

# Optimal Income Taxes, Transfers and Inequality for an Atkinson-Based Social Welfare Function

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## Abstract

The paper extends Atkinson's measure of inequality to incorporate endogenous determination of incomes, with labour supplies determined in response to the tax system. On the basis of solutions for optimal income taxes using double limit analysis and derived from the social welfare function in this extension, the paper considers how the optimal income tax, transfers and inequality vary with the parameter used in the measure of inequality, and how taxation affects inequality. A measure of inequality is proposed that decomposes inequality into wage, labour supply, taxation and utility factors.

Keywords: optimal taxation, inequality, social welfare, transfers  
JEL Codes: H21, D31, C63, D63

## 1 Introduction

This paper examines implications of optimal income taxation for transfers, social welfare and inequality arising from Anthony Atkinson's measure of inequality (Atkinson, 1970; see also Serge-Christophe Kolm, 1969, as well as discussions of inequality measurement in Stephen Jenkins and Philippe Van Kerm, 2000; Frank Cowell, 1995, 2000; Cowell and Emmanuel Flachaire, 2015; and Jacques Silber, 1999). Atkinson's measure is based on the principle of transfers (that a transfer of a marginal unit of income from a richer to a poorer person should raise the sum of utilities) attributed to A.C. Pigou (1912) and Hugh Dalton (1920; see the discussion by Atkinson and Andrea Brandolini, 2015). Atkinson's measure depends on an inequality aversion parameter  $e$  that incorporates society's distributional objectives. An informative characterization of Atkinson's measure of inequality is that the parameter  $e$  depends on the loss in income that one

would accept in transferring an amount of income from a richer person to a poorer person. This transfer of income cannot be carried out by simply raising taxes on the richer person and reducing them on the poorer person since these changes would affect other people through their labour responses. Instead, marginal transfers of income are brought about by combinations of increases and decreases in marginal tax rates that generate inefficiencies. As a result, these inefficiencies provide a potential economic source for the losses in transferring income considered in Atkinson's measure. The possibility of redistribution of income through shifting of tax collections among income intervals raises the question of the form of a tax system consistent with the value judgments of the Atkinson inequality measure and the extent to which such a tax could reduce inequality. The results in this paper are generated by analytic results in optimal income taxation as well as specific examples now made possible by developments in the solution of optimal income taxes in the general case.

While retaining some essential features of the Dalton and Atkinson measures of inequality, the application of optimal income taxation to the study of inequality requires departures from both. Dalton's measure of inequality, based on utilitarianism, is given by one minus the ratio of average utility to the utility of the average income. Atkinson's measure does not make explicit reference to a specific utility function and is given by one minus the ratio of average income to an "equally distributed equivalent income" that would yield the same level of social welfare if all incomes were equally distributed (see Frank Cowell, 1995, pp. 44-47, for a discussion of the differences between the Dalton and Atkinson measures of inequality). The income comparisons in Atkinson's measure are consistent with a constant elasticity utility function of income that yields constant relative risk aversion. Both Dalton and Atkinson consider social welfare and inequality as depending only on incomes or functions of incomes, whereas the optimal income tax literature requires a utility function of leisure as well as consumption (Amartya Sen, 1992, p. 29, has suggested using utilities in place of income in Atkinson's inequality measure). The optimal income tax calculated in this paper maximizes a Constant Elasticity of Substitution (CES) function of individual utilities, the same functional form used by Atkinson in his measure of inequality. It departs from Dalton's measure by allowing for nonutilitarian social welfare.

Application of optimal income taxation to Atkinson's inequality measure also requires extension of the tax literature that originated with Mirrlees (1971) and with extensions by Diamond (1998) and Emmanuel Saez (2001). The analytic conclusions of this literature have previously been based on restrictive assumptions of a semi-linear utility function, absence of income effects, unlimited labour supply, or constant elasticities of substitution. The double limit analysis applied in this paper (Sattinger, 2017) does not impose these restrictions and determines the second order differential equation for an optimal income tax for arbitrary (but well-behaved) utility and social welfare functions, allowing the determination of the optimal income tax for an Atkinson-based measure of social welfare. This paper differs from the earlier paper that developed double limit analysis by applying it to a CES social welfare function, analyzing the

consequences of transfers, and developing the consequences of optimal income taxation for inequality.

Amartya Sen (1992, Chapter 1) has criticized the use of utilitarian social welfare functions, defined as the sum of individual utilities (see also Mirrlees, 1982; Rawls, 1971, Chapter 1; Roemer, Chapter 4; Decancq Koen, Marc Fleurbaey and Erik Schokkaert, 2015; and Tuomala, 2016, pp. 25-34). Sen argues that utilitarian social welfare functions disregard differences in income that contribute to inequality. In an example considered by Sen, one individual obtains higher utility from a given level of income than the other, so that with equal incomes utility would be unequal. Equalizing the marginal utilities of income to maximize a utilitarian social welfare function would redistribute income so that the differences in utility would be even greater. A question considered in this paper is whether the criticism of utilitarian social welfare functions extends to optimal income taxation based on utilitarian social welfare functions, and whether this criticism extends further to an optimal income tax based on a social welfare function related to Atkinson's measure of inequality. This paper shows that deviations from a utilitarian social welfare function generate systematic changes in optimal income taxes, transfers and the measurement of inequality. However, since individuals are assumed to have the same utility function, Sen's criticisms are only partially addressed.

The next section shows how optimal income taxes can be derived for the social welfare function related to Atkinson's inequality measure. Section 3 on transfers shows that marginal transfers cause losses in utility for nonutilitarian social welfare functions as well as decreases in efficiency in the form of labour supply responses to substitution effects. The section also provides an analytic result on the direction of shifts in tax collections as Atkinson's parameter increases. Section 4 shows how social welfare, inequality and production change as Atkinson's parameter increases using solutions that hold tax revenue fixed. The section also compares the separate effects of changes in the inequality parameter and changes in the income tax on social welfare and inequality using a measure based on utilities in direct analogy to Atkinson's measure. The section shows that the income tax that maximizes social welfare does not minimize inequality. The section also considers an alternative measure that decomposes inequality into wage, labour supply, taxation and utility factors. Section 5 considers implications of the results for the principle of transfers, utilitarianism, and the measurement of inequality, as well as opportunities for future work.

## 2 Optimal Income Tax

### 2.1 Derivation

Atkinson's measure of income inequality in the continuous case is given by

$$I = 1 - \frac{\left( \int_{y_{\min}}^{y_{\max}} z^{1-\rho} f_y[z] dz \right)^{1/(1-\rho)}}{\bar{y}} \quad (1)$$

where  $f_y$  is the probability density function of incomes,  $\bar{y}$  is average income and  $\rho$  is used instead of the parameter  $e$  from Atkinson's original exposition.

This is a measure of inequality but not social welfare since doubling incomes should raise social welfare but would leave the measure of inequality the same. The parameter  $\rho$  is related to the transfers among income levels, combined with income losses, that would leave inequality unchanged. Now consider replacing income in this inequality measure by utility, with the intention of constructing a social welfare function that can be used to determine an optimal income tax. As in the optimal income tax literature, suppose that individuals have exogenously distributed productivities that determine their wages  $w$  and suppose the probability density function of wages is given by  $f[w]$  on the interval from  $w_{\min}$  to  $w_{\max}$ . A worker's income is given by labour supply  $h$  times the wage rate  $w$  so that  $y = wh$ . Let  $t[y]$  be the level of taxes paid by the worker with income  $y$ . Suppose the worker's utility is a function of leisure,  $1 - h$ , and after-tax income,  $y - t[y]$ :  $u[1 - h, y - t[y]]$ . Maximizing utility with respect to  $h$  (with  $y = wh$ ) yields a first order condition on  $h$  that often (but not always) determines labour supply as an analytic function of the after-tax wage rate,  $(1 - t'[y])w$ , and after-tax income,  $y - t[y]$ :  $h[(1 - t'[y])w, y - t[y]]$ . Using  $u[1 - h, y - t[y]]$  in place of income in (1) yields a social welfare measure based on utilities instead of income:

$$\begin{aligned} SWF &= \left( \int_{w_{\min}}^{w_{\max}} (u[1 - h, y[z] - t[y[z]]])^{1-\rho} f[z] dz \right)^{1/(1-\rho)} \\ &= \left( \int_{w_{\min}}^{w_{\max}} u^{1-\rho} f[z] dz \right)^{1/(1-\rho)} \end{aligned} \quad (2)$$

where the arguments of  $u$  are dropped to simplify notation. Aggregate taxes are

$$\hat{T} = \int_{w_{\min}}^{w_{\max}} t[y[z]] f[z] dz$$

The double limit analysis developed in Sattinger (2017) can be extended to this social welfare function. This analysis applies the standard perturbation method by considering an increase in the marginal tax rate by a factor of  $k_1$  in the interval  $y_1$  to  $y_1 + \epsilon$ , corresponding to the wage interval  $w_1$  to  $w_1 + w_1\epsilon$ . Then the double limit analysis applies l'Hospital's rule to calculate the ratio of change in social welfare to the change in tax revenue in the limit as  $k_1$  approaches 1 and as  $\epsilon$  approaches zero. The derivation for the social welfare function in (2) differs from the derivation in Sattinger (2017) because of the form of the social welfare function and is presented in Appendix 1. For a perturbation generated by an increase in the marginal tax rate at  $w_1$ , the ratio of change in social welfare to change in tax revenue is

$$\begin{aligned}
& \lambda\{w_1, w_{\max}\} \tag{3} \\
&= \frac{-SWF^\rho \int_{w_1}^{w_{\max}} u^{-\rho} u_2 f[z] dz}{\frac{-w_1^2 h_1 t'[y_1]}{h+w_1 h_1(1-t'[y_1])} f[w_1] + \int_{w_1}^{w_{\max}} \frac{1+z^2 h_1 t''[y[z]]-zh_2}{1+z^2 h_1 t''[y[z]]-zh_2(1-t'[y[z]])} f[z] dz}
\end{aligned}$$

