to quoted bid-ask prices and show the existence of a solution in a broad class of distortion risk measures, following the frameworks of [Cherny and Madan 2010] and [Bannör and Scherer 2011a]. We present an approximation of distortion risk measures by a piecewise linear approximation of concave distortions. This is used to construct a tractable non-parametric calibration procedure to bid-ask prices based on piecewise linear concave distortion functions. For an empirical proof of concept, we calibrate quoted bid-ask prices non-parametrically and w.r.t. parametric families and obtain a jump-linear structure. To conclude, we suggest using the parametric family of ess sup-expectation convex combinations as a suitable family of distortion functions, allowing for fast and efficient calibration.

Keywords

Distorted probabilities, distortion function, non-parametric calibration, conic finance, convex risk measure, spectral risk measure, bid-ask spread
1 Introduction

Since the rise of risk-neutral valuation techniques, approaches to bid-ask price modeling enjoy increasing attention by the derivatives community. Classical results interpret bid-ask prices as a consequence of incomplete markets, interpreting the ask (resp. bid) price as the supremum (resp. infimum) of the expectation w.r.t. all equivalent martingale measures. Meanwhile, the seminal paper [Artzner et. al. 1999] established the fruitful theory of convex risk measures (a standard reference is [Föllmer and Schied 2004]), which turned out to be interesting from both a mathematical and a practical point of view. Since the supremum w.r.t. measures which are absolutely continuous w.r.t. the real-world measure can be interpreted as convex risk measures, several papers endorse to calculate bid-ask prices by the usage of convex risk measures (as a natural generalization to incomplete markets pricing), either in an incomplete markets setting (see [Carr et. al. 2001, Xu 2006]), due to abstract considerations (see [Bion-Nadal 2009]), or in a model/parameter uncertainty setting (see [Cont 2006, Bannör and Scherer 2011a]).

Unfortunately, a concrete implementation of the developed theory immediately invokes the question which risk measure to choose for calculating bid-ask prices. Thus, in most papers, the choice of risk measure is regarded as somewhat subjective.

A more “objective” way of choosing the risk measure is obtained by calibrating model bid-ask prices to given market prices, similar to the standard calibration to mid market prices. [Cherny and Madan 2010] use some parametric calibration scheme to calibrate their conic finance framework to bid-ask prices, focusing on parametric classes of distortion functions. However, restricting bid-ask pricing to a narrow class of parameterized distortion risk measures (characterized by a class of concave distortion functions) may exclude better fitting alternatives, since the shape of the distortion functions is predefined by its parametrization.

In this paper, we close this gap and discuss how one can re-engineer distortion risk measures from quoted bid-ask prices. Thus, we first provide existence theorems, ensuring that the bid-ask calibration problem can be solved in proper domains of distortion functions. In contrast to [Cherny and Madan 2010], we suggest a non-parametric approach for obtaining the market-implied distortion risk measure for pricing, not restricting ourselves to a specific parametric shape of the distortion function. This ansatz is based on a piecewise linear approximation. Our non-parametric approach provides more flexibility for solving the bid-ask calibration problem and is simple to implement. Furthermore, it enables us to empirically observe the shape of possible market-implied distortion functions. We present an empirical analysis in the parameter risk setting established in [Bannör and Scherer 2011a, Bannör and Scherer 2011b], comparing our non-parametric calibration to a calibration within the AVaR- and minmaxvar-families. Interestingly, in the non-parametric approach we obtain a characteristic pattern with a jump close to zero and a linear behavior afterwards. This gives rise to the introduction of an alternative parametric class of distortion functions, the esssup-expectation convex combinations, which allow for much faster calibration than the AVaR-/minmaxvar-families. Further-
more, we enhance the existence theorems to larger families of distortion functions with discontinuities at zero, containing the newly introduced family.

Our paper is organized as follows: First, we recall the necessary background concerning distortion functions and Choquet integrals and summarize some suggestions to bid-ask pricing employing distortion functions. Second, we state the bid-ask calibration problem and prove the existence of a solution in the set of distortion risk measures (under mild technical restrictions, e.g. Lipschitzianity of the corresponding distortion function). Third, we present a numerical ansatz for the calibration problem based on a piecewise linear approximation of distortion functions, enabling us to reduce the bid-ask calibration problem to a constrained optimization problem on the unit cuboid. Finally, we compare the calibration performance of our non-parametric approach to a parametric calibration à la [Cherny and Madan 2010] and introduce a new family of distortion functions based on our calibration result.

2 Background and notation

2.1 Preliminaries from the theory of distortion risk measures

In this section, we recall the crucial definitions and theorems related to distortion risk measures. A standard reference for convex risk measures is [Föllmer and Schied 2004], for details about Choquet integration we refer to [Denneberg 1994].

Definition 2.1 (Distortion function)
Let \( \gamma : [0, 1] \to [0, 1] \). \( \gamma \) is called a distortion function iff \( \gamma \) is monotone, \( \gamma(0) = 0 \), and \( \gamma(1) = 1 \).

The interpretation of distortion functions is as follows: Given a probability space \((\Omega, \mathcal{F}, P)\), instead of measuring the probability of a set \( A \) classically via \( P(A) \), we alternatively consider the distorted probability \( \gamma \circ P \) w.r.t. some distortion function \( \gamma \). Obviously, the set function \( \gamma \circ P : \mathcal{F} \to [0, 1] \) is not a probability measure any more for general \( \gamma \), but preserves monotonicity w.r.t. the set order: \( A \subset B, A, B \in \mathcal{F} \) implies \( \gamma(P(A)) \leq \gamma(P(B)) \). This type of set functions is often referred to as “capacities” (cf. [Föllmer and Schied 2004]). The ordinary Lebesgue integral w.r.t. probability measures can be extended to an integral w.r.t. monotone set functions, the Choquet integral.

Definition 2.2 (Choquet integral w.r.t. distorted probabilities)
Let \((\Omega, \mathcal{F}, P)\) be a probability space and \( \gamma \) a distortion function. Let furthermore \( X : \Omega \to \mathbb{R} \) be a bounded random variable. The Choquet integral of \( X \) w.r.t. \( \gamma \circ P \) is then defined as

\[
\int_{\Omega} X \, d(\gamma \circ P) = \int_{-\infty}^0 \gamma(P(X > x)) - 1 \, dx + \int_0^{\infty} \gamma(P(X > x)) \, dx.
\]
2.1 Preliminaries from the theory of distortion risk measures

The Choquet integral is a monotone, translation invariant, positively homogeneous, and cocomonotonic additive functional and coincides with the Lebesgue integral w.r.t. $P$ for $\gamma = \text{id}_{[0,1]}$. Furthermore, it can be shown (cf. [Denneberg 1994]) that if $\gamma$ is concave, the Choquet integral is a subadditive functional. Thus, Choquet integrals w.r.t. concave distorted probabilities fulfill the axioms of convex risk measures (even coherent risk measures) and are often referred to as distortion risk measures. Some authors (e.g. [Acerbi 2002]) treat so-called spectral risk measures instead, which is a somewhat equivalent approach to distortion risk measures (cf. [Föllmer and Schied 2004, Gzyl and Mayoral 2008]). In this work we always use the notion of distortion risk measures, due to the convenient computability of the Choquet integral.

