An Empirical Study
of Stock and American Option Prices

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Abstract

This paper describes an empirical study of the daily price of U.S. stocks and American options with high trading volume. Asset portfolios and a nonparametric measure of aggregate volatility are taken as proxies for the systematic determinants of prices. The joint dynamics of these proxies as well as equity returns and their volatilities are estimated without any parameterization of their historical dynamics. The estimation is in accordance with the theory of no-arbitrage and accounts for the option early exercise. The study finds that the heaviest discounts of asset payoffs are the ones for extreme values of common volatility, followed in sequence by momentum, market and size factors. In addition, the paper illustrates the benefits provided by the theory of no-arbitrage for the nonparametric estimation of the historical joint dynamics of risk factors, equity returns and their volatilities.

Keywords: Risk premia, American option, nonparametric estimation, Fama/French portfolios, realized volatilities.

JEL Codes: C14, C52, G13.
Understanding the formation of prices is a central issue in the study of financial markets. In practical counterparts of standard consumption-based discount factor models, observable factors (risk factors) are taken as approximating variables for aggregate consumption or marginal utility growth, and then in turn as systematic determinants of financial asset prices. This paper considers highly liquid equity and option markets and takes the Fama/French (FF) market, value/growth (Basu, 1977, and Fama and French, 1993, 1998), size (Banz, 1981) and momentum (Jagadeesh, 1990, Jagadeesh and Titman, 1993, and Carhart, 1997) portfolios and a measure of aggregate volatility (Goyal and Santa-Clara, 2003, Ang, Hodrick, Xing and Zhang, 2006, 2009) as proxies for systematic price determinants. The market and the aggregate volatility factors describe the behavior of the markets to a large extent. Value, size and momentum factors account for the related empirical anomalies. Aim of this paper is describing how the risk factors evolve over time and measuring the fear of the representative investor about future realizations of the risk factors. The econometric model introduced in the paper is characterized by three major features. First, it enhances the identification of the risk premia parameters by considering individual equity options alongside equities. Parameters in rational expectations models are often affected by problems of weak identification, and their traditional estimates are not very useful, being unstable and characterized by large confidence intervals. They are usually sensitive to changes in the basic asset returns, instruments and model parameterization. Tests about their value based on standard asymptotic approximations are often unreliable in finite samples. For example, it is often hard to determine with accuracy the value of certain preference parameters, such as the risk-aversion coefficient. Second, since the style of the individual equity options is American, the econometric model accounts for the early exercise premium, which can be a considerable part of the option price. Third, the econometric model avoids the risk of a wrong specification of the dynamics and dependences of risk factors, equity returns and their volatilities. In this way there is no risk of misspecified dynamics and dependencies, which is particularly serious in studies involving American option prices, since they can strongly depend on the used asset pricing model.

The heaviest discounts of asset payoffs are the ones for extreme values of common volatility, followed in sequence by momentum, market and size factors. Moreover, no-arbitrage restrictions in both the equity and option markets improves the efficiency in the estimation of the risk premia. Differences in forecast power between the methods that account and do not account for no-arbitrage restrictions in the equity and option markets show the improvement in the estimation of the historical joint dynamics of risk factors, equity returns and their volatilities.

Both the ICAPM (Merton, 1973) and the APT (Ross, 1976) assume the existence of multiple risk factors
related to changes in the investment opportunity set. Many asset pricing models take the ICAPM as a theoretical support for the introduction of ad hoc risk factors (Fama, 1991). Maio and Santa-Clara (2012), who define this practice as a “fishing license”, show that the market, value/growth, size, momentum factors (i.e., the factors in the models of Fama and French, 1993, and Carhart, 1997) are the most consistent with the restrictions imposed by the ICAPM of Merton (1973) inside a large group of multifactor asset pricing models. Connor, Hagmann and Linton (2012), without relying in the linear market hypothesis, show that momentum and volatility risk factors are “as important or more important than size and value in explaining equity return comovements.” Alongside the mentioned factors, this paper considers individual equity returns and their volatilities as state variables. Instead of looking for additional risk factors, the paper treats any further impact on prices as given by exogenous variables and displayed in the dynamics of the individual equity returns and their volatilities. Therefore, if an additional risk factor impacts the historical dynamics of prices, its effect is included in the dynamics of the individual equity returns and their volatilities.