The first order condition for maximizing social welfare is that the trade-off  $\lambda\{w_1, w_{\max}\}$  should take the same value for all boundaries  $w_1$ , and should equal  $\lambda\{w_{\min}, w_{\max}\}$  and  $\lambda\{w_{\min}, w_2\}$  for all values of  $w_2$  greater than  $w_{\min}$ . As  $w_1$  approaches the upper boundary of  $w_{\max}$ , the numerator and the second term in the denominator approach zero. For the ratio to have a finite value of  $\lambda\{w_1, w_{\max}\}$ , it is necessary for the first term in the denominator (arising from an individual's substitution response  $h_1$  to an increase in the marginal tax rate) to approach zero also. Then either  $t'[y_1]$  must approach zero or else  $f[w_1]$  must approach zero (Proposition 2 in Sattinger, 2017).<sup>1</sup>

To derive the differential equation, rewrite (3) as

$$\begin{aligned}
& \lambda\{w_1, w_{\max}\} \left( + \int_{w_1}^{w_{\max}} \frac{\frac{-w_1^2 h_1 t'[y_1]}{h+w_1 h_1(1-t'[y_1])} f[w_1]}{(1+z^2 h_1 t''[y[z]]-zh_2)} f[z] dz \right) \\
&= -SWF^\rho \int_{w_1}^{w_{\max}} u^{-\rho} u_2 f[z] dz
\end{aligned}$$

Taking the derivative of both sides with respect to  $w_1$  (with  $\lambda\{w_1, w_{\max}\}$  and  $SWF$  constant) yields

$$\begin{aligned}
& \lambda\{w_1, w_{\max}\} \tag{4} \\
&= \frac{-SWF^\rho u^{-\rho} u_2 f[w_1]}{\frac{d}{dw_1} \left( \frac{-w_1^2 h_1 t'[y_1]}{h+w_1 h_1(1-t'[y_1])} f[w_1] \right) - \frac{(1+w_1^2 h_1 t''[y_1]-w_1 h_2)}{1+w_1^2 h_1 t''[y_1]-w_1 h_2(1-t'[y_1])} f[w_1]}
\end{aligned}$$

where  $u$  and  $u_2$  are functions of the wage rate, income, tax rate and labour supply at  $w_1$ . The expression in (4) can be solved for  $t''[y_1]$  to yield a second order differential equation that can be solved. With values of  $\lambda\{w_1, w_{\max}\}$  and  $SWF$  determined, a specific solution of the differential equation will depend on two constants of integration or alternatively the initial values of  $t'[y]$  and  $t[y]$  at some point  $y$ . Following the procedures in Sattinger (2017), the initial values

<sup>1</sup>Because of the chain rule for taking derivatives, the CES social welfare function in (2) yields the same functional form for trade-offs in (3) as  $\int_{w_1}^{w_{\max}} G[u]f[z]dz$ , where  $G[u] = u^\gamma/\gamma$ ,  $0 < \gamma < 1$ , and has a constant Coefficient of Relative Risk Aversion. If  $\lambda$  and  $\rho$  are used to determine an optimal income tax for (2), then  $\lambda SWF^\rho$  and  $\gamma = 1 - \rho$  yield the same solution using the social welfare function with  $G[u]$ , everything else the same. It is possible that this social welfare function can be used to derive solutions with values of  $\rho$  greater than one, but this derivation is not pursued here.

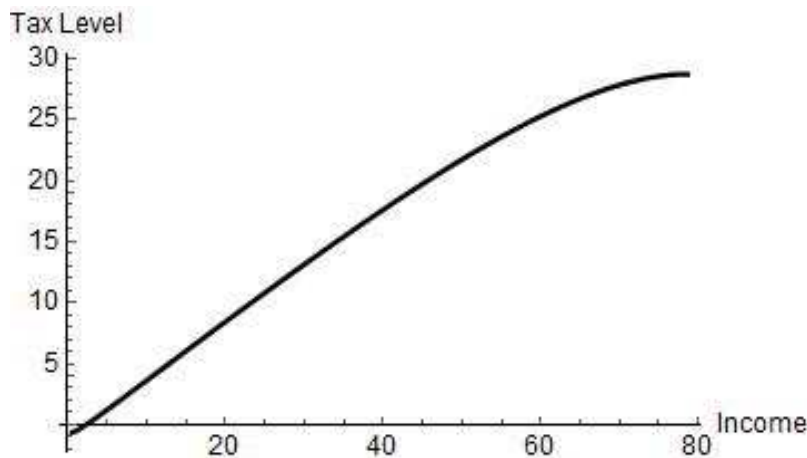


Figure 1: Tax Level,  $\rho = .2$

can be determined by setting the marginal tax rate equal to zero at the highest and lowest incomes. Then at the highest income, the average tax rate can be varied (thereby determining  $t[y_{\max}]$ ) until  $t'[y_{\min}] = 0$ .

## 2.2 Effects of $\rho$ on Optimal Tax

Figures 1 and 2 show the optimal income tax that maximizes the social welfare function in (2) with  $\rho = .2$  and other particular assumptions.<sup>2</sup> Figure 1 shows the tax level at each income while Figure 2 shows the marginal and average tax rates. In this example, the average tax rate starts out negative (since tax levels are negative at low income), rises rapidly and then declines slowly at high income levels. Figure 3 shows that in this particular case, labour supply rises with income over most of the income interval.

While these figures describe the solution at different incomes, they do not indicate the tax conditions for the majority of workers. Figure 4 shows the marginal and average tax rates by wage percentile of worker. This figure shows that average and marginal taxes rise for almost all workers as wage and income increase. The crossover income, where the average tax rate is maximized and equals the marginal tax rate, is 37.64, where 99.85 percent of the workers earn less.

Setting  $\rho = 0$ , the social welfare function in (2) reduces to a utilitarian sum of utilities. It is then possible to compare the optimal income tax for the Atkinson-based SWF with a utilitarian social welfare function, controlling for the level of

<sup>2</sup>The example assumes a lognormal distribution of wages in the interval  $[1,100]$  with parameters  $\mu = 1$  and  $\sigma = 1$ , Cobb-Douglas utility with exponent of leisure equal to .3, and  $\lambda\{1,100\} = -.35$ . The solution yields tax revenue of .434. Appendix 2 provides the Mathematica derivation of the figures in the paper.