One of the well-known examples for a distortion risk measure is the celebrated risk measure Average-Value-at-Risk (AVaR), treated, e.g., in [Acerbi and Tasche 2002].

Example 2.3 (Average-Value-at-Risk)
For a probability space $(\Omega, \mathcal{F}, P)$, a given significance level $\alpha \in (0,1]$, and an integrable $X \in \mathcal{L}^1(P)$, the $\alpha$-Value-at-Risk is defined as the upper $\alpha$-quantile $\text{VaR}_\alpha(X) := q_X^P(1-\alpha)$ and the $\alpha$-Average-Value-at-Risk is defined by

$$\text{AVaR}_\alpha(X) := \frac{1}{\alpha} \int_0^\alpha \text{VaR}_\beta(X) \, d\beta.$$ 

It can easily be shown that the $\text{AVaR}_\alpha$ can be represented as a Choquet integral w.r.t. a distorted probability. The corresponding distortion function is given by

$$\gamma_\alpha(u) = \begin{cases} 
\frac{u}{\alpha}, & u \in [0,\alpha] \\
1, & \text{otherwise}
\end{cases}$$

and $\gamma_\alpha$ is obviously concave.

As a more elaborate example, [Cherny and Madan 2009] introduce several parametric families of concave distortion functions and calculate positions w.r.t. the induced distortion risk measures. The following $\min\text{maxvar}$-family of concave distortions is introduced in [Cherny and Madan 2009] and successfully employed in [Cherny and Madan 2010].

Example 2.4 ($\min\text{maxvar}$-family of concave distortions)
Let $\psi_x(y) : [0,1] \to [0,1], x \in \mathbb{R}_{\geq 0}$, defined by

$$\psi_x(y) := 1 - \left(1 - y^{\frac{1}{x+1}}\right)^{x+1}.$$ 

It can easily be verified that $\psi_x$ is a concave distortion function.

\footnote{Choosing $x \in \mathbb{N}$ allows for the following interpretation of the $\min\text{maxvar}$-distortion functions: If a random variable $X \in \mathcal{L}^1(P)$ has the same distribution as $\min\{Z_1,\ldots,Z_{x+1}\}$, $Z_1,\ldots,Z_{x+1}$ i.i.d., then

$$\int X \, d(\psi_x \circ P) = \int \max\{Z_1,\ldots,Z_{x+1}\} \, dP$$

follows.}
2.2 Bid-ask pricing

Several authors (see, e.g., [Carr et. al. 2001, Xu 2006, Bion-Nadal 2009, Cherny and Madan 2010, Bannör and Scherer 2011a]) have suggested modeling bid-ask prices with convex risk measures. All approaches have in common that a certain convex risk measure $\Gamma := \Gamma_{\text{ask}}$ is employed for the calculation of ask prices. Furthermore, the dual functional $\Gamma_{\text{bid}}$, defined by $\Gamma_{\text{bid}}(X) := -\Gamma_{\text{ask}}(-X)$, is usually used for calculating bid prices. In our case, we stick to this kind of “bid-ask symmetry”.

[Carr et. al. 2001] and [Cherny and Madan 2010] generalize the classical results from incomplete markets and deliver an explanation for bid-ask spreads. In incomplete markets with real-world measure $P$, a set of “stress-test measures” $Q$ is selected from the set of all equivalent martingale measures and the ask price of a contingent claim $X$ is determined as the supremum of the expectations w.r.t. the stress-test measures$^2$:

$$\Gamma(X) := \sup_{Q \in Q} \mathbb{E}_Q[X].$$

In particular, [Cherny and Madan 2010] suggest choosing the set of stress test measures by an acceptability index. Their result consists of using coherent risk measures induced by a parametric family of concave distortion functions $(\gamma_\lambda)_{\lambda \in \Lambda}$ with

$$\Gamma_\lambda(X) := \int X \, d(\gamma_\lambda \circ P).$$

This approach is described in [Cherny and Madan 2010] as “conic finance” and we will adopt this name.

A different approach to bid-ask prices, also relying on convex risk measures, is discussed in [Bannör and Scherer 2011a]: Here, bid-ask spreads are explained in the context of parameter uncertainty of a parametric family of martingale measures $(Q_\theta)_{\theta \in \Theta}$. In presence of a distribution $R$ on the parameter space $\Theta$, one can define for contingent claims $X$ and law-invariant, normalized convex risk measures $\rho$ the parameter risk-captured price, which may serve as an ask price, by

$$\Gamma(X) := \rho(\theta \mapsto \mathbb{E}_\theta[X]),$$

denoting the risk-neutral price w.r.t. a fixed parameter $\tilde{\theta} \in \Theta$ by $\mathbb{E}_{\tilde{\theta}}[X]$.

Obviously, we can restrict the choice of risk measures to distortion risk measures, which are clearly law-invariant and normalized. Thus, using a concave distortion $\gamma$, the ask price turns into

$$\int_{\Theta} \mathbb{E}_\theta[X] \left( \gamma \circ R \right)(d\theta).$$

$^2$[Cont 2006] discuss a similar supremum-based approach not restricting to an incomplete markets setting.
In what follows, we discuss the calibration to bid-ask prices that are created in a distortion risk measure-driven environment. Interestingly, it can be shown that the conic finance approach of [Cherny and Madan 2010] can be embedded into the parameter risk framework, since it restricts parameter risk to the market price of risk. To cover both ideas in a unified way, we formulate the framework in a general manner and write \( f(X) \) (\( f \) being a linear map w.r.t. some matching vector spaces) as a symbol for the integrand under the Choquet integral w.r.t. the distorted probability and \( R \) as symbol for the probability measure. Hence, the generic ask pricing formula is given by

\[
\int f(X) \, d(\gamma \circ R).
\]  

(1)

In the parameter risk-capturing framework described in [Bannör and Scherer 2011a], \( f(X) = \mathbb{E}_\theta[X] \) and \( R \) is the probability measure on the parameter space \( \Theta \). In the conic finance environment as described in [Cherny and Madan 2009], we identify \( f(X) = X \) and \( R \) is simply the real-world measure \( P \).

### 3 The bid-ask calibration problem

For classical linear pricing systems, the calibration of liquid instruments \( C_1, \ldots, C_M \) to their quoted mid prices \( \overline{C}_\text{bid}^\text{mid}, \ldots, \overline{C}_\text{ask}^\text{mid} \) is a common way of obtaining the model’s unobservable parameters; thus obtaining a market-implied distribution (from a previously selected family) for the expectation. Consequently, when extending to non-linear pricing systems as briefly summarized in Section 2.2, it is natural to think about obtaining a market-implied convex risk measure from quoted bid-ask prices. As a starting point, [Cherny and Madan 2010] develop closed-form solutions for the calculation of bid and ask prices in their conic finance approach. Using these closed-form representations, they calibrate to given bid-ask prices employing the \textit{minmaxvar}-family of distortion functions.