The analysis focuses on the common stocks of the DJIA’s components traded at NYSE and on some American options written on them traded in centralized markets. The DJIA’s components are providers of industrial, consumer and IT goods and financial and entertainment services. Their activities cover a large part of the U.S. economy. The study considers the period from January 2006 to August 2008. Both a bullish and (relatively) stable and a bearish and (relatively) volatile periods for the U.S equity market are considered. These periods are before and after July 2007, when a market downturn realizes. The analysis ends with the stock market crash in the fall of 2008, when shocks in recession indicators severely modify the process of price formation and the average risk-aversion.

The disparity between trading frequency in centralized derivative and equity markets forecloses the possibility of using synchronous trading prices. Therefore, the study focuses on option mid-quotes and share prices at market close. Moreover, a series of realizations of risk factors and stock returns and volatilities is necessary to attain nonparametric estimates of their dynamics. The FF portfolios and measures of realized volatilities are then considered. The daily interest rate obtained by 1-month T-bills is taken as the reference risk-free rate used by the market to discount for time and compute excess stock returns.

The estimation of a reduced form specification of a Stochastic Discount Factor (SDF) is repeated at any business day in July and August 2008. At any day, the estimation is performed by a Generalized Method of Moments (GMM, Hansen, 1982, and Hansen and Singleton, 1982) and an Extended Method of Moments (XMM, Gagliardini, Gouriéroux and Renault, 2011, and Gagliardini and Ronchetti, 2012). The XMM is a

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1This downturn is partially due to the growing widespread concerns about prices and ratings of financial assets, liquidity risk and counterparty risk.
method to estimate accurately the discount for risk using both equity and option data. This method enhances
the identification of the SDF parameters. For example, Gagliardini, Gouriéroux and Renault (2011) show
that when an asset return follows a Cox, Ingersoll and Ross (1985) process under the historical probability
measure process, and both asset return and volatility are priced, the no-arbitrage restrictions in the option
market provide the identification of the SDF parameters. Other estimation methodologies aimed at coping
with the problem of weak identification of the model parameters do not lead to very precise estimates and
indicate very different levels of risk aversion dependently on the asset pricing model (see, e.g., Stock and
Wright, 2000, and Yogo, 2004).

The GMM can consider only the model restrictions that come from equity and bond markets, while the
XMM can add also the restrictions that come form derivative markets. The former technique is applied to a
time series of T-bill and equity prices. The latter technique is applied to both the mentioned time series and a
cross-section composed by option mid-quotes and T-bill and equity prices. Each estimation method provides
a consistent SDF estimator that best satisfies, on the basis of a particular criterion, the empirical counterparts
of a set of no-arbitrage restrictions. Repeating the estimation on different subsets of data allows to recover
the empirical distribution of the SDF estimates and measure the dispersion in parameter estimation. From
the analysis of the empirical distribution of the GMM and XMM estimates it appears that the information on
the SDF contained in equity prices and option quotes is not redundant and that the efficiency in the estimation
of the model parameters is improved by considering both the information in the equity and option markets.

The model is based on a non-parametric autoregressive specification of the joint dynamics of the risk
factors and equity returns and volatilities. With a parameterization of the joint dynamics of risk factors
and/or stock returns and their volatilities, as done in most of the asset pricing literature, the functional
form of their conditional probability distribution is restricted to be the same for different economic settings.
Several are the advantages of such a parameterization, as getting analytical (or closed-form) expressions of
some financial quantities (e.g., asset prices or portfolio hedges) or obtaining an higher estimation efficiency
when the model is correctly specified. These benefits come at the cost of constraining the functional form of
the conditional probability distribution to be invariant in every economic setting. However, in real markets,
some conditional higher marginal or cross-moments of the variables contribute significantly to the moment-
generating function only in extreme conditions. For example, higher moments of the volatility factor become
significant only when the equity market is in extreme conditions and investors are particularly agitated.
Similarly, the value of some higher moments of equity returns and their volatilities become substantial just
when the firm is in distress. Moreover, all these moments may follow complex non-linear dynamics (see
also the discussion in Hansen, 1994).\footnote{Regime-switching models can account for these dynamics, at the additional risk of on an erroneous specification of the probabilistic law of the transition between regimes.}