Marginal, Average Tax Rates

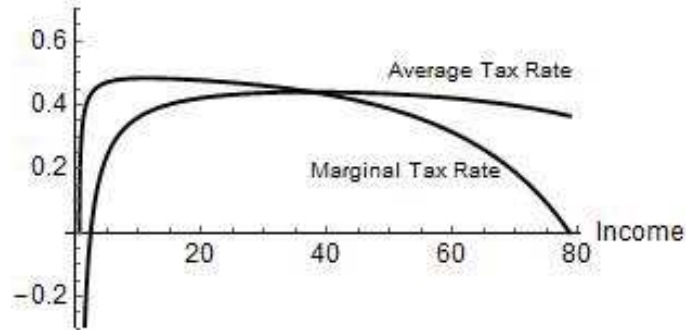


Figure 2: Marginal and Average Tax Rates,  $\rho = .2$

Labor Supply

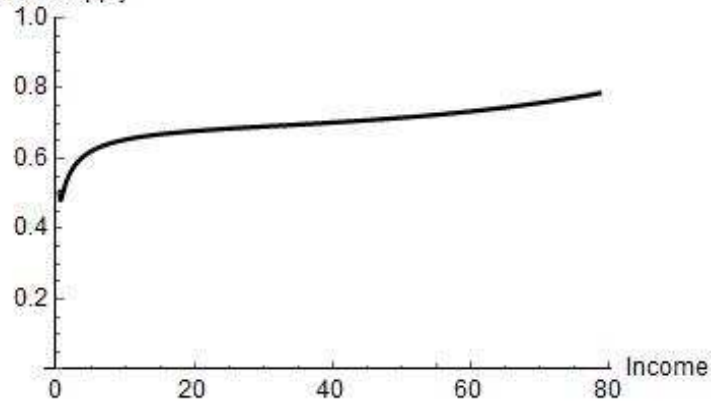


Figure 3: Labour Supply,  $\rho = .2$

Marginal, Average Tax Rates

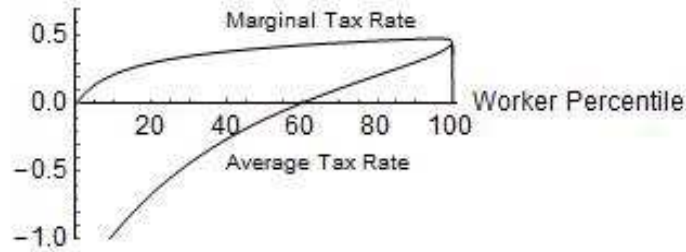


Figure 4: Marginal and Average Tax Rates by Worker Percentile

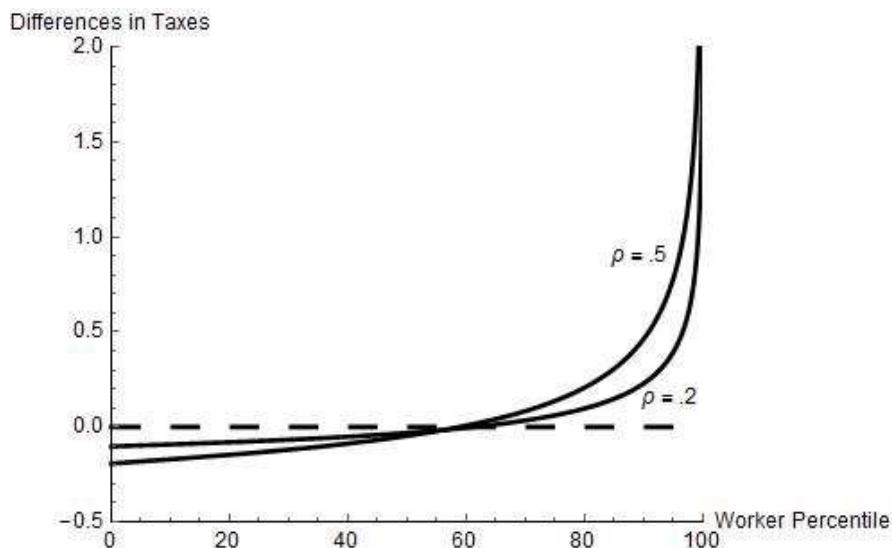


Figure 5: Tax Levels Compared to Utilitarian Social Welfare Function

tax revenues. Since the comparison arises from a change in  $\rho$ , with tax revenues the same, the conclusion from the analysis above applies: the optimal tax for  $\rho > 0$  will differ from the solution for  $\rho = 0$  (the utilitarian solution) by moving tax collections from lower income to higher income individuals. Figure 5 confirms this result, showing that when  $\rho > 0$ , taxes decline for lower income individuals and increase for higher income individuals. The figure shows differences in taxes between solutions for  $\rho = .2$  and  $\rho = .5$  in comparison to the utilitarian solution with  $\rho = 0$  (with aggregate tax revenue remaining the same). With the higher value of  $\rho$  (corresponding to greater sensitivity to inequality in the Atkinson measure), the shifts in tax collections from lower to higher income individuals are even greater.

Figures 6 and 7 expand on the comparisons. Figure 6 compares labour supplies, showing that labour supply decreases for almost all workers, and rises only for the very highest income workers. The labour supply responses are generated by income and substitution effects, arising from changes in the tax level and in the marginal tax rate, respectively. For substitution effects, an increase in the marginal tax rate reduces labour supply while a reduction raises labour supply. At lower income levels, the decline in labour supply is caused by the combined effects of increases in after-tax income from reduced taxes and greater increases in the marginal tax rate, operating through the substitution effect. For higher incomes, labour supply declines because the income effect dominates the substitution effect generated by a falling marginal tax rate. At the very highest income level, the falling marginal tax rate combines with a reduced income to generate an increase in labour supply. Figure 7 shows what



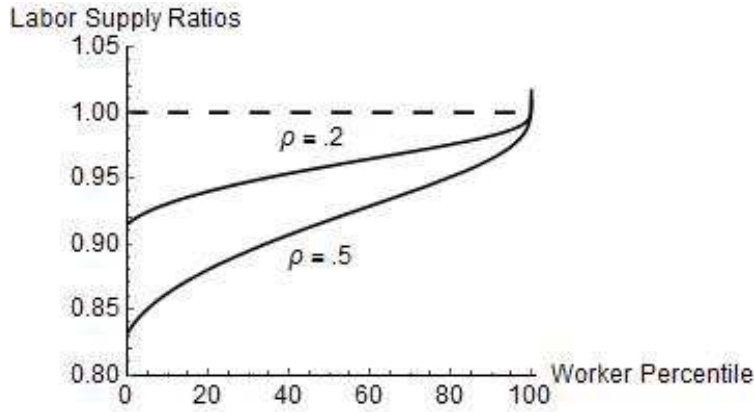


Figure 6: Labour Supplies Compared to Utilitarian Social Welfare Function

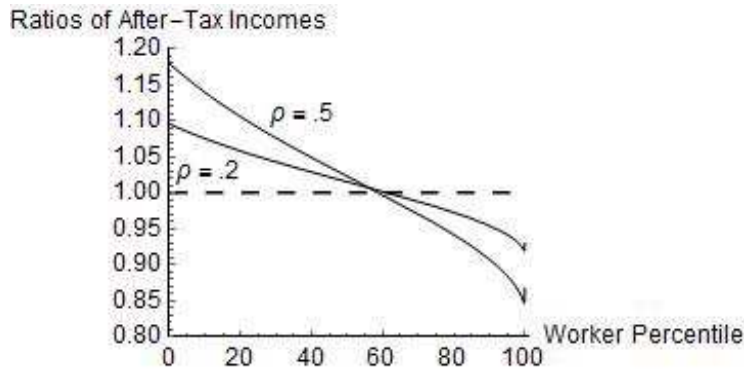


Figure 7: After-Tax Incomes Compared to Utilitarian Social Welfare Function

happens to after-tax income after labour supply and income adjust to the change in the tax system. After-tax income increases for lower income (and lower wage) workers, up to about the 63 percentile level, and declines for higher income workers. The ratio is not monotonically declining since the ratio increases for the very highest income individuals from a greater labour supply. In spite of this upward turn in the ratio, after-tax incomes decline for the highest income individuals in comparison with the utilitarian solution.

These results confirm that an optimal income tax that optimizes an Atkinson-based CES social welfare function generates distributions of taxes and after-tax incomes that differ systematically from results derived using a simple utilitarian social welfare function.

Figure 8 demonstrates a significant feature of optimal taxation for increasing values of  $\rho$ . The figure shows the marginal tax rates by worker percentile for

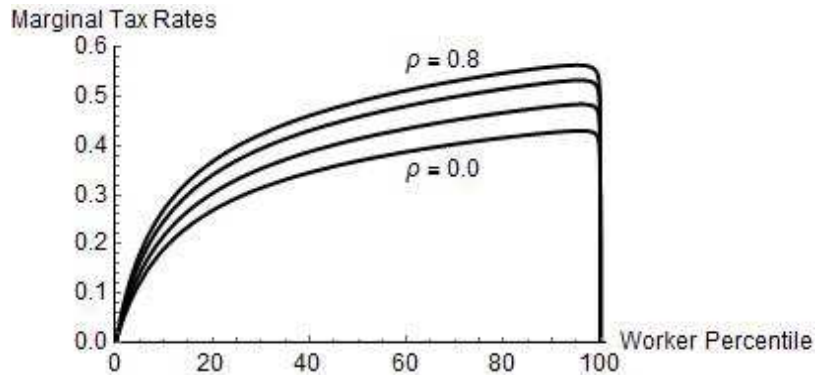


Figure 8: Marginal Tax Rates for  $\rho = 0, .2, .5,$  and  $.8$

optimal income taxes yielding the same tax revenue for  $\rho = 0, 0.2, 0.5,$  and  $0.8$ . As  $\rho$  increases, the marginal tax rate increases for all workers except the workers with the minimum and maximum wage rates.

### 3 Transfers

#### 3.1 Principle of Transfers

The principle of transfers as developed by Pigou and Dalton is that a transfer of a marginal dollar from a richer person to a poorer person must increase the sum of utility levels. The principle allows the development of criteria for redistribution that improve social welfare without knowing the specific utility function beyond the conditions that it is increasing and concave (Atkinson, 1970, p. 245). The desirability of transferring income from richer to poorer must be compared with the costs of doing so. Since plucking a marginal dollar from a richer person and giving it to a poorer person is not an admissible policy, it is necessary to consider the costs of transferring income that arise from feasible public policies. With transfers carried out by changes in the tax system, the income consequences of a transfer are measured by after-tax income. As a result of an optimal income tax, no further transfers of after-tax income from richer to poorer are desirable. With a utilitarian social welfare function ( $\rho = 0$ ), moving a dollar of after-tax income from a richer to a poorer person does not change the sum of utilities or social welfare (because the trade-off between social welfare and tax revenue is the same at all income levels).<sup>3</sup> With a nonutilitarian social welfare function ( $\rho > 0$ ), there is again no net change in social welfare, but equal changes in

<sup>3</sup>Moving a marginal dollar of tax collections from a poorer person does not in general raise their after-tax income by one dollar since they adjust their labor supply and income to the change in taxes. However, the change in utility and social welfare is the same before and after this response.

social welfare imply a smaller gain in utility at the poorer income than the loss at the richer income, as demonstrated in the following section. Despite this seeming contradiction with the principle of transfers, it remains correct that a transfer of a dollar of after-tax income from a richer to a person, if it could be carried out without any additional consequences or costs, would raise the sum of utilities and social welfare. The consequences considered in the next two sections therefore do not indicate a conflict with the principle of transfers.