In this section, we restate the bid-ask calibration problem in a formal way, using the generic language of (1), and show that there exists a solution in a mildly restricted class of distortion functions, not restricting to a fixed parametric shape of the distortion function. As a corollary, we show the existence of a solution of the bid-ask calibration problem in the AVaR- and \textit{minmaxvar}-family of distortion functions.

We start with a general formulation of the bid-ask calibration problem.

**Problem 3.1 (Bid-ask calibration problem)**

Let \( C_1, \ldots, C_M \) be contingent claims (e.g. vanillas) with given market bid-ask quotes \((\overline{C}_\text{bid}^1, \overline{C}_\text{ask}^1), \ldots, (\overline{C}_\text{bid}^M, \overline{C}_\text{ask}^M)\). Let furthermore \( \eta : \mathbb{R}_{\geq 0}^M \to \mathbb{R}_{\geq 0} \) be an error function\(^4\) measuring the pricing error between model prices and market quotes.

\(^3\)Due to the general formulation, we omit the area of integration in the generic ask pricing formula (1), one could write \( \text{dom}(f(X)) \).

\(^4\)We call \( \eta \) an error function (in the spirit of [Bannör and Scherer 2011b]) iff \( \eta(0, \ldots, 0) = 0 \) and \( \eta \) is componentwise non-decreasing.
A convex risk measure $\tilde{\Gamma}$ solves the (symmetric) bid-ask calibration problem on a domain $G$, if $\tilde{\Gamma}$ minimizes the function

$$\Gamma \mapsto \eta\left(\left| -\Gamma\left(-C_1\right) - \tilde{C}_1^{\text{bid}} \right|, \ldots, \left| -\Gamma\left(-C_M\right) - \tilde{C}_M^{\text{bid}} \right|, \left| \Gamma\left(C_1\right) - \tilde{C}_1^{\text{ask}} \right|, \ldots, \left| \Gamma\left(C_M\right) - \tilde{C}_M^{\text{ask}} \right| \right)$$

over the set of admissible functionals $G$.

The formulation of the bid-ask calibration (Problem 3.1) is quite general. Since the class of convex risk measure functionals can be very large and a numerical solution may be difficult to obtain, we restrict ourselves to the set of distortion risk measures. As described in the previous section, they represent a very tractable and rich class of convex risk measures. The tractability of distortion risk measures stems from the convenient representation as a Choquet integral w.r.t. a distorted probability. We formulate our results in the generic language of (1).

The first theorem provides the existence of a best fit to market prices in broadly defined subclasses of distortion risk measures. In short, Problem 3.1 is solvable.

**Theorem 3.2 (Existence of a solution to the bid-ask calibration problem)**

Let $K > 0$, $\eta$ be a continuous error function, $f(X)$ be $\mathbb{R}$-a.s. bounded and define $G_K := \{ \gamma : [0,1] \to [0,1] : \gamma \text{ Lipschitz, concave distortion function with Lipschitz constant } K \}$. Furthermore, write $\Gamma_\gamma(X) := \int f(X) d(\gamma \circ R)$ for $\gamma \in G_K$. Then the bid-ask calibration problem has a solution in $G_K$, i.e. there exists a minimizing distortion function $\tilde{\gamma} \in G_K$ s.t. the function

$$\tilde{\eta} : \gamma \mapsto \eta\left(\left| -\Gamma\gamma\left(-C_1\right) - \tilde{C}_1^{\text{bid}} \right|, \ldots, \left| -\Gamma\gamma\left(-C_M\right) - \tilde{C}_M^{\text{bid}} \right|, \left| \Gamma\gamma\left(C_1\right) - \tilde{C}_1^{\text{ask}} \right|, \ldots, \left| \Gamma\gamma\left(C_M\right) - \tilde{C}_M^{\text{ask}} \right| \right)$$

is minimized in $\tilde{\gamma}$ in the following sense:

$$\tilde{\eta}(\tilde{\gamma}) \leq \tilde{\eta}(\gamma) \quad \text{for all } \gamma \in G_K.$$

**Proof**

Step 1: $G_K$ is compact in the uniform topology.

Let $(\gamma_N)_{N \in \mathbb{N}}$ be a uniformly convergent sequence in $G_K$ with $\gamma := \lim_{N \to \infty} \gamma_N$. It is easy to see that $\gamma$ is monotone and concave. Furthermore, $\gamma(0) = 0$ and $\gamma(1) = 1$. Thus, $\gamma$ is a concave distortion function. It is also Lipschitz with Lipschitz constant $K > 0$: Choose $\varepsilon > 0$, then there exists some $N \in \mathbb{N}$ s.t. $\|\gamma - \gamma_N\|_\infty < \varepsilon/2$. Thus, by $\gamma_N \in G_K$ and the triangular inequality,

$$|\gamma(x) - \gamma(y)| \leq |\gamma(x) - \gamma_N(x)| + |\gamma_N(x) - \gamma_N(y)| + |\gamma_N(y) - \gamma(y)| < K|x - y| + \varepsilon$$

for all $x, y \in [0,1]$. Hence, $G_K$ is closed. Since $G_K$ is uniformly bounded and equicontinuous by construction, the Arzelà-Ascoli theorem yields that $G_K$ is compact.
Step 2: The map \( \gamma \mapsto \int f(X) \mathrm{d}(\gamma \circ R) \) is \( \| \cdot \|_{\infty} \)-continuous.

Let \( \gamma_N \to \gamma \) and recall the definition of the Choquet integral (e.g. [Denneberg 1994]). Dominated convergence (applicable due to the boundedness of \( f(X) \)) immediately delivers

\[
\int f(X) \mathrm{d}(\gamma_N \circ R) = \int_{-\infty}^{0} \gamma_N(R(f(X) > s)) - 1 \, ds + \int_{0}^{\infty} \gamma_N(R(f(X) > s)) \, ds \\
\to \int_{-\infty}^{0} \gamma(R(f(X) > s)) - 1 \, ds + \int_{0}^{\infty} \gamma(R(f(X) > s)) \, ds \\
= \int f(X) \mathrm{d}(\gamma \circ R),
\]

hence \( \gamma \mapsto \int f(X) \mathrm{d}(\gamma \circ R) \) is continuous.

Since \( \eta \) is a continuous error function and \( G_K \) is compact, there exists a \( \tilde{\gamma} \in G_K \) s.t. \( \eta(\tilde{\gamma}) \leq \eta(\gamma) \) for all \( \gamma \in G_K \). □

The assumption of the distortion functions being Lipschitz is not too restrictive for practical purposes: Every concave distortion function on \([0, 1]\) is Lipschitz on \([\varepsilon, 1]\) with constant \(1/\varepsilon\) for every \(\varepsilon \in (0, 1)\) (furthermore, it can only have a jump in zero and is continuous on \((0, 1)\)). In case of a numerical treatment, it is therefore sufficient to focus on Lipschitz functions.

**Remark 3.3**

1. One can show that the AVaR- and minmaxvar-families described in Examples 2.3 and 2.4 are Lipschitz when the inducing parameter set is mildly restricted. Thus, these examples are captured by our calibration framework.

2. If we scrutinize the proof, in the second part we have only used that \( \gamma_N \to \gamma \) pointwise, in particular, \( \gamma \mapsto \int X \mathrm{d}(\gamma \circ R) \) is sequentially continuous w.r.t. the pointwise topology.