In parametric asset pricing models, conditions implied by the absence of arbitrage opportunities (absolute continuity in continuous-time models) binds the parameterization under the historical and risk-neutral probability measures. In these models the identification of the parameters is provided by specific restrictions. In particular, some risk premia parameters are often identified by no-arbitrage restrictions in the equity or option market. If a no-arbitrage restriction in a single market identifies a parameter, its value can be inferred only considering that market. While the equity and option trading process is not redundant (see, e.g., Easley, O’Hara and Srinivas, 1998, Chakravarty, Gulen and Mayhew, 2004), there is no empirical evidence that the information content about risk premia in the two markets is non-overlapping. Since in this paper no parametric specification of the dynamics of the considered financial variables is introduced, the model does not impose the identification of a particular risk premium as given by a particular model restriction. Therefore, the model allows to exploit the information about any risk premium coming from any single market.

Two distinct nonparametric estimation approaches to the historical joint dynamics of risk factors, equity returns and their volatilities are considered. First, the dynamics are estimated by a kernel estimator applied to time series of the variables just mentioned. Second, the dynamics are estimated under the hypothesis of no-arbitrage in the T-bill, equity and option markets. On each day, the results obtained by imposing the absence of no-arbitrage have an higher predictive power, measured by considering the actual joint realization of the variables on the following day. As illustration, some estimates of the conditional skewness and kurtosis of the return on the IBM equity, which is a DJIA’s component, and the conditional correlation of this return and its volatility are reported. These moments are conditional to the historical values of the information set and are estimated by both the estimators of the dynamics of the considered variables.

In most of the quantitative studies grounded in the theory of no-arbitrage, the dynamics of asset returns, volatilities and early exercise premia are described by parametric models. As a consequence, the results of these studies are subject to the risk of a wrong parametric specification of the dynamics of the considered variables. This risk is particularly serious in studies of option prices, since option prices implied by different models can greatly differ (see, e.g., Green and Figlewski, 1999, and Hull and Suo, 2002). In order to avoid the risk of assuming an erroneous dynamics of the considered variables, the empirical study reported in this paper employs nonparametric estimation methods. These methods permit the description of the main empirical features of a stochastic process without assuming a parametric model, but they provide satisfactory
results only when applied to relatively large samples.  

The early exercise possibility of the American options is explicitly considered, since the early exercise premium can be several hundred bps of the option price. For example, among the quantitative studies of the early exercise premium, the studies of Zivney (1991) and Duiker and Miller (2003), based on just arbitrage relationships and without adopting any parametric model, find that the early exercise premia for S&P 100 and S&P 500 index options are up to about 6% and 11% of the call and put option prices, respectively. For individual equity options this premium is likely to be even greater. Higher the volatility of the underlying asset, higher is the probability of the option early exercise, and individual stock prices are indeed usually more volatile than index levels.

Exogenous factors can induce sub-optimal option exercise. For example, the sudden need to buy or sell equities after an influential announcement can induce an anticipated option exercise. Moreover, Poteshman and Serbin (2003) find that sometimes customers of full-service and discount brokers exercise American options in an irrational way, while firm proprietary traders never do so. Duffie, Liu and Poteshman (2005) show that an irrational exercise can be triggered by exceptional levels of the underlying stock price. Barraclough and Whaley (2012) consider the contribution of frictions to the option exercise decision. The study reported in this paper is focused exclusively on actively traded options, and then not yet exercised. Considering departures from rationality and frictions would require further assumptions on data generating process and arbitrage opportunities. In order to avoid the additional component of model risk that these assumptions would bring, frictions are neglected and option prices are considered as free of arbitrage possibilities and set by investors that are rational in aggregate. The high trading volume of the considered options justifies this choice.

Section 1 contains the description of the asset pricing model and the implications of the absence of arbitrage opportunities. Section 2 introduces the estimators of the model parameters. The criteria to select the data and the characteristics of the sample are discussed in Section 3. Section 4 contains the description and interpretation of the estimates of the SDF and some properties of the joint historical dynamics of equity returns and their volatilities. Section 5 summarizes the results. Further details on the model and the estimators are gathered in Appendixes A and B.