### 3.2 Consequences for Utility

A feature of the original Atkinson measure of inequality is that  $\rho$  can be derived from a thought experiment in which a unit of income is moved from one income level to a lower income, and one then determines how much of an income loss makes the transfer just desirable in analogy to transporting water in a leaky bucket (Okun, 1975). In the Atkinson-based measure, a dollar of tax collections can be moved from one income level to another, and there would be no loss of social welfare because of the optimal income tax result that the trade-off between social welfare and tax collections is the same at all income levels. However, with the Atkinson-based CES measure, equal social welfare changes imply unequal utility changes if  $\rho > 0$ . The direct analogy with the original Atkinson measure would be to consider net changes in utility. The optimization strategy in this paper makes it possible to analyze the losses from shifting a dollar from one income level to another. The trade-off between social welfare and tax revenue for an interval can be derived using (3) and is given by

$$\begin{aligned} & \lambda\{w_1, w_2\} \\ & -SWF^\rho \int_{w_1}^{w_2} u^{-\rho} u_2 t'[y_1] f[z] dz \\ = & \frac{\frac{w_2^2 h_1 t'[y_2]}{h+w_2 h_1 (1-t'[y_2])} f[w_2] - \frac{w_1^2 h_1 t'[y_1]}{h+w_1 h_1 (1-t'[y_1])} f[w_1]}{\int_{w_1}^{w_2} \left( \frac{(1+z h_1 t''[y[z]] - z h_2)}{1+z h_1 t''[y[z]] - z h_2 (1-t'[y[z]])} \right) f[z] dz} \end{aligned}$$

Then the trade-off at an income level can be derived by considering the limit of the trade-off above as  $w_2$  approaches  $w_1$  (essentially by expressing  $w_2$  as  $w_1 + \epsilon$  and taking the limit as  $\epsilon$  approaches zero). The result is identical to the differential equation arising from (4) and can be expressed as

$$\lambda = \frac{-SWF^\rho u^{-\rho} u_2}{\frac{d}{dw_1} \left( \frac{-w_1^2 h_1 t'[y_1] f[w_1]}{h+w_1 h_1 (1-t'[y_1])} \right) - \frac{1+w_1^2 h_1 t''[y_1] - w_1 h_2}{1+w_1^2 h_1 t''[y_1] - w_1 h_2 (1-t'[y_1])}} \quad (5)$$

where  $\lambda$  is the common trade-off in the first order condition and  $u^{-\rho}$  and  $u_2$  are the values of these functions at  $w_1$ . Let  $u\{1\}$  and  $u\{2\}$  be the utilities at two income levels  $y_1$  and  $y_2$ , with  $y_2 > y_1$ . Consider the movement of an equal amount of tax revenue from one income interval to another. Since the trade-offs between social welfare and tax revenue are the same at both income levels, the

changes in social welfare would be the same. Then setting the two changes in social welfare equal,

$$SWF^\rho u\{1\}^{-\rho} u_2\{1\} = SWF^\rho u\{2\}^{-\rho} u_2\{2\} \quad (6)$$

where  $u_2\{1\}$  and  $u_2\{2\}$  are the marginal utilities of income at income levels  $y_1$  and  $y_2$ , respectively.<sup>4</sup> Then

$$\frac{u\{1\}^{-\rho}}{u\{2\}^{-\rho}} = \left(\frac{u\{1\}}{u\{2\}}\right)^{-\rho} = \left(\frac{u\{2\}}{u\{1\}}\right)^\rho = \frac{u_2\{2\}}{u_2\{1\}} > 1$$

If  $u\{2\}/u\{1\} = 2$ , then  $u_2\{2\}/u_2\{1\} = 2^\rho$ . If tax collection is increased by one dollar for a worker with utility  $u\{2\}$  and reduced by one dollar for a worker with utility  $u\{1\}$ , the higher income individual would lose  $u_2\{2\}$  of utility, and the lower income individual would gain  $u_2\{1\}$ . Thus there would be a net loss of utility equal to

$$u_2\{1\} - u_2\{2\} = u_2\{1\}(1 - 2^\rho) < 0$$

In the example worked out here, a dollar is moved (via tax collections) from a higher income individual to a lower income individual, and there is a resulting net loss in utility. This is formally analogous to the Atkinson thought experiment but differs in that it is utility lost instead of income.

### 3.3 Consequences for Efficiency

Transfers carried out using the tax system require changes in marginal tax rates and have consequences for the efficiency of taxation. These consequences occur because the mix of substitution effects and other taxes vary by income level. The following table shows the results of raising tax revenues by a marginal unit at income levels 1 and 4 using the solution for  $\rho = .2$  shown in Figures 1-3.<sup>5</sup>

Table 1: Transfer Consequences for Substitution Effects

Income Level	$y = 1$	$y = 4$
Change in Substitution Effects	0.1969	-0.3126
Change in Other Taxes	0.8031	1.3126

In this table, the change in substitution effects arises from the first term in the denominator of (5) and is generated by the tax consequences of the labour response to a change in the marginal tax rate at  $y = 1$  or  $y = 4$ . The change in other taxes arises from the second term in (5). This term corresponds to the second term in the denominator of (3), generated by the tax consequences for individuals at higher income levels caused by a change in the marginal tax rate at the income for wage  $w_1$ . Since the changes add up to one (per marginal unit of

<sup>4</sup>To find the numerical value of  $SWF^\rho u\{1\}^{-\rho} u_2\{1\}$ , the perturbation that generates this change in social welfare must be multiplied by the factor that would yield a unit change in tax revenue. This factor is given by  $1/(d\hat{T}/dk_1)$ , where  $d\hat{T}/dk_1$  is the numerical value of the denominator in (5).

<sup>5</sup>Appendix 3 provides the Mathematica derivations of the table entries in the paper.

tax revenue) at each income level, shifting a dollar of tax collections from  $y = 1$  to  $y = 4$  would have no effect on aggregate tax revenue. However, the shift increases the losses from substitution effects by 0.5095 (equal to  $-0.3126$  minus  $0.1969$  since taxes are reduced at  $y = 1$ ), and these losses are made up dollar for dollar by an increase in other taxes of 0.5095 (equal to  $1.3126$  minus  $0.8031$ ). The increase in losses from substitution effects reflects a loss in the tax system's efficiency. Marginal tax rates introduce a wedge between what a worker receives by providing an extra unit of labour,  $(1 - t'[y])w$ , and what an employer pays for that labour, generating inefficiencies common to most taxation. Transfers of income from higher income to lower income workers increase these inefficiencies whenever the loss from substitution effects per unit increase in net tax revenue declines as income goes up, which occurs for the optimal tax solution considered here. The increase in substitution effects from transfers affects production as shown in Figure 6. At greater values of  $\rho$ , higher marginal tax rates shift income from higher to lower income individuals and reduce labour supplies because of the substitution effects. The reduced labour supplies and resulting losses in production reflect the greater inefficiencies corresponding to higher marginal tax rates.

### 3.4 Consequences of Increasing $\rho$

The transfer properties of the Atkinson-based CES measure can be used to draw analytic inferences about the consequences of an increase in  $\rho$  for the distribution of taxes and after-tax income. The inferences arise by considering whether a marginal dollar of tax collection reduces social welfare more for a lower income when the parameter rises from  $\rho_1$  to  $\rho_2$  while the tax remains optimal for  $\rho_1$ . To establish notation for this comparison, suppose  $t_{\rho=\rho_i}$  is the optimal tax for  $\rho_i$ , with  $\rho_2 > \rho_1$ . As before, let  $y_1$  and  $y_2$  be the two income levels, with  $y_2 > y_1$ , and let  $u\{i\}$  and  $u_y\{i\}$  be the utility and marginal utility of after-tax income for an individual observed to earn income  $y_i$  when the optimal income tax is  $t_{\rho_1}$ . Let  $SWF_{\rho_2}\{t_{\rho=\rho_1}\}$  be the level of social welfare generated by the tax  $t_{\rho=\rho_1}$  when the parameter is  $\rho_2$  instead of  $\rho_1$  (the value of this term will not matter since it will drop out). Since the tax remains  $t_{\rho=\rho_1}$  and the social welfare parameter  $\rho_2$  does not affect individual labour supply decisions, individual incomes and taxes would be the same as if the parameter  $\rho_1$  were being used. Then from (5), the loss in social welfare from increasing the tax collection by a marginal dollar for an individual with income  $y_i$  (with tax  $t_{\rho_1}$  but calculating social welfare using  $\rho_2$ ) is

$$(SWF_{\rho_2}\{t_{\rho=\rho_1}\})^{\rho_2} u\{i\}^{-\rho_2} u_y\{i\}$$

The ratio of losses in social welfare is

$$\frac{(SWF_{\rho_2}\{t_{\rho=\rho_1}\})^{\rho_2} u\{1\}^{-\rho_2} u_y\{1\}}{(SWF_{\rho_2}\{t_{\rho=\rho_1}\})^{\rho_2} u\{2\}^{-\rho_2} u_y\{2\}} = \left(\frac{u\{1\}}{u\{2\}}\right)^{\rho_1 - \rho_2} \frac{u\{1\}^{-\rho_1} u_y\{1\}}{u\{2\}^{-\rho_1} u_y\{2\}}$$

From the equal trade-off condition in (6), which holds when the parameter is  $\rho_1$ ,

$$\frac{u\{1\}^{-\rho_1} u_y\{1\}}{u\{2\}^{-\rho_1} u_y\{2\}} = 1$$

Then the ratio of losses in social welfare is

$$\left(\frac{u\{1\}}{u\{2\}}\right)^{\rho_1 - \rho_2} = \left(\frac{u\{2\}}{u\{1\}}\right)^{\rho_2 - \rho_1} > 1 \quad (7)$$

Since the social welfare loss is greater at lower income levels, it is desirable to shift tax collections to higher income intervals when  $\rho$  increases.