We can immediately conclude the following corollary.

**Corollary 3.4**

Let \( G \subseteq G_K \) for some \( K > 0 \) (\( G_K \) as above) be \( \| \cdot \|_{\infty} \)-closed. The bid-ask calibration problem then has a solution in \( G \).

**Proof**

Since \( G \) is \( \| \cdot \|_{\infty} \)-closed and \( G_K \) is \( \| \cdot \|_{\infty} \)-compact, \( G \) is \( \| \cdot \|_{\infty} \)-compact. Thus, the same arguments as above provide the existence of a minimizing element \( \tilde{\gamma} \in G \). □

In particular, Corollary 3.4 can be applied to ensure the existence of a solution of the bid-ask calibration problem in some parametric family. Here, we show – under mild technical restrictions – the existence of a solution in the popular AVaR- and minmaxvar-families.
4 A non-parametric calibration scheme for bid-ask price

Example 3.5
1. Let $\varepsilon > 0$ and define the $\varepsilon$-bounded AVaR-family

$$G_\varepsilon := \left\{ \gamma : [0,1] \to [0,1] : \exists \alpha \in [\varepsilon,1] : \gamma|_{[0,\alpha]}(y) = \frac{y}{\alpha} \text{ and } \gamma|_{[\alpha,1]}(y) = 1 \right\}.$$ 

Obviously, $\gamma_N \to \gamma$ implies $\alpha_N \to \alpha$, thus $\gamma_N \in G_\varepsilon$ for all $N \in \mathbb{N}$ implies $\gamma \in \mathbb{N}$, thus $G_\varepsilon$ is closed. Corollary 3.4 shows that the bid-ask calibration problem has a solution in $G_\varepsilon$.

2. Let $K \in [0, \infty)$ and define

$$G_K := \left\{ \gamma : [0,1] \to [0,1] : \exists L \in [0,K] : \gamma(y) = 1 - (1 - y \frac{1}{L+1})^{L+1} \right\},$$

i.e. $\gamma \in G_K$ is a minmaxvar-type distortion function with parameter $L \in [0,K]$. If $(\gamma_N)_{N \in \mathbb{N}}$ is a sequence in $G_K$ that converges uniformly, it is easy to show that $L_N \to L$, in particular, $\lim_{N \to \infty} \gamma_N \in G_K$. Thus, $G_K$ is $\| \cdot \|_\infty$-closed, applying Corollary 3.4 shows that the bid-ask calibration problem has a solution in $G_K$.

4 A non-parametric calibration scheme for bid-ask price

In this section, we present an alternative to the calibration of parametric families à la [Cherny and Madan 2010]. We estimate the distortion function in a non-parametric way, using a piecewise linear approximation which eventually reduces the bid-ask calibration problem to a constrained optimization problem on the unit cuboid. We argue that a non-parametric calibration scheme may be useful to obtain the shape of a market-implied distortion function: Empirical results on market-implied distortion functions are very rare, yet. Thus, a parametric approach may not capture the distortions appropriately that are quoted on the market. Furthermore, our approach can be used to find and justify some parametric shape of the market-implied distortion function from empirical observations.

4.1 General results

We start with the following lemma, ensuring that a piecewise linear approximation of a distortion function is a sensible approximation and does not produce much different distorted expectations.

Lemma 4.1 (Piecewise linear approximation)

Let $\gamma : [0,1] \to [0,1]$ be a continuous concave distortion function. Let $N \in \mathbb{N}$ and choose $0 := y_0 < y_1 < \cdots < y_N := 1$. Denote $n(x) := \max\{n \in \{0,\ldots,N\} : x \geq y_n\}$ and define

$$\gamma_N(x) := \begin{cases} \gamma(y_{n(x)}) + \frac{\gamma(y_{n(x)+1}) - \gamma(y_{n(x)})}{y_{n(x)+1} - y_{n(x)}}(x - y_{n(x)}) & , x \in [0,1) \\ 1 & , x = 1. \end{cases}$$ (2)
4.1 General results

Then $\gamma_N$ is a piecewise linear, concave distortion function and $\gamma_N \to \gamma$ uniformly, if the mesh of the decompositions $\mathcal{Y}_N := \{y_0, \ldots, y_N\}$ converge to zero, i.e. if $\max_{n \in \{1, \ldots, N\}} \{y_n - y_{n-1}\} \to 0$ for $N \to \infty$.

**Proof**

Let $N \in \mathbb{N}$. Obviously, $\gamma_N$ is piecewise linear, monotone, and $\gamma_N(0) = 0$, $\gamma_N(1) = 1$. Thus, $\gamma_N$ is a piecewise linear distortion function. Furthermore, $\gamma_N$ is concave, since the a.e. defined derivative is decreasing. The convergence property follows from the proof that the set of piecewise linear functions is uniformly dense in the space of continuous functions on the unit interval $C[0,1]$ (cf. [Shilov 1996, Section 1.23]). □

In particular, for all $K > 0$: $G_{lin}^K := \{\gamma \in G_K : \gamma$ is piecewise linear$\}$ is dense in $G_K$. Thus, approximation in $G_{lin}^K$ is a sensible way for any distortion function in $G_K$.

Furthermore, the continuity of the error function assures that integrals w.r.t. piecewise linear concave distortion functions converge as well, so optimizing in the piecewise concave distortion functions may not provide significant disadvantages in accuracy compared to optimizing in $G_K$.

As a second result, we note that piecewise linear concave distortion functions are fully characterized by an $N$-tuple of real numbers (with constraints). Considering the decomposition $0 = y_0 < y_1 < \cdots < y_N = 1$ and piecewise linearity on $[y_{n-1}, y_n]$ for all $n = 1, \ldots, N$, a piecewise concave distortion function $\gamma$ is fully characterized by the difference vector $\Delta \gamma \in \mathbb{R}^N$, where

$$\Delta \gamma_n := \gamma(y_n) - \gamma(y_{n-1}), \quad n = 1, \ldots, N.$$  

Monotonicity, concavity, and $\gamma(0) = 0$, $\gamma(1) = 1$ immediately deliver for the discrete derivatives

$$\frac{\Delta \gamma_1}{y_1 - y_0} \geq \frac{\Delta \gamma_2}{y_2 - y_1} \geq \cdots \geq \frac{\Delta \gamma_N}{y_N - y_{N-1}} \geq 0 \quad \text{and} \quad \sum_{n=1}^{N} \Delta \gamma_n = 1. \quad (3)$$

In particular, solving the bid-ask calibration problem can be traced back to finding the best matching $(\Delta \gamma_1, \ldots, \Delta \gamma_N)$ fulfilling the conditions in (3). The key result for calculating distorted expectations is presented in the following theorem.