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3Nonparametric estimation techniques are largely used in finance. See, e.g., Cai and Hong (2009) for a review.
4See also Shastri and Tandon, 1986, and Whaley, 1986, for parametric settings.
1 Model

This section illustrates the representation of the dynamics of the considered financial variables and the risk premia. Section 1.1 deals with the variables that summarize the state of equity and option markets. Section 1.2 describes the market expectations of future payoffs. Section 1.3 discusses the way American option prices are generated. Section 1.4 describes the model restrictions.

1.1 State variables

The equity and option markets are composed by $L$ firm equities and options written on them. The dynamics of the equity and option prices are described by the joint dynamics of both common risk factors for the equity and option markets and individual equity returns and their volatilities. The former factors explain the common behavior of equity returns and their volatilities, while the latter variables account for possible dependencies of prices on firm-specific characteristics and additional exogenous risk factors. Dependences and dynamics of individual equity returns and their volatilities are firm-specific and change in different economic settings. For example, other things being equal, lower is the diversification of the economic activities of the individual firm, generally higher is the impact of idiosyncratic volatility on the individual equity (Merton, 1987, and Barberis and Huang, 2001), and smaller is a firm, less are its possibilities to survive bad economic situations (Chan and Chen, 1991). Moreover, prices are affected by firm-specific characteristics, such as the relative supply of equity, marginal changes in the company capitalization rate and forecasts of its economic activities. The individual stock returns are observed, while for the true value of the other variables we have some proxies.

The FF market, value/growth (Basu, 1977, and Fama and French, 1993, 1998), size (Banz, 1981) and momentum (Jagadeesh, 1990, Jagadeesh and Titman, 1993, and Carhart, 1997) portfolios and a measure of aggregate volatility (Goyal and Santa-Clara, 2003, Ang, Hodrick, Xing and Zhang, 2006, 2009, and Connor, Hagmann and Linton, 2012) are taken as proxies for the risk factors that determine equity returns and their volatilities. On business day $t$, the portfolio values are indicated by $Z_{1,t}, \ldots, Z_{4,t}$, respectively, and the aggregate volatility by $V_t$. All the risk factors are serially independent, with the exception of the common volatility factor, which is autocorrelated.\(^5\)

**Assumption 1** The variables $Z_{1,t}, \ldots, Z_{4,t}, V_t$ are not multicollinear.

\(^5\)The model can be adapted to account for different autocorrelated risk factors. However, since the model is nonparametric w.r.t. their dynamics, only a low number of distinct autocorrelated risk factors can avoid the curse of dimensionality.
On business day $t$, the value of the daily return on the traded equity of the $i$-th firm and its volatility are denoted by $r_{i,t}$ and $\sigma_{i,t}$, respectively, for any $i = 1, \ldots, L$.

**Assumption 2** The variables $r_{i,t}$ and $\sigma_{i,t}$, for any $i = 1, \ldots, L$, do not Granger-cause the variables $Z_{1,\tau}, \ldots, Z_{4,\tau}, V_\tau$, at any day $t$ and $\tau$.

**Assumption 3** There is no Granger causality of the volatility $\sigma_{i,t-1}$ on the return $r_{j,t}$ and the volatility $\sigma_{j,t}$, for any $j = 1, \ldots, L \neq i$.

Let us collect all the systematic risk factors and all the equity returns and their volatilities at business day $t$ in vector $F_t := [Z_t V_t r_{1,t} \sigma_{1,t} \ldots r_{L,t} \sigma_{L,t}]'$ and let us refer to them as state variables. The dynamics of the economic system is described by means of the transition density for the process of vector $F_t$.

**Assumption 4** The joint process for the variables included in vector $F_t$ is stationary, time-homogeneous and Markovian of order 1.

**Assumption 5** Among the components of vector $F_{t-1}$, just the common volatility factor $V_{t-1}$ and the individual equity return volatilities $\sigma_{1,t-1}, \ldots, \sigma_{L,t-1}$ can Granger-cause vector $F_t$.

Assumption 5 implies the lack of Granger causality of the equity return $r_{j,t}$ on $r_{j,t+1}$ that is in line with the lack of empirical serial correlation for daily returns of liquid equity stocks. Figure 1 displays the Granger causality linkages between the risk factors and equity return and their volatilities at different times.