## 4 Measurement of Inequality

### 4.1 Inequality Based on Utilities

An important feature of the Atkinson measure of inequality is that it is based on a comparison between the average income and the value of income that would equal the function of observed incomes if all individuals had that value of income. An analogous measure of inequality can be obtained using the utilities in the Atkinson-based CES measure of social welfare. To consider the general case, let  $u_g[w]$  be the utility obtained by an individual with wage  $w$ , where income, tax level, marginal tax rate and labour supply correspond to the values at  $w$ . Then define the CES social welfare based inequality as

$$I_{CES} = 1 - \frac{u^*}{\bar{u}} \quad (8)$$

where  $\bar{u}$  is the average utility:

$$\bar{u} = \int_{w_{\min}}^{w_{\max}} u_g[z] f[z] dz$$

and  $u^*$  is the utility such that

$$\begin{aligned} u^* &= \left( \int_{w_{\min}}^{w_{\max}} (u^*)^{1-\rho} f[z] dz \right)^{\frac{1}{1-\rho}} \\ &= \left( \int_{w_{\min}}^{w_{\max}} u_g[z]^{1-\rho} f[z] dz \right)^{\frac{1}{1-\rho}} \end{aligned} \quad (9)$$

In analogy to Atkinson's equally distributed equivalent level of income in his measure of inequality,  $u^*$  is the equally distributed equivalent level of utility, or simply the equivalent utility.<sup>6</sup>

<sup>6</sup>The measure  $I_{CES}$  differs from Dalton's inequality index, which would be one minus the average utility divided by the utility at average income (Cowell, 1995, p. 46).

An initial question is whether the optimal income tax minimizes this measure of inequality. Let  $t_{\rho=\rho_1}$  be the optimal income tax for  $\rho = \rho_1$ . Then (9) will be the social welfare  $SWF$ , which is maximized by the optimal income tax. Since  $u^*$  is maximized, one may expect that the measure of inequality  $I_{CES}$  in (8) is minimized, but this expectation is incorrect. With the income tax that maximizes the social welfare function, it is always possible to go beyond the transfer consistent with the parameter  $\rho$  by transferring more tax collections from the poor to the rich. The greater transfers would reduce inequality and social welfare. This trade-off is demonstrated later in this section in Table 4.

A second question is how the measure of inequality changes as the parameter  $\rho$  increases, with the income tax adapting to maximize the social welfare for the higher value of  $\rho$ . The results on transfers in Section 3 indicate that as  $\rho$  increases, greater transfers of tax collections from lower to higher income individuals (so that after-tax incomes would be shifted from richer to poorer individuals) would raise social welfare. This section presents an alternative analytic approach based on deviations from an optimal income tax. As before, let  $t_{\rho=\rho_1}$  be the optimal income tax that maximizes the CES social welfare function when  $\rho = \rho_1$ . Let  $\delta\rho$  be a small positive change in  $\rho$  and consider the optimal income tax  $t_{\rho=\rho_1+\delta\rho}$  that maximizes social welfare for  $\rho = \rho_1 + \delta\rho$  while leaving tax revenue unchanged. The change in the tax level at income  $y$  from the change in tax schedule from  $t_{\rho=\rho_1}[y]$  to  $t_{\rho=\rho_1+\delta\rho}[y]$  can be written as

$$\delta t_{\rho=\rho_1}[y] = t_{\rho=\rho_1+\delta\rho}[y] - t_{\rho=\rho_1}[y]$$

In Euler's solution for the functional in the calculus of variations, this deviation from  $t_{\rho=\rho_1}[y]$  (leaving the boundary conditions satisfied) would have no first-order effects on the functional being maximized, the social welfare function, while leaving tax revenue the same. The change in social welfare from the increase in  $\rho$  can now be considered in two steps. The first change is calculated by taking the derivative of social welfare with respect to  $\rho$ , holding the tax function  $t_{\rho=\rho_1}$  (and therefore individual choices of labour supply and income) the same. The second change is calculated by working out how labour supplies and incomes respond to the change in taxes, and then how these responses affect social welfare. However, since the change in taxes  $\delta t_{\rho=\rho_1}[y]$  is a deviation from  $t_{\rho_1}[y]$ , the first order effects on social welfare are zero, and can be disregarded along with the second step. (This is essentially an envelope theorem argument applied to the optimization of a functional instead of a function.) The following result then arises by using just the first step (the full derivation is provided in the appendix).

$$\begin{aligned}
& \frac{d}{d\rho} \left( \int_{w_{\min}}^{w_{\max}} u_g[z]^{1-\rho} f[z] dz \right)^{\frac{1}{1-\rho}} \\
&= \frac{SWF}{1-\rho} \left( \text{Log}[SWF] - \frac{\int_{w_{\min}}^{w_{\max}} \text{Log}[u_g[z]] u_g[z]^{1-\rho} f[z] dz}{SWF^{1-\rho}} \right)
\end{aligned} \tag{10}$$

Table 2 presents social welfare and inequality calculations generated by optimal income taxes for different values of  $\rho$ , holding tax revenue the same and using the parameters and assumptions for the solutions in Figure 7. In the table, as  $\rho$  increases, social welfare decreases, the equivalent utility (equal to social welfare) also decreases, and inequality increases. The table confirms the consequences of using a higher value of  $\rho$  to evaluate social welfare and determine the optimal income tax. In the case with  $\rho = 0$ , corresponding to the utilitarian social welfare function, social welfare is simply the average utility, so that the inequality measure is zero. As  $\rho$  increases beyond the utilitarian case, corresponding to higher values of Atkinson's  $e$ , the social welfare function decreases and inequality increases as a result of a greater sensitivity to differences in utility, despite the adjustment in optimal income taxes. The decline in average utilities confirms the results of Section 3 on transfers. As  $\rho$  increases, it becomes more desirable to increase transfers from higher to lower income individuals, as demonstrated in (7). The greater transfers reduce aggregate and average utility, as indicated by the decline in  $\bar{u}$ .

The decline in production and increase in average tax rate are consistent with the inefficiencies generated by greater substitution effects as  $\rho$  increases. Since lower income individuals have a higher proportion of changes in substitution effects per dollar of additional tax revenue, transfers of income to lower income individuals raise the level of substitution effects, as shown in Table 1, Section 3.2. Inefficiencies from greater substitution effects then reduce production through reduced labour supply and require higher average tax rates to generate the same aggregate tax revenue. The calculation of  $dSWF/d\rho$  from (10) is consistent with the decline in social welfare as  $\rho$  increases. Using the value of  $dSWF/d\rho$  at  $\rho = 0$ , the predicted decline in social welfare would be 0.126 times .2, or 0.0252, compared to the actual decline of  $1.485 - 1.463 = .022$ . For the change from  $\rho = 0.2$  to  $\rho = 0.5$  and from  $\rho = 0.5$  to  $\rho = 0.8$ , the predicted declines are 0.0297 and 0.0216, compared to the actual declines of 0.025 and 0.013, respectively.



Table 2: Outcomes for Different Values of  $\rho$ 

	$\rho = 0.0$	$\rho = 0.2$	$\rho = 0.5$	$\rho = 0.8$
<i>SWF</i> , Social Welfare	1.485	1.462	1.436	1.416
Average Utility, $\bar{u}$	1.485	1.483	1.478	1.472
Equivalent Utility, $u^*$	1.485	1.462	1.436	1.416
<i>ICES</i> , Inequality	0.0	.0139	.0280	.0380
Production	3.235	3.145	3.050	2.982
Tax Revenue	0.434	0.434	0.434	0.434
Average Tax Rate	0.134	0.138	0.142	0.145
$dSWF/d\rho$	-0.126	-0.100	-0.076	-0.061

Table 3 shows the separate effects of changing the parameter  $\rho$  used to calculate social welfare and adjusting the optimal income tax to parameter  $\rho$ . Each row shows different social welfare calculations for the same optimal income tax. Since the tax is the same in a given row, labour responses, production, utilities and tax revenues are the same, but the level of social welfare varies with the parameter  $\rho$  used in the calculation. The diagonal values in this table (1.485, 1.462, 1.436 and 1.416) are the same values that appear for social welfare in Table 2. Consider the change in social welfare in moving from  $\rho = 0$  to  $\rho = .2$ . Holding the tax the same (at the income tax that is optimal for  $\rho = 0$ ), social welfare declines to 1.460 when  $\rho = .2$  in the calculation. Holding the parameter  $\rho = .2$  the same in calculating social welfare, the change in the optimal income tax to the tax that is optimal for  $\rho = .2$  raises social welfare from 1.460 to 1.462. The change in the optimal income tax reverses only part of the decline in social welfare that occurs in moving from  $\rho = 0$  to  $\rho = .2$ . The table also confirms that the optimal income tax is indeed optimal: social welfare for a given value of  $\rho$  is maximized by the income tax that is optimal for that value of  $\rho$ .