**Theorem 4.2 (Expectation w.r.t. a piecewise linear concave distorted probability)**

Let $N \in \mathbb{N}$, $\gamma_N$ be a piecewise linear concave distortion function with decomposition $0 = y_0 < y_1 < \cdots < y_N = 1$, and difference vector $(\Delta \gamma_1, \ldots, \Delta \gamma_N) \in [0,1]^N$. Let furthermore $f(X) \in L^\infty(\mathbb{R})$ and $y_0, \ldots, y_N$ are points of continuity of $F_{f(X)}$. Then the distorted expectation of $f(X)$ w.r.t. $\gamma_N \circ R$ can be calculated via

$$\int X \, d(\gamma_N \circ R) = \sum_{n=1}^{N} \Delta \gamma_n \mathbb{E}[f(X) \mid f(X) \in [\text{VaR}_{y_n}(f(X)), \text{VaR}_{y_{n-1}}(f(X))]].$$

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4.1 General results

**Proof**
Let $N \in \mathbb{N}$. Define $\tilde{\gamma}_n := \Delta \gamma_n/(y_n - y_{n-1})$ and $f_n(y) := \min\{y_n, \max\{y_{n-1}, y\}\} - y_{n-1}$ for $n \in \{1, \ldots, N\}$. With some elementary substitutions it can be verified that $\gamma_N$ as in (2) can be represented as

$$
\gamma_N(y) = \sum_{n=1}^{N} \tilde{\gamma}_n f_n(y).
$$

In particular, we obtain for $X \in L^\infty(R)$ by applying linearity of Lebesgue-Stieltjes integrals in the integrators

$$
\int f(X) \, d(\gamma_N \circ R) = - \int_{-\infty}^{\infty} x \left( \gamma_N \circ F_{-f(X)} \right) (dx)
$$

$$
= - \int_{-\infty}^{\infty} x \left( \sum_{n=1}^{N} \tilde{\gamma}_n \left( f_n \circ F_{-f(X)} \right) \right) (dx)
$$

$$
= - \sum_{n=1}^{N} \tilde{\gamma}_n \int_{-\infty}^{\infty} x \left( f_n \circ F_{-f(X)} \right) (dx).
$$

Scrutinizing $f_n \circ F_{-f(X)}$, one observes that $f_n \circ F_{-f(X)}$ is constant on $[-\infty, q_{y_{n-1}}(-f(X))] \cup [q_{y_n}(-f(X)), \infty]$, denoting by $q_{\alpha}(-f(X))$ the lower $\alpha$-quantile of $-f(X)$, due to the definition of $f_n$. Therefore, the integration can be restricted to $[q_{y_{n-1}}, q_{y_n}]$. Hence,

$$
- \frac{1}{y_n - y_{n-1}} \int_{-\infty}^{\infty} x \left( f_n \circ F_{-f(X)} \right) (dx)
$$

$$
= - \frac{1}{y_n - y_{n-1}} \int_{q_{y_{n-1}}}^{q_{y_n}} x \left( F_{-f(X)} \right) (dx)
$$

$$
= - \mathbb{E} \left[ -f(X) \big| \begin{array}[]{c} -f(X) \\ x \in [q_{y_{n-1}}(-f(X)), q_{y_n}(-f(X))] \end{array} \right]
$$

$$
= \mathbb{E} \left[ f(X) \big| f(X) \in [\text{VaR}_{y_n}(f(X)), \text{VaR}_{y_{n-1}}(f(X))] \right]
$$

delivers the desired result, since $q_{\alpha}(-f(X)) = - \text{VaR}_{\alpha}(f(X))$. \qed

The Choquet integral of $f(X)$ w.r.t. the distorted probability $\gamma_N \circ R$ can be used to calculate ask prices. To get a similarly convenient result for bid prices, we find for $-f(X)$:

$$
- \left( \int -f(X) \, d(\gamma_N \circ R) \right) = \sum_{n=1}^{N} \Delta \gamma_n \mathbb{E} \left[ f(X) \big| f(X) \in [q_{y_{n-1}}(f(X)), q_{y_n}(f(X))] \right].
$$

In particular, the above formulae yield a tractable setting for solving the bid-ask calibration problem within the piecewise linear concave distortion functions, given a decomposition $\mathcal{Y} = \{y_0, \ldots, y_N : 0 = y_0 < \cdots < y_N = 1\}$ of the unit interval. Solving the bid-ask
calibration problem corresponds to finding a vector $\Delta \gamma \in \mathbb{R}^N$ satisfying the constraints in (3) and minimizing some distance between model bid-ask prices and quoted market bid-ask prices.

**Algorithm 4.3 (Bid-ask calibration to piecewise concave distortions on a fixed grid)**

Let $(\bar{C}^{\text{bid}}_1, \bar{C}^{\text{ask}}_1), \ldots, (\bar{C}^{\text{bid}}_M, \bar{C}^{\text{ask}}_M)$ be given bid-ask market quotes of contingent claims $C_1, \ldots, C_M$ and $\eta : \mathbb{R}_{>0}^M \to \mathbb{R}_{>0}$ an error function. The bid-ask calibration problem can be solved by the following algorithm:

1. Choose $N \in \mathbb{N}$.
2. Choose a decomposition $\mathcal{Y} = \{y_0, \ldots, y_N : 0 = y_0 < \cdots < y_N = 1\}$ of the unit interval.
3. Calculate
   \[
   \mathbb{E}\left[f(C_j) \middle| f(C_j) \in [q_{y_{n-1}}(f(C_j)), q_{y_n}(f(C_j))]\right]
   \]
   and
   \[
   \mathbb{E}\left[f(C_j) \middle| f(C_j) \in [\text{VaR}_{y_n}(f(C_j)), \text{VaR}_{y_{n-1}}(f(C_j))]\right]
   \]
   for $n = 1, \ldots, N$, $j = 1, \ldots, M$.
4. Solve the constrained optimization problem

   \[
   \min_{\Delta \gamma \in \mathbb{R}^N_{>0}} \eta \left( \sum_{n=1}^{N} \Delta \gamma_n \mathbb{E}[f(C_1)|f(C_1) \in [q_{y_{n-1}}, q_{y_n}]] - \bar{C}^{\text{bid}}_1, \ldots, \right.
   \]
   \[
   \left. \sum_{n=1}^{N} \Delta \gamma_n \mathbb{E}[f(C_M)|f(C_M) \in [q_{y_{n-1}}, q_{y_n}]] - \bar{C}^{\text{bid}}_M, \right.
   \]
   \[
   \sum_{n=1}^{N} \Delta \gamma_n \mathbb{E}[f(C_1)|f(C_1) \in [\text{VaR}_{y_{n-1}}, \text{VaR}_{y_n}]] - \bar{C}^{\text{ask}}_1, \right.
   \]
   \[
   \left. \sum_{n=1}^{N} \Delta \gamma_n \mathbb{E}[f(C_M)|f(C_M) \in [\text{VaR}_{y_{n-1}}, \text{VaR}_{y_n}]] - \bar{C}^{\text{ask}}_M \right) \right)
   \]

subject to

\[
\sum_{n=1}^{N} \Delta \gamma_n = 1, \Delta \gamma \geq 0,
\]

\[
\left( \frac{\Delta \gamma_2}{y_2 - y_1} - \frac{\Delta \gamma_1}{y_1 - y_0}, \ldots, \frac{\Delta \gamma_N}{y_N - y_{N-1}} - \frac{\Delta \gamma_{N-1}}{y_{N-1} - y_{N-2}} \right) =: D(\Delta \gamma) \leq 0.
\]