The causal relationships between contemporaneous and lagged equity returns and volatilities are widely studied in the literature, and economic interpretations of the observed correlations between them are offered. The comovement between an individual equity stock return and its volatility, which empirically is observed to be negative, is explained on the basis of opposite causal relationships by the leverage effect and the delayed volatility feedback. In the leverage effect hypothesis, a drop in the equity value, associated to a negative stock return, makes the company more leveraged, since its debt-to-equity price ratio increases. As a result, the stock gets riskier and more volatile (Black, 1977, Christie, 1982, and Nelson, 1991). In the volatility feedback hypothesis, as the idiosyncratic volatility increases, investors require an higher stock return, which realizes by a drop in the equity value, and then by a negative stock return (Poterba and Summers, 1986, French, Schwert and Stambaugh, 1987, Campbell and Hentschel, 1992, and Bekaert and Wu, 2000). Autoregressive conditional heteroskedastic models account for the observed persistence of the volatility factor and idiosyncratic volatilities (Engle, 1982, and Bollerslev, 1986). Furthermore, stochastic stock returns and volatilities are used as state variables in standard stochastic volatility models used for asset pricing. For

1.2 SDF

Investors require a compensation for future values of the risk factors that are different than expected. To provide incentives to invest, prices are generated by turning the historical probability measure induced by the state variables into an equivalent (i.e. with equal support) risk-neutral one. The link between the historical and any risk-neutral probability density of the state variables is provided by an SDF. Agents have access to risk-free investments at the daily rate $r_{f,t+1}$ from business day $t$ to business day $t+1$, and they know this rate at $t$. While the markets for stocks, T-Bills and options are not assumed to be complete and the common SDF to be unique, a set of asset pricing specifications with the following SDF from business day $t$ to business day $t+1$ is supposed to be admissible with the observed prices:

$$M_{t+1}(\theta) = \exp \left( -r_{f,t+1} - \eta - \sum_{j=1}^{4} \theta_j Z_{j,t+1} - \theta_5 V_{t+1} \right),$$  \hspace{1cm} (1)

for the unknown SDF parameter vector $\theta = [\eta \ \theta_1 \ldots \ \theta_5]'$. The SDF $M_{t+1}$ is parameterized as a function of the state variables at day $t+1$. Parameter $\eta$ measures the fixed discount related to the rate of time preference. Parameter $\theta_j$ measures the discount for future realizations of the $j$-th systematic risk factor, for any $j = 1, \ldots, 5$. In particular, parameter $\theta_5$ accounts for future values of the market volatility. When markets are volatile, often stock returns are negative. Investors require then a compensation for an higher level of the volatility factor than expected. In more volatile times consumption and the number of investment opportunities are indeed generally lower (Campbell, 1993).

Let us now describe some further notation used in the paper. The volatility risk factor and the individual equity return and its volatility are collected in vector $X_{i,t} := [V_t \ r_{i,t} \ \sigma_{i,t}]'$ and just the last two variables in vector $Y_{i,t} := [V_t \ \sigma_{i,t}]'$. Under Assumption 3 the conditioning set of the transition density $\phi_i$ for process $(X_{i,t})$ simplifies to just vector $Y_{i,t}$. The probability density for the vector $X_{i,t+1}$ to assume value $x$ after a business day with value $y$ for vector $Y_{i,t}$ is then denoted by $\phi_i(x|y)$. Let us indicate by $\psi(z_1, \ldots, z_4|v)$ the conditional probability density function for the realizations $z_1, \ldots, z_4$ of the risk factors $Z_{1,t}, \ldots, Z_{4,t}$, given the contemporaneous value $v$ for vector $V_t$. Let us indicate by $E_{\phi_i} [\cdot | Y_{i,t} = y]$ and $E_{\psi} [\cdot | V_t = v]$ the
conditional expectation operators based on the conditional probability density functions \( \phi_i \), given vector \( Y_{i,t} = y \), and \( \psi \), given \( V_t = v \) respectively.

**Assumption 6** There exist two scalars \( \alpha \) and \( \beta \) such that

\[
E_{\psi} \left[ \exp \left( -\sum_{j=1}^{4} \theta_j Z_{j,t} \right) \middle| V_t = v \right] = \exp (-\alpha - \beta v),
\]

for any value \( v \) of the conditioning common volatility factor.