Table 3: Social Welfare by Tax and Parameter

	$\rho = 0$	$\rho = .2$	$\rho = .5$	$\rho = .8$
$t_{\rho=0}$ , Tax Optimal for $\rho = 0$	1.485	1.460	1.427	1.397
$t_{\rho=.2}$ , Tax Optimal for $\rho = .2$	1.483	1.462	1.434	1.408
$t_{\rho=.5}$ , Tax Optimal for $\rho = .5$	1.478	1.460	1.436	1.415
$t_{\rho=.8}$ , Tax Optimal for $\rho = .8$	1.472	1.457	1.435	1.416

Table 4 shows the separate effects of the parameter  $\rho$  and the tax on the measure of inequality given by *ICES* in (8). Inequality always rises as  $\rho$  increases, holding the tax the same (i.e., moving across the table). Except for the utilitarian case with  $\rho = 0$ , inequality always falls as the parameter  $\rho$  for which the tax system is optimal increases, holding the social welfare parameter the same (i.e., moving down the table). As noted above in this section, it is not the case that the tax that maximizes social welfare for a given parameter value also minimizes inequality for that parameter value. For example, if  $\rho = .5$ , using the tax that is optimal for  $\rho = .8$  will yield a lower level of inequality (.0249 versus .0280).

Maximization of social welfare is not the same as minimization of inequality: using a more redistributive tax system than the optimal income tax can lower inequality, placing social welfare and equality in some degree of conflict for

non-marginal changes in the tax system.

Table 4: Inequality by Tax and Parameter

	$\rho = 0$	$\rho = .2$	$\rho = .5$	$\rho = .8$
$t_{\rho=0}$ , Tax Optimal for $\rho = 0$	0	0.0164	0.0391	0.0595
$t_{\rho=.2}$ , Tax Optimal for $\rho = .2$	0	0.0139	0.0331	0.0505
$t_{\rho=.5}$ , Tax Optimal for $\rho = .5$	0	0.0118	0.0280	0.0427
$t_{\rho=.8}$ , Tax Optimal for $\rho = .8$	0	0.0105	0.0249	0.0380

## 4.2 Inequality Decomposition

Although  $I_{CES}$  is a measure of inequality that is directly analogous to Atkinson's measure, it has several shortcomings. First, it cannot be calculated from observable data since it relies on utilities. Second, the Atkinson measure compares a distribution to an equal distribution in calculating social welfare and inequality, whereas the optimal income tax context assumes exogenous inequality in productivities and wages. Third,  $I_{CES}$  is not decomposable into the factors that determine inequality. This section proposes an alternative measure that follows the generation of inequality through four steps, with the first three steps generating an observable measure and the fourth step based on an assumed utility function. These measures are decomposable since the products of the factors yields the measures.<sup>7</sup> The first measure is observable because it does not depend on the measurement of utility and could be calculated using suitable data.

**Definition 1** *The Observable After-Tax Inequality Measure is*

$$I_{AT} = 1 - y_{AT}^* / \bar{y}_{AT} \quad (11)$$

where  $\bar{y}_{AT}$  is average after-tax income and

$$y_{AT}^* = \left( \int_{w_{\min}}^{w_{\max}} (y[z] - t[y[z]])^{1-\rho} f[z] dz \right)^{\frac{1}{1-\rho}} \quad (12)$$

**Definition 2** *The Utility After-Tax Inequality Measure is*

$$I_{ATu} = 1 - y_{ATu}^* / y_{AT}^* \quad (13)$$

where  $y_{AT}^*$  is given in (12) and  $y_{ATu}^*$  is the level of after-tax income such that the utility at that after-tax income,  $u_{AT}[y_{ATu}^*]$ , equals social welfare:

$$u_{AT}[y_{ATu}^*] = \left( \int_{w_{\min}}^{w_{\max}} u_{AT}[y[z] - t[y[z]]]^{1-\rho} f[z] dz \right)^{\frac{1}{1-\rho}}$$

where  $u_{AT}[y - t[y]]$  is the utility corresponding to after-tax income  $y - t[y]$ .

<sup>7</sup>Decomposability usually refers to how the measure of inequality is related to inequality between and among subgroups of individuals, or to inequality in sources of income.

The steps in the calculation of these two measures are as follows.

1. The wage factor is

$$\mathcal{F}_1[\rho] = \frac{w^*}{\bar{w}}$$

where  $\bar{w}$  is the average wage

$$\bar{w} = \int_{w_{\min}}^{w_{\max}} z f[z] dz$$

and

$$w^* = \left( \int_{w_{\min}}^{w_{\max}} z^{1-\rho} f[z] dz \right)^{\frac{1}{1-\rho}} \quad (14)$$

If all wages were equal, the wage factor would be one. For  $0 < \rho < 1$ , unequal wages generate a wage factor that is less than one. The difference between the wage factor and one measures the inequality in the exogenous distribution of wages using the parameter  $\rho$  from Atkinson's measure.

2. Let

$$y^* = \left( \int_{w_{\min}}^{w_{\max}} y[z]^{1-\rho} f[z] dz \right)^{\frac{1}{1-\rho}}$$

The labour supply factor is

$$\mathcal{F}_2[\rho] = \frac{h^*}{\bar{y}/\bar{w}}$$

where  $h^*$  equals  $y^*/w^*$ . If all wages were equal (so that the wage factor would be one), all labour supplies would be equal too, and  $h^*$  would equal average labour supply,  $\bar{h}$ . The labour supply factor would be one since then  $h^*/(\bar{y}/\bar{w}) = \bar{w}\bar{h}/\bar{y}$ . If instead wages are unequal, individuals will respond by supplying different amounts of labour, and the labour supply factor will be less than one. The inequality generated by the labour supply factor is not exogenous but contributes to inequality through endogenous responses to wage differences if more productive (higher wage) individuals supply more labour. As a result of a labour supply factor less than one, there will be greater inequality in incomes than in wage rates. Note that the product of the wage factor and labour supply factor is

$$\frac{w^*}{\bar{w}} \frac{h^*}{\bar{y}/\bar{w}} = \frac{y^*}{\bar{y}}$$

3. The taxation factor arises because after-tax incomes are generally more equally distributed than pre-tax incomes. The taxation factor is

$$\mathcal{F}_3[\rho] = \frac{y_{AT}^*/\bar{y}_{AT}}{y^*/\bar{y}}$$

where  $\bar{y}_{AT}$  is average after-tax income and

$$y_{AT}^* = \left( \int_{w_{\min}}^{w_{\max}} (y[z] - t[y[z]])^{1-\rho} f[z] dz \right)^{\frac{1}{1-\rho}}$$

As before, if wage rates were equal,  $y^*/\bar{y}$  would be one, all individuals would receive the same after-tax income, and  $y_{AT}^*$  would equal average after-tax income  $\bar{y}_{AT}$ . Then the taxation factor would equal one, and taxation would not reduce inequality since there would be no inequality. If wages are unequal, after-tax incomes will be more equally distributed than pre-tax incomes, and the taxation factor would be greater than one. The observable product of the first three factors is

$$\prod_{i=1}^{i=3} \mathcal{F}_i[\rho] = \frac{w^*}{\bar{w}} \frac{h^*}{\bar{h}} \frac{y_{AT}^*/\bar{y}_{AT}}{y^*/\bar{y}} = \frac{y^*}{\bar{y}} \frac{y_{AT}^*/\bar{y}_{AT}}{y^*/\bar{y}} = \frac{y_{AT}^*}{\bar{y}_{AT}} \quad (15)$$

One minus this observable product yields the Observable After-Tax Inequality Measure in (11).

4. The utility factor modifies the product of the first three factors to incorporate the consequences of the utility function for the measurement of inequality. The utility factor is

$$\mathcal{F}_4[\rho] = \frac{y_{ATu}^*}{y_{AT}^*}$$

where  $y_{ATu}^*$  is determined in (12). Then the product of the four factors is the observable product of the three factors times the utility factor:

$$\prod_{i=1}^{i=4} \mathcal{F}_i[\rho] = \frac{y_{AT}^*}{\bar{y}_{AT}} \frac{y_{ATu}^*}{y_{AT}^*} = \frac{y_{ATu}^*}{\bar{y}_{AT}}$$

One minus the product of the four factors yields the Utility After-Tax Inequality Measure in (13). The utility factor arises because utilities will not be proportional to after-tax incomes, so that utilities could be more equally or less equally distributed than after-tax incomes.