Thus, the core of the bid-ask calibration problem is reduced to a non-linear constrained optimization problem in the compact convex space

\[
G := \left\{ \Delta \gamma \in \mathbb{R}^N : \Delta \gamma \geq 0, D(\Delta \gamma) \leq 0, \sum_{n=1}^{N} \Delta \gamma_n = 1 \right\}.
\]
4.1 General results

Algorithm 4.3 treats the optimization on a fixed decomposition $\mathcal{Y} = \{y_0, \ldots, y_N\}$ of the unit interval. Obviously, the methodology can be enhanced by varying over the decompositions as well, which also delivers a constrained optimization problem. The problem accompanying optimization over decompositions is performance: Varying decompositions considerably slow down the optimization procedure, since all conditional expectations

$$E\left[f(C_j) \mid f(C_j) \in [q_{y_{n-1}}(f(C_j)), q_{y_n}(f(C_j))]\right]$$

and

$$E\left[f(C_j) \mid f(C_j) \in [\text{VaR}_{y_n}(f(C_j)), \text{VaR}_{y_{n-1}}(f(C_j))]\right]$$

have to be recalculated in every optimization step, while they only have to be calculated once when fixing the decomposition $\mathcal{Y} = \{y_0, \ldots, y_N : 0 = y_0 < \cdots < y_N\}$.

One possibility to accelerate and simplify the bid-ask calibration is equidistant spacing of $y_0 < \cdots < y_N$:\footnote{Actually, this can be relaxed further: If $y_0 < \cdots < y_N$ are symmetrically spaced around 0.5, the same acceleration argument can be done.} Step 3 of Algorithm 4.3 then reduces to the calculation of

$$E\left[f(C_j) \mid f(C_j) \in [q_{y_{n-1}}(f(C_j)), q_{y_n}(f(C_j))]\right]$$

for $n = 1, \ldots, N$, $j = 1, \ldots, M$ and the concavity constraint $D(\Delta \gamma) \leq 0$ simplifies to

$$\Delta^2 \gamma := (\Delta \gamma_2 - \Delta \gamma_1, \ldots, \Delta \gamma_N - \Delta \gamma_{N-1}) \leq 0.$$

Furthermore, a closer look on the optimization problem in Algorithm 4.3 suggests the following strategies:

**Remark 4.4 (Numerical treatment)**

1. Note that if the objective function of the optimization problem is continuously differentiable in $\Delta \gamma$, the bid-ask calibration to piecewise linear concave distortions fulfills the Karush-Kuhn-Tucker conditions for nonlinear optimization (see, e.g., [Boyd and Vandenberghe 2004]). Thus, Algorithm 4.3 can be solved by using Lagrange multipliers.

2. If we use the Euclidean distance (or some strictly monotone transformation of it) as an error function (e.g. the popular RMSE error function), the bid-ask calibration to piecewise linear concave distortions reduces to a linearly constrained least-squares optimization problem which is well treated in literature (see, e.g., [Hanson and Haskell 1982, Hanson 1986]).
4.2 Application to parameter risk-captured prices

A framework for the calculation of parameter risk-captured prices is developed in [Bannör and Scherer 2011a]. The central assumption in this framework is the presence of a distribution $R$ on the parameter space $\Theta$. In some cases of historical estimation of parameters (e.g. correlation estimation as in [Bannör and Scherer 2011a]), the distribution $R$ on $\Theta$ is given by the distribution of the parameter’s estimator. In other cases, e.g. the calibration to market prices, the distribution $R$ may be recovered from the calibration to mid prices by an algorithm as described in [Bannör and Scherer 2011b]. While in the papers [Bannör and Scherer 2011a, Bannör and Scherer 2011b] the choice of risk measure was regarded to be subjective, Algorithm 4.3 allows us to calibrate parameter risk-captured prices to bid-ask prices using a broad and flexible class of risk measures, represented by piecewise linear concave distortion functions.

Using the suggested procedure for obtaining a distribution $R$ on the parameters, when calibrating to market prices of vanillas, the result is a three-step calibration scheme:

First, we calibrate to mid market prices and obtain a parameter $\theta_0 \in \Theta$. Second, using the parameter $\theta_0$, we construct the distribution $R$ on the parameter space $\Theta$ as suggested in [Bannör and Scherer 2011b]. Finally, we calibrate to bid-ask prices using Algorithm 4.3 to obtain the best matching concave distortion function $\gamma_0$.

The choice of the error function is also somewhat delicate: [Detlefsen and Härdle 2007] observe that different choices of error functions deliver different calibration results. Furthermore, [Guillaume and Schoutens 2011] scrutinize different calibration methods (historical calibration, calibration to vanillas, etc.), restricted to the Heston model. We omit further interdependencies to those issues in the current investigation and concentrate on one error function and one calibration method.

5 Application to data

In the previous section, we have presented and discussed a non-parametric calibration scheme based on piecewise linear concave distortion functions to bid-ask prices. In this section, we apply our piecewise linear calibration scheme and compare it to a parametric calibration scheme à la [Cherny and Madan 2010], using the introduced parametric families of AVaR- and minmaxvar-type distortion functions. Therefore, we calibrate a

---

6In [Bannör and Scherer 2011b], the distribution $R$ is constructed by transforming the error function to mid prices $\eta_{\text{mid}}$ via a decreasing function $h$ s.t. $\int_{\Theta} h(\eta_{\text{mid}}(\theta)) \, d\theta = 1$. The choice of $h$ gauges the weight being assigned to the parameters - the lower the pricing error, the higher the weight. Using our solution of the bid-ask calibration problem, we obtain different market-implied concave distortion functions $\gamma$ for different transformation functions $h$. We argue that fixing the transformation function $h$ (e.g. due to experience of traders) and solving for the market-implied concave distortion may be a sensible way of treating the bid-ask calibration problem in our parameter risk-capturing setting. Otherwise (solving for the transformation function and the concave distortion) our results may become difficult to interpret, due to the partial exchangeability of $h$ and $\gamma$.  

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5.1 Parameter uncertainty setup and calibration

Barndorff-Nielsen–Shephard model with parameter risk (cf. [Bannör and Scherer 2011b]) to quoted bid-ask prices of vanillas. As a result, we obtain a very characteristic shape of the piecewise linear approximation. From our empirical results, we consequently suggest using a different parametric family of concave distortion functions: The \( \text{ess sup} \)-expectation convex combinations, simply interpolating between the essential supremum and the vanilla expectation w.r.t. the parameter distribution \( R \). This family turns out to match the results from the piecewise linear calibration approach and allows for a fast and efficient calibration.

Our set of data is the DAX option surface, consisting of 501 bid and ask vanilla prices as of December 2, 2011, with different maturities and strikes. For simplicity, we assume no bid-ask spreads in the DAX spot price and EUR interest rates.