Under Assumption 6 the historical conditional expectation of a generic function \( \zeta \) of just vector \( X_{i,t} \) is completely determined by the transition density \( \phi_i \) of the process \((X_{i,t})\) (see Appendix A):

\[
E \left[ M_{t+1}(\theta) \zeta(X_{i,t+1}) \middle| F_t \right] = E_{\phi_i} \left[ \exp \left( -r_{f,t+1} - \theta_5 V_{t+1} \right) \zeta(X_{i,t+1}) \middle| V_t, \sigma_{i,t} \right],
\]

where the expectation on the LHS is w.r.t. the transition density of vector \( F_t \), and the expectation on the RHS is w.r.t. the transition density \( \phi_i \). While retaining a non-parametric setting, Assumption 6 lowers the dimensionality of the problem and make the computation of expected payoffs feasible.

The SDF in Equation (1) is exponential-affine in the systematic risk factors. An exponential-affine specification of the SDF is common in the asset pricing literature. A first example is in equilibrium models. In the standard CCAPM model (Lucas, 1978), when the representative agent has a power or CARA utility function, the implicit SDF is exponential-affine in inflation rate and a function of consumption. A second example is in option pricing. In continuous time, when coupled with affine specifications of the differential equation for the Markov process of the state variables, an exponential-affine specification of the SDF offers analytical tractability. When the state variables follow a jump-diffusion dynamics, the computations of some of their transforms is feasible in closed form (Hull and White, 1987, Heston, 1993, Duffie, Pan and Singleton, 2000, and Duffie, Filipovic and Schachermayer, 2003). Similar manageability benefits are offered also in discrete time (Gouriéroux, Monfort and Polimenis, 2006, and Gouriéroux and Monfort, 2007).

The 1-month T-bill is assumed non-defaultable in the short term, and the market discounts future returns on risky investments w.r.t. its rate. Therefore, the chosen parametrization of the SDF given in Equation (1) involves the excess return on the market \( Z_{1,t} \). Investors cannot directly trade assets with a reference level of return variance, so that the SDF is parametrized as a function of variances and not excess variances w.r.t. risk-free variance levels. Daily volatility swap rates can be taken as risk-free levels of volatility.\(^6\) These

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\(^6\)The payoff of a volatility swap contract is the volatility risk premium converted to monetary units.
rates are derived from volatility swap contracts, which are OTC derivatives, or approximated by using some option portfolios (Carr and Wu, 2009). Volatility swap contracts are less liquid than the assets considered in this paper, and approximations due to the estimation of volatility swap rates by option portfolios are avoided in the study.

1.3 American options

In the rest of the paper the process of price formation for these assets does not depend on the specific firm. Therefore, the process for just a firm is described and the firm index is dropped in the notation. Let us then simplify the notation as follows \( r_t = r_{i,t}, \sigma_t = \sigma_{i,t}, X_t = X_{i,t}, Y_t = Y_{i,t}, \phi = \phi_i. \)

Let us consider the value of an American put option written on an individual equity and with strike price \( K. \) At the option expiration, its value is \( (K - S)^+ = \max [K - S, 0], \) if the price of the underlying stock is \( S. \) Before this date, the value of the option is the maximum between the gain for the holder if the option is exercised and the value of the contract if the option is kept alive. The former is the early exercise payoff \( (K - S)^+ \). The latter is the expectation of the option value on the successive business day, discounted by time and risk and conditional on the available information. This representation of the put option price is used in lattice methods (see, e.g., Cox, Ross and Rubinstein, 1979, Boyle, 1988, and Ritchken and Trevor, 1999), regression-based Monte Carlo methods (Longstaff and Schwartz, 2001) and other iterative integration methods (Sullivan, 2000). The notation for the option prices used in this paper highlights the dependence of the contract price on the model parameters. The price of an American put option is denoted by \( P(h, K, S, y; \theta, \phi) \), for time-to-maturity \( h \), strike price \( K \), underlying stock price \( S \), vector \( y = [v \sigma]' \) for volatility factor \( v \) and individual equity return volatility \( \sigma \), and when the SDF parameter vector has value \( \theta \) and the probability density \( \phi \) is taken as the historical transition density of the process \( (X_t) \). A similar notation is adopted for call options, so that the price of an American call option with the same contract characteristics and computed for the same variables and model parameters is denoted by \( C(h, K, S, y; \theta, \phi) \).