Table 5 shows the values of the four factors and the resulting products and inequality measures using the solutions for  $\rho > 0$ . The wage factor is generated by the exogenous distribution of wage rates. For the same distribution of wage rates, the wage factor varies depending on the value of the parameter  $\rho$  used to evaluate differences. Higher values of  $\rho$  yield a measure of inequality that is more sensitive to differences in wage rates, generating a lower factor (i.e., as  $\rho$  increases, the wage rate in (14) decreases relative to the average wage, which does not depend on  $\rho$ ). The same reasoning does not apply to the other factors, since both the numerators and denominators of the factors are affected

by  $\rho$ . The labour supply factor also declines as  $\rho$  increases, since  $h^*$  declines more than  $\bar{y}/\bar{w}$  declines. Taxation reduces inequality, so the factor is greater than one. The product of the first three factors is less than one and declines with  $\rho$ , so that the measure of inequality  $I_{AT}$  increases with  $\rho$ . Inclusion of the fourth factor, for utility, does not change this pattern.

Table 5: Decomposition of Inequality

	$\rho = .2$	$\rho = .5$	$\rho = .8$
Wages, $w^*/\bar{w}$	0.925	0.827	0.746
labour Supply, $h^*/(\bar{y}/\bar{w})$	0.982	0.943	0.899
Taxation, $(y_{AT}^*/\bar{y}_{AT})/(y^*/\bar{y})$	1.058	1.178	1.328
Product of Factors, $y_{AT}^*/\bar{y}_{AT}$	0.961	0.919	0.890
Inequality using $I_{AT}$	0.039	0.081	0.110
Utility, $y_{ATu}^*/y_{AT}^*$	0.957	0.983	1.001
Product of All Factors, $y_{ATu}^*/\bar{y}_{AT}$	0.920	0.904	0.891
Inequality using $I_{ATu}$	0.080	0.096	0.109

Table 6 uses the information on the factors to show their relative contributions to inequality. Dividing up the factors into percentage determinants of inequality would be misleading since the taxation factor is greater than one. Instead, a simple expression arises by calculating the percentage change in the product of factors arising from a given factor, calculated as  $(1 - 1/\mathcal{F}_i[\rho])100\%$ . For example, for the wage factor, with  $\rho = .2$ , the percentage change in the product of factors is  $(1 - 1/0.925)100\% = -8.11\%$ . Changing the wage factor from one to 0.925 therefore reduces the product of the factors by 8.11%, holding everything else the same, and raises the measure of inequality, whether using  $I_{AT}$  or  $I_{ATu}$ . Comparing these contributions to products, the exogenous distribution of wage rates is the major contributor to inequality, with labour supply differences contributing substantially less. The optimal income tax counteracts a substantial portion of this exogenous inequality but not all of it. The utility factor has a minor impact on the product of factors used to calculate  $I_{ATu}$ , and the impact can be positive or negative. If this specific case characterizes the effects of the utility function on inequality, a calculation based on observable values,  $I_{AT}$ , would not be very different from a calculation based on utilities. With appropriate data (wage rates reflecting productivities, labour supply levels determined by worker choice, and after-tax incomes), the first three factors and the inequality measure  $I_{AT}$  could be calculated across economies or over time to determine how taxation has affected the level of inequality. This empirical application as well as development of further properties of the decomposable measure  $I_{AT}$  will be left to later work.

Table 6: Percentage Contributions to Products

	$\rho = .2$	$\rho = .5$	$\rho = .8$
Wages	-8.11%	-20.88%	-34.13%
Labour Supply	-1.88%	-5.99%	-11.29%
Taxation	5.49%	15.11%	24.72%
Utility	-4.46%	-1.75%	0.10%

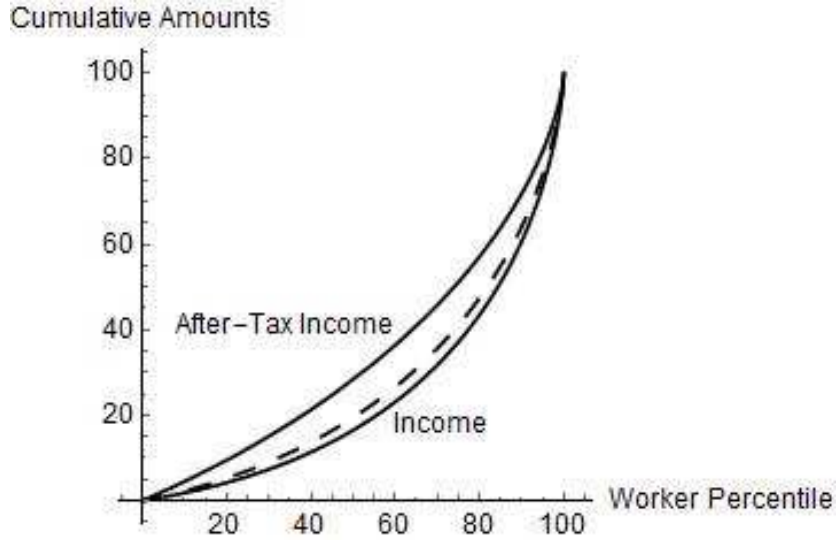


Figure 9: Lorenz Curves for Wages, Incomes and After-Tax Incomes

In addition to decomposing the sources of inequality, it is possible to show how endogenous labour supplies and taxation affect the distribution of after-tax labour incomes. Figure 9 uses Lorenz curves to compare the distributions of wages, incomes and after-tax incomes using the optimal income tax solution for  $\rho = .2$ . The dashed curve in the middle shows cumulative wages, the curve to the right shows income (the effect of endogenous labour supply), and the curve to the left shows cumulative after-tax income.

## 5 Conclusions

Using utilities instead of incomes, the optimal income tax related to Atkinson's measure of inequality can be determined. Choosing a higher value of the parameter  $\rho$  (corresponding to Atkinson's  $\epsilon$ ) generates changes in the optimal income tax that transfer tax collections from poorer to richer individuals, thereby transferring income from richer to poorer. Like the leaky bucket in Atkinson's analogy, using the tax system to transfer income generates losses. While the solution strategy for the optimal income tax implies that the ratio of change in social welfare to change in tax revenue must be the same for all, equal changes in social welfare imply unequal changes in utilities when the social welfare function is nonutilitarian. Then a transfer of income from a richer to a poorer individual implies a greater loss in utility from the richer person than the gain in utility for the poorer person. Since the ratio of substitution effects to tax revenue is higher for a poorer person, transfers also imply a net increase in substitution

effects that reduce labour supply and production as shown in Figure 6 and Table 2. The optimal income tax that maximizes the CES social welfare function associated with Atkinson’s measure of inequality does not minimize the corresponding measure of inequality  $I_{CES}$  given in (8). Starting with the optimal income tax for a given parameter value (for example  $\rho = .2$ ), shifting to the optimal income tax for a higher value of  $\rho$  transfers more income from richer to poorer individuals, reducing the social welfare function but reducing inequality further. This occurs because average utility declines more than social welfare when the tax for a higher parameter value is used.

The development of optimal income taxation by Mirrlees and others allows the determination of taxation based on a social welfare function instead of general principles of equal sacrifice, horizontal equity or vertical equity (Richard Musgrave and Peggy Musgrave, 1980, Part 3; Mirrlees et al, 2011). The methods developed in this paper make it possible to analyze the contribution of a tax system to measures of inequality in after-tax income.<sup>8</sup> Using the measure of inequality given by  $I_{AT}$ , taxation reduces inequality more than the labour supply responses increase it, but only partly reduces the inequality generated by exogenous wages and productivity. The factorization method would allow the determination of the inequality generated by observed taxation systems as well as a comparison of alternative taxation systems, whether optimal or observed. A further consequence of the application of optimal income taxation to the social welfare function associated with Atkinson’s measure is that it generates a specific trade-off between production and inequality operating through the increasing transfers induced by more redistributive tax systems. In Table 2, as the income tax changes from the system that is optimal for  $\rho = .2$  to the system that is optimal for  $\rho = .5$  (holding  $\rho$  fixed at  $.2$ ), inequality as measured by  $I_{CES}$  declines from  $.0139$  to  $.0116$  while production declines from  $3.145$  to  $3.050$ .

Optimal income tax solutions for different values of  $\rho$ , keeping tax revenues the same, allow a determination of the consequences of departing from a utilitarian social welfare function generated by  $\rho = 0$ . Nonutilitarian social welfare functions with  $\rho > 0$  systematically affect taxation, after-tax incomes and labour supply as shown in Figures 5, 6 and 7. Although the CES functional form is sensitive to differences in utility when  $\rho > 0$ , it generates the same optimal taxes as a sum of functions of utility as indicated in Section 2.1. Then the CES social welfare function only departs from the utilitarian social welfare function by placing transformations on utilities. Changing the weighting of different utilities in this way does not fully address objections to utilitarianism but shows that the optimal income tax responds by increasing transfers from richer to poorer individuals at the cost of reducing total utilities and increasing inefficiencies. Relating optimal income taxation to the criticisms of utilitarianism is limited by the requirement that the tax be a function of income alone so that it cannot take into account individual conditions.