We use the Barndorff-Nielsen–Shephard model, for details see [Barndorff-Nielsen and Shephard 2001, Cont 2004]. The risk-neutral dynamics of the log-index price \( (X_t)_{t \geq 0} \) in the Barndorff-Nielsen–Shephard model are given by

\[
\text{d}X_t = \left( r - \frac{\sigma^2 t}{2} + \frac{\lambda c \rho}{\alpha + \rho} \right) \text{d}t + \sigma_t \text{d}W_t - \rho \text{d}Z_t,
\]

\[
\text{d}\sigma^2_t = -\lambda \sigma^2_t \text{d}t + \text{d}Z_t,
\]

with parameters \( r, S_0, \sigma^2_0, \lambda, c, \rho, \alpha > 0 \), \( (W_t)_{t \geq 0} \) is a Brownian motion, and \( (Z_t)_{t \geq 0} \) is a compound Poisson process with exponentially distributed jump size, i.e. \( Z_t = \sum_{j=1}^{N_t} U_j \) with a Poisson process \( (N_t)_{t \geq 0} \) with intensity \( c > 0 \) and \( (U_j)_{j \in \mathbb{N}} \) are exponentially i.i.d. with parameter \( \alpha > 0 \). \( (W_t)_{t \geq 0} \) and \( (Z_t)_{t \geq 0} \) are independent. Since we assume the DAX spot price \( S_0 \) and risk-free rate \( r \) to be given, the unspecified parameters with exposure to parameter risk are gathered in the quintuple \( (\sigma^2_0, c, \alpha, \lambda, \rho) \).

For calculating vanilla prices, we use the Fourier pricing method of [Carr and Madan 1999, Raible 2000] and calculate call prices for the moneyness dimension simultaneously via FFT. As an error function for both the initial calibration to mid prices and the calibration to bid-ask prices, we use the popular RMSE without any weighting (for the impact of using different error functions, we refer to [Detlefsen and Härdle 2007]), standardized by the mean option price, similar to the setting in [Bannör and Scherer 2011b]. Thus, our error function is

\[
\eta(\theta) = \sqrt{\frac{1}{M} \sum_{m=1}^{M} \left( \mathbb{E}_\theta [C^m] - C^\text{mid}_m \right)^2}.
\]

5.1 Parameter uncertainty setup and calibration

We follow [Bannör and Scherer 2011b] and obtain a parameter risk distribution \( R \) on \( \Theta \) by discretizing \( \Theta \) and weighting the parameters with their error to market prices, transformed and normed by a decreasing function \( h \) s.t. the sum of all parameters equals one. We hereby incorporate all parameters (on a discrete grid) up to an aggregate
5.2 Results

market error of 3.5% and weight them by transforming the error function with the normal transformation function

\[ h_N^\lambda(t) := c \cdot \exp \left( -\frac{(t - t_0)^2}{2\lambda^2} \right), \]

where \( \lambda = 0.005, c > 0 \) matching, and \( t_0 = 1.63\% \) denoting the minimal aggregate market error. Doing so, we obtain a discrete distribution on \( \Theta \) with a support of 9,430 parameter vectors with an aggregate market error of less than 3.5%.

With a distribution \( R \) on \( \Theta \) at hand, we calibrate to bid and ask prices in various ways: First, we calibrate to bid-ask prices with our non-parametric piecewise linear approximation scheme described in Algorithm 4.3. As a unit interval decomposition, we use 100 and 1,000 equidistant nodes; for optimization purposes, we use again the mean-standardized RMSE as in the mid prices calibration. Second, we compare our result to a parametric calibration using parametric families of distortion functions. We hereby employ the popular AVaR- and minmaxvar-families for calibration and observe differences in calibration performance.

5.2 Results

As a first result, we obtain that the calibration performances of the piecewise linear and the parametric approaches do not differ significantly, the standardized RMSEs of all approaches are close to each other:

<table>
<thead>
<tr>
<th>Distortion framework</th>
<th>RMSE/mean to bid-ask prices</th>
<th>CPU time</th>
</tr>
</thead>
<tbody>
<tr>
<td>piecewise linear 1,000 nodes</td>
<td>1.64%</td>
<td>301.11 sec</td>
</tr>
<tr>
<td>piecewise linear 100 nodes</td>
<td>1.64%</td>
<td>4.26 sec</td>
</tr>
<tr>
<td>minmaxvar-family</td>
<td>1.65%</td>
<td>3.17 sec</td>
</tr>
<tr>
<td>AVaR-family</td>
<td>1.64%</td>
<td>3.73 sec</td>
</tr>
</tbody>
</table>

Actually, the distortion functions that result from the calibration differ in shape (cf. Figure 2), in case of the AVaR-calibration due to its specific shape. The piecewise linear calibration result is clearly the most flexible method and does not exhibit too strong performance drawbacks compared to parametric calibration when choosing a reasonable number of nodes.

As a very remarkable result, one observes that both the 100 and 1,000 nodes approximation follow the same pattern: After a sharp increase close to zero, we can observe linear growth in the argument of the distortion function. This observation is quite robust: When using different transformation functions for creating the distribution on \( \Theta \) and incorporating more parameters, this pattern remains the same. Even when using a different model (e.g. the Heston model), a similarly shaped distortion function is obtained. Furthermore, the result is stable w.r.t. different choices of starting vectors in the
5.2 Results

optimization procedure, so it is also unlikely that it results from numerical instabilities. Hence, our results motivate to introduce another parametric family of concave distortion functions, see Definition 5.1.

**Definition 5.1 (ess sup-expectation convex combinations)**

Let \( \lambda \in [0, 1] \). The distortion function

\[
\gamma_\lambda(u) := \begin{cases} 
0, & u = 0 \\
\lambda + (1 - \lambda)u, & u \in (0, 1]
\end{cases}
\]

is called the \textit{ess sup}-expectation convex combination risk measure with weight \( \lambda \) on the essential supremum.

The name of this family stems from the behavior of the Choquet integral w.r.t. a \( \gamma_\lambda \)-distorted probability: If we have a probability measure \( P \) and a bounded random variable \( X \), the Choquet integral w.r.t. \( \gamma_\lambda \circ P \) is a convex combination of the essential supremum of \( X \) and the ordinary expectation w.r.t. \( P \), weighting the essential supremum with \( \lambda \) and the expectation with \( 1 - \lambda \). We can easily show the more general result:

**Proposition 5.2**

Let \((\Omega, \mathcal{F}, P)\) be a probability space, \( \gamma \) a concave distortion function with a jump in zero, \( X \) an integrable random variable that is \( P \)-a.s. bounded from above, and let \( \lambda = \lim_{v \downarrow 0} \gamma(v) \) be the height of the jump. Then the function

\[
\gamma_{\text{cont}}(u) = \lim_{v \downarrow u} \gamma(v) - \lambda \frac{u}{1 - \lambda}
\]

is a continuous concave distortion function and

\[
\int_{\Omega} X \, d(\gamma \circ P) = \lambda \text{ess sup} \, X + (1 - \lambda) \int_{\Omega} X \, d(\gamma_{\text{cont}} \circ P).
\]

In particular, for every concave distortion function with a jump in zero the Choquet integral w.r.t. \( \gamma \circ P \) is representable as a convex combination of the essential sup and a continuous concave distortion function \( \gamma_{\text{cont}} \).