The price \( P \) of an American put option is linearly homogeneous in the stock price and it is expressed as the product of the stock price and the American put option-to-stock price ratio \( p \) (see Merton, 1973b, 1990, and Gagliardini and Ronchetti, 2012, for parametric and nonparametric settings, respectively). Since function \( p \) has less arguments than \( P \), this representation reduces the dimensionality of the pricing problem and speeds up the computation of an option for any value of the model parameters. More precisely, the American option price \( P \) depends separately on the strike price \( K \) and the underlying asset price \( S \), while,
other things being equal, the American put option-to-stock price ratio depends solely on the moneyness strike \( k = K/S \):

\[
P(h, K, S, y; \theta, \phi) = S p(h, k, y; \theta, \phi).
\] (2)

Gagliardini and Ronchetti (2012) show that a backward dynamic programming iteration can be applied to the put option-to-stock price ratio \( p \). At the option maturity, i.e. for \( h = 0 \), the value of this ratio, for any value of its remaining arguments, is just the exercise-to-stock price ratio:

\[
p(0, k, y; \theta, \phi) = (k - 1)^+.
\] (3)

All other things being equal, on any business day before the option maturity, i.e. for \( h \geq 1 \), when the risk-free rate is \( r_f \), the American put option-to-stock price ratio is the maximum between the exercise-to-stock price ratio and a discounted expected value of the American put option-to-stock price ratio one business day ahead:

\[
p(h, k, y; \theta, \phi) = \max \left[ (k - 1)^+, E_{\phi} \left[ M_{t+1}(\theta)e^{r_{t+1}}p(h - 1, ke^{-r_{t+1}}, V_{t+1}, \sigma_{t+1}; \theta, \phi) \bigg| Y_t = y \right] \right],
\] (4)

where \( E_{\phi} \left[ \cdot \bigg| Y_t = y \right] \) is the conditional expectation w.r.t. the historical transition density \( \phi \) of the process \((X_t)\), given the value \( y \) of the vector composed by volatility factor and individual equity return volatility.

The operator \( \max \left[ \ldots \right] \) in the RHS of Equation (4) acts on the counterparts of the early exercise payoff and continuation value of an American put option in ratio terms. The daily cum-dividend gross return \( e^{r_{t+1}} \) on the individual equity stock in the continuation value-to-underlying asset price ratio accounts for the fact that we deal with option-to-stock price ratios and not with prices. The continuation value-to-underlying asset price ratio can therefore be considered as the expected value of the American put option-to-stock price ratio for the probability measure under which \( 1/S_t \) is a martingale (see Gagliardini and Ronchetti, 2012). Similar equations and definitions hold for the American call option price \( C \) and the American call option-to-stock price ratio \( c \).

### 1.4 No-arbitrage restrictions

At a business day with value \( y^* = [v^* \sigma^*]' \) for volatility factor and individual equity return volatility, we observe the price of \( M \) American put options and \( N \) American call options, written on the same stock and different in terms of strike price and/or maturity. Moreover, let us denote the true value of parameter \( \theta \) by
\[ \theta_0 := [\eta_0 \ \theta_{1,0} \ \ldots \ \theta_{4,0} \ \theta_{5,0}]^T \] and by \( \psi_0 \) the true value of the conditional probability density \( \psi \). If there is no arbitrage opportunity and the specification of the asset pricing model is correct, the option prices computed for the true value \((\theta_0, \phi_0)\) of the model parameters coincide with the observed option prices. Any calibration method is based on this match. From this relation and the homogeneity property of the American option price, expressed in Equation (2) for a put option, we get some equations between observed and model-implied values. Let us consider the model-implied put option-to-stock price ratio evaluated at the values \( k_p^j \) and \( h_p^j \) of moneyness strike and time-to-maturity of the \( j \)-th observed put option and for the value \( y^* \) of volatility factor and individual equity return volatility. If the model is correctly specified, this model-implied ratio, computed using the true value \((\theta_0, \phi_0)\) of the model parameters, coincides with the observed American put option-to-stock price ratio \( p_j \):

\[
p(h_p^j, k_p^j, y^*; \theta_0, \phi_0) = p_j,
\]

for \( j = 1, \ldots, M \). A similar match is satisfied for the \( i \)-th observed American call option-to-stock price ratio \( c_i \), with moneyness strike \( k_c^i \) and time-to-maturity \( h_c^i \):

\[
c(h_c^i, k_c^i, y^*; \theta_0, \phi_0) = c_i,
\]

for \( i = 1, \ldots, N \). Equations (5) and (6) provide the restrictions for the option market, and since the put-call parity does not hold for American options (for whom only a weaker put-call relationship holds) there is no redundancy between them.