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<sup>8</sup>See the analysis of taxation and inequality in Nanak Kakwani, 1980, chapters 11 and 12; Lambert, 2001, Chapters 7 through 9; Lambert, 1985, 1993; and Tapan Mitra and E. Ok, 1996.

An issue in optimal income taxation that is relevant to the measurement of inequality and social welfare concerns the optimal tax when individuals are heterogeneous with respect to parameters affecting income or utility (Jacquet et al, 2013; Jacquet and Lehmann, 2015). For example, individuals may vary by preferences for leisure (or costs of working) or health needs, or may choose not to participate in the labour market. Questions arise in the determination of an optimal income tax when the contributions of different individuals to social welfare cannot be compared. Inequalities in utilities would arise for individuals with the same income. A measure of inequality that excludes individuals outside the labour market would not represent inequality in the population. These issues raise further questions in the relation between taxation and inequality.

The methods of analysis developed in this paper suggest opportunities for further applications. The same methods could be applied to alternative measures of inequality such as the Gini coefficient. This application would determine the effects of taxation on the Gini coefficient and a comparison of taxes that are optimal for the social welfare functions related to different inequality measures. Implications of the principle of diminishing transfers, proposed by Kolm (1976, p. 417), could be derived by finding the optimal income tax for Kolm's absolute measure (Kolm, 1976, p. 419). The properties of the inequality measure  $I_{AT}$  need to be determined, in particular the relationship among the successive Lorenz curves generated at each step after applying a factor. An empirical implementation of the measure including the wage, labour supply and taxation factors would provide an assessment of the role of tax systems in affecting inequality as an alternative to traditional methods related to horizontal and vertical equity and tax incidence. By demonstrating how optimal income taxation can be related to the measurement of inequality, this paper opens new areas in the study of public finance and social welfare.

## References

- [1] Atkinson, Anthony B. (1970). "On the Measurement of Inequality," *Journal of Economic Theory*, Vol. 2, 244-263.
- [2] Atkinson, Anthony B., and Brandolini, Andrea (2015). "Unveiling the Ethics Behind Inequality Measurement: Dalton's Contribution to Economics," *The Economic Journal*, 125th Anniversary Issue, Vol. 125 (583), 209-220.
- [3] Cowell, Frank A. (1995). *Measuring Inequality*, second edition. Prentice-Hall, London.
- [4] Cowell, Frank A. (2000). "Measurement of Inequality," Chapter 2 in A.B. Atkinson and F. Bourguignon (eds), *Handbook of Income Distribution*, Volume 1. Elsevier, Amsterdam.
- [5] Cowell, Frank A. and Flachaire, Emmanuel (2015). "Statistical Measures for Distributional Analysis," Chapter 6 in Anthony Atkinson and François



- Bourguignon (eds), *Handbook of Income Distribution*, Volume 2A, Elsevier, Amsterdam, 359-465.
- [6] Dalton, H. (1920). "The Measurement of the Inequality of Incomes", *Economic Journal*, Vol. 30, 348-361.
- [7] Decancq, Koen, Fleurbaey, Marc and Schokkaert, Erik (2015), "Inequality, Income, and Well-Being," Chapter 2 in Anthony Atkinson and François Bourguignon (eds), *Handbook of Income Distribution*, Volume 2A, Elsevier, Amsterdam, 67-140.
- [8] Deutsch, Joseph and Silber, Jacques (1999). "Inequality Decomposition by Population Subgroups and the Analysis of Interdistributional Inequality," Chapter 13 in Jacques Silber (ed), *Handbook on Income Inequality Measurement*, Kluwer, Boston, 363-397.
- [9] Diamond, Peter (1998). "Optimal Income Taxation: An Example with a U-Shaped Pattern of Optimal Marginal Tax Rates", *American Economic Review*, Vol. 88, 83-95.
- [10] Hemming, R. (1980). "Income Tax Progressivity and Labor Supply," *Journal of Public Economics*, Vol. 14, 95-100.
- [11] Jacquet, Laurence, Lehmann, Etienne, and Van der Linden, Bruno (2013). "Optimal Redistributive Taxation with Both Extensive and Intensive Responses," *Journal of Economic Theory*, Vol. 148, 1770-1805.
- [12] Jacquet, Laurence and Lehmann, Etienne (2015). "Optimal Income Taxation when Skills and Behavioral Elasticities are Heterogeneous," CESifo Working Paper 5265.
- [13] Jenkins, Stephen P., and Van Kerm, Philippe (2009). "The Measurement of Economic Inequality," Chapter 3 in Wiemer Salvedra, Brian Nolan and Tim Smeeding (eds), *The Oxford Handbook of Economic Inequality*, Oxford University Press, Oxford.
- [14] Kakwani, Nanak C. (1977). "Measurement of Tax Progressivity: An International Comparison," *Economic Journal*, Vol. 87, 71-80.
- [15] Kakwani, Nanak C. (1980). *Income Inequality and Poverty; Methods of Estimation and Policy Applications*. Oxford University Press, New York.
- [16] Kolm, Serge-Christophe (1969). "The Optimal Production of Social Justice," in J. Margolis and H. Guitton, eds, *Public Economics*. Macmillan, London.
- [17] Kolm, Serge-Christophe (1976). "Unequal Inequalities I," *Journal of Economic Theory*, Vol. 12(3), 416-442.
- [18] Lambert, Peter J. (1985). "The Redistributive Effect of Taxes and Benefits," *Scottish Journal of Political Economy* Vol. 32, 39-54.

- [19] Lambert, Peter J. (1993). "Inequality Reduction Through the Income Tax," *Economica* Vol. 60, 357-365.
- [20] Lambert, Peter J. (1999). "Redistributional Effects of Progressive Income Taxes," Chapter 16 in Jacques Silber (ed), *Handbook on Income Inequality Measurement*, Kluwer, Boston, 485-509.
- [21] Lambert, Peter J. (2001). *The Distribution and Redistribution of Income*, third edition. Manchester University Press, Manchester.
- [22] Lerman, Robert I. (1999). "How Do Income Shares Affect Income Inequality?," Chapter 12 in Jacques Silber (ed), *Handbook on Income Inequality Measurement*, Kluwer, Boston, 341-362.
- [23] Maasoumi, Esfandiar (1999), "Multidimensioned Approaches to Welfare Analysis," Chapter 15 in Jacques Silber (ed), *Handbook on Income Inequality Measurement*, Kluwer, Boston, 437-484.
- [24] Mirrlees, James A. (1971). "An Exploration in the Theory of Optimal Income Taxation", *Review of Economic Studies*, Vol. 38, 175-208.
- [25] Mirrlees, James A. (1982). "The Economic Uses of Utilitarianism," in Amartya Sen and Bernard Williams (eds), *Utilitarianism and Beyond*. Cambridge University Press, Cambridge, Chapter 3, 63-84.
- [26] Mirrlees, James A., Adam, Stuart, Besley, Timothy, Blundell, Richard, Bond, Stephen, Chote, Robert, Gammie, Malcolm, Johnson, Paul, Myles, Gareth, and Poterba, James (Eds.) (2011). *Tax by Design: The Mirrlees Review*, Chapter 3, "The Taxation of Labour Earnings," Institute for Fiscal Studies, Oxford University Press, Oxford.
- [27] Mitra, T. and Ok, E. (1996). "Personal Income Taxation and the Principle of Equal Sacrifice Revisited," *International Economic Review* Vol. 37, 925-948.
- [28] Musgrave, Richard A., and Musgrave, Peggy B. (1980). *Public Finance in Theory and Practice*. McGraw-Hill, New York.
- [29] Okun, A.M. (1975). *Equality and Efficiency. The Big Tradeoff*. The Brookings Institution, Washington.
- [30] Pigou, A.C. (1912). *Wealth and Welfare*. London: Macmillan.
- [31] Rawls, John (1971). *A Theory of Justice*. Belknap Press of Harvard University Press, Cambridge.
- [32] Roemer, John E. (1996). *Theories of Distributive Justice*. Harvard University Press, Cambridge.
- [33] Saez, Emmanuel (2001). "Using Elasticities to Derive Optimal Income Tax Rates," *Review of Economic Studies*, Vol. 68, 205-229.

- [34] Sandmo, A. (1983). "Progressive Taxation, Redistribution and Labor Supply," *Scandinavian Journal of Economics*, Vol. 85, 311-323.
- [35] Sattinger, Michael (2017). "Double Limit Analysis of Optimal Personal Income Taxation," forthcoming, *Oxford Economic Papers*.
- [36] Sen, Amartya (1992). *Inequality Reexamined*. Harvard University Press, Cambridge.
- [37] Sen, Amartya (1997). *On Economic Inequality; Expanded Edition*. Clarendon Press, Oxford.
- [38] Silber, Jacques, editor (1999). *Handbook of Income Inequality Measurement*, Springer Science and Business Media. Kluwer, Boston.
- [39] Slemrod, Joel (1994). *Tax Progressivity and Income Inequality*. Cambridge University Press, Cambridge.
- [40] Tuomala, Matti (2016). *Optimal Redistributive Taxation*. Oxford University Press, Oxford.