We call \( \gamma_{\text{cont}} \) the continuous part of \( \gamma \) and \( \lambda \) the jump part of \( \gamma \). It may be noted that the continuous part \( \gamma_{\text{cont}} \) is not unique, but we denote with it the construction in Proposition 5.2. For \( \lambda = 1 \), we trivially set \( \gamma_{\text{cont}} = 0 \).

**Proof (of Proposition 5.2)**

It is easy to show that \( \gamma_{\text{cont}} \) is a continuous and concave distortion function (if \( \lambda = 1 \), the proposition is trivially true). Abbreviate \( x_0 := \text{ess sup} \, X \) (first assuming \( x_0 \geq 0 \),
5.2 Results

then the definition of the Choquet integral immediately yields

\[
\int_{\Omega} X \, d(\gamma \circ P) = \int_{-\infty}^{0} \gamma(P(X > x)) \, dx + \int_{0}^{\infty} \gamma(P(X > x)) \, dx
\]

\[
= \int_{-\infty}^{x_0} \gamma(P(X > x)) \, dx + \int_{0}^{\infty} \gamma(P(X > x)) \, dx
\]

\[
= \int_{-\infty}^{x_0} (1 - \lambda)\gamma^\text{cont}(P(X > x)) + \lambda \, dx + \\
\int_{x_0}^{\infty} (1 - \lambda)\gamma^\text{cont}(P(X > x)) + \lambda \, dx
\]

\[
= \lambda x_0 + (1 - \lambda) \int_{\Omega} X \, d(\gamma^\text{cont} \circ P).
\]

Similar for \(x_0 < 0\).

In particular, if we use \(\gamma_\lambda\) from Definition 5.1 as distortion function, we are interpolating between the essential supremum and the expectation (since the continuous part of \(\gamma_\lambda\) is just the identity function on \([0, 1]\)) when calculating the Choquet integral. Since our piecewise linear calibration results in Figure 1 look similar to an ess sup-expectation convex combination, they deliver an appealing interpretation for trading: When setting the bid-ask prices by means of the calibration risk framework of [Bannör and Scherer 2011b], we get bid-ask spreads that are fairly in line with the market when calculating the worst case of all parameters, the expectation w.r.t. the delivered distribution, and simply interpolating between them, using the weight of the worst case as a “risk-aversion parameter”. Strikingly, this simple approach ties with the more sophisticated AVaR- and minmaxvar-methodologies in calibration performance and is much easier and faster to implement: In our setting, we just have to calculate the supremum, the infimum, and the expectation, which can efficiently be done using vectorized programming systems (e.g. MATLAB). Afterwards, the optimization procedure only incorporates three values per vanilla option and is much faster compared to other parametric distortion function types or the 100 nodes piecewise linear function:

<table>
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<td>3.73 sec</td>
</tr>
<tr>
<td>ess sup-exp-family</td>
<td>1.64%</td>
<td>0.21 sec</td>
</tr>
</tbody>
</table>

Furthermore, we obtain a firm economic interpretation of the expectation w.r.t. the distribution: The expectation can be interpreted as the “true mid price” (which is not observable in the market, since we only get bid and ask quotes), with a (non-symmetric) risk premium relative to the parameter risk which is expressed by the essential supremum (for ask prices) and infimum (for bid prices). This is in line with [Cont 2006] and
6 Conclusion

[Lindström 2010], who argue separately for the supremum and the expectation incorporating model/parameter risk.

Since our empirical observations motivate the introduction of discontinuous distortion functions, we finally generalize the existence theorem for a solution to the bid-ask calibration problem to discontinuous distortion functions.

**Theorem 5.3**

Let \( f(X) \) be bounded, \( \eta \) a continuous error function, \( K > 0, \varepsilon \in (0, 1) \), and define

\[
G^{\text{jump}, \varepsilon}_K := \{ \gamma : [0, 1] \to [0, 1] : \gamma \text{ is a distortion function with jump height } \leq 1 - \varepsilon \text{ and } \gamma^{\text{cont}} \in G_K \}.
\]

Then there is a solution of the bid-ask calibration problem in \( G^{\text{jump}, \varepsilon}_K \).

**Proof**

Let \( \gamma \in G^{\text{jump}}_K \) and take the representation \( \gamma = (\lambda, \gamma^{\text{cont}}) \) from Proposition 5.2. The mapping

\[
(\lambda, \gamma^{\text{cont}}) \mapsto \lambda \text{ess sup } f(X) + (1 - \lambda) \int f(X) \, d(\gamma^{\text{cont}} \circ P)
\]

is continuous w.r.t. the product topology on \([0, 1 - \varepsilon] \times G_K\). Furthermore, Tychonoff’s theorem ensures that \([0, 1 - \varepsilon] \times G_K\) is compact. Hence, a minimizing element exists. \(\square\)

6 Conclusion

In this paper, we state the bid-ask calibration problem in a generic manner and provide existence theorems for its solution in the set of distortion risk measures; mildly restricted to certain concave distortions. Our results apply to both the conic finance approach presented in [Cherny and Madan 2010] and the parameter risk-captured price framework of [Bannör and Scherer 2011a]. Furthermore, we provide a non-parametric calibration scheme to bid-ask prices based on a piecewise linear approximation of concave distortion functions. This scheme turns out to be efficient to implement and reduces in prominent special cases to the well-known linearly constrained linear least-squares problem. Applying our piecewise linear calibration scheme in the parameter risk-captured price framework from [Bannör and Scherer 2011a, Bannör and Scherer 2011b], we compare our model bid-ask spreads to bid-ask spreads provided by parametric bid-ask calibration results as in [Cherny and Madan 2010] and obtain a new parametric family of concave distortions. The parametric family of ess sup-expectation convex combinations matches well with our observed non-parametric calibration results and allows for an efficient and accurate calibration.
References


References


Figure 1  Different calibration results for a (non-parametric) piecewise linear calibration with 1 000 and 100 nodes, a parametric calibration to the AVaR-distortion function family, and a parametric calibration to the minmaxvar-family. While the piecewise linear calibration results with different nodes are very similar and have a characteristic jump close to zero and afterwards a linear behavior, we get very different results for the minmaxvar- and AVaR-results. In the upper probability regions, the minmaxvar-result allocates more probability than the piecewise linear result, while the AVaR allocates even more probability. In the lower probability region, we get more similarity for minmaxvar- and AVaR-results, but a very different pattern to the jump in the piecewise linear calibration results. Since the calibration results seem to be pretty similar, the higher probability allocation to high-probability events like in the AVaR- and minmaxvar-case does not seem to impact the calibration result too much.
Figure 2 Bid-ask spreads for differently calibrated distortions vs. real-world bid-ask spreads, expressed in implicit volatilities for a short-term and a long-term maturity. Since the BNS model with its downward jumps emphasizes the put side of the smile, we exhibit more parameter risk on the left wing and those larger bid-ask spreads can be captured well by all models. For the same reason, the right wing vol spread is matched less accurate, in particular for short-term maturities, but AVaR- and piecewise linear distortions match it more efficiently than the calibrated minmaxvar-distortion. One has to encounter that parameter risk may not be the main driver for far-OTM call prices, other effects like illiquidity seem to be more predominant.