Equations (5) and (6) and System (7) are imposed in arbitrage-free option pricing models. For example, let us consider a standard binomial tree for the risk-neutral dynamics of a share price with null risk-free rate and dividend yield, and with the share price as unique state variable. In consecutive days, the share price can move from \( S \) to \( Su \), with probability \( \tilde{p} \), or to \( Sd \), with probability \( 1 - \tilde{p} \). Arbitrage opportunities on the share are excluded by imposing \( u\tilde{p} + d(1 - \tilde{p}) = 1 \). This restriction plays the role of the first equation in System (7) with parameters \( \tilde{p}, u, d \). We then calibrate these parameters to the market price of a cross-section of financial derivatives written on the share. This calibration is based on the match between observed and model-implied asset prices similarly to what stated by Equations (5) and (6).

Although the model restrictions hold always, we observe the price of traded options only for certain combinations of values of time-to-maturity, moneyness strike, and conditioning volatility factor and individual equity return volatility. Therefore, on a given date we can construct a time series of arbitrage-free prices only for the T-bill and the equities. Intuitively, in the period covered by this time series, large part
of the possible realizations of the volatility factor and individual equity return volatility are observed. Differently, since the time-to-maturity of the traded options changes over time, we have at disposal option data only for some sets of values of the volatility factor and individual equity return volatility, moneyness strike and time-to-maturity. Therefore, on the one hand, we can build empirical counterparts of the model restrictions for the equity market and SDF that hold for almost every value of the volatility factor and individual equity return volatility. On the other hand, we can build empirical counterparts of the model restrictions for the option market that hold just for some combinations of the volatility factor and individual equity return volatility, moneyness strike and time-to-maturity. This is the reason why the restrictions for bond and stock price in System (7) are introduced for any value $y$ of the volatilities, and the restrictions for the options in Equations (5) and (6) only for the combinations $(h_j^p, k_j^p, y^*)$ and $(h_i^c, k_i^c, y^*)$ of the observed values of time-to-maturity, moneyness strike, volatility factor $v^*$ and individual equity return volatility $\sigma^*$.

Considering the T-bill and equity prices as consistent with the absence of arbitrage opportunities and Assumption 6, we have the following restrictions over the one-day horizon:

$$\begin{align*}
    & \mathbb{E}_{\phi_0} \left[ M_{t+1}(\theta_0)e^{r_{t+1}} \mid Y_t = y \right] = 1, \\
    & \mathbb{E}_{\phi_0} \left[ M_{t+1}(\theta_0)e^{r_{f,t+1}} \mid Y_t = y \right] = 1,
\end{align*}$$

respectively, for any value $y$ of volatility factor and individual equity return volatility. The first equation represents the martingale property for the equity price. The second equation imposes the normalization of the SDF together with Assumption 6 (see Appendix A).

## 2 Estimation

In this section the estimation approaches to the true values $\theta_0$, $\psi_0$ and $\phi_0$ of the SDF parameter vector, the conditional probability density function of variables $Z_{1,t}, \ldots, Z_{4,t}$ given $Y_t$, and the transition density for process $(X_t)$ are described. First, the densities $\psi_0$ and $\phi_0$ are estimated by nonparametric kernel estimators. Second, the vector $\theta_0$ is estimated in two ways, by using a time series of T-bill and equity prices and by using the same time series and a cross-section of option mid-quotes and T-bill and equity prices. Third, transition density $\phi_0$ is re-estimated by considering all the model restrictions described in Section 1. Let us consider some standard assumptions on the serial dependence of process $(X_t)$ to ensure convergence of the estimators (see, e.g., Bosq, 1998).

**Assumption 7** Under the historical probability measure, the serial dependence between $X_t$ and $Y_{t-j}$ de-
cays geometrically fast as the positive integer time lag \( j \) increases.

The nonparametric kernel estimator \( \hat{\phi} \) of the historical transition density for process \( (X_t) \), for a time series sample of length \( T \), is defined as

\[
\hat{\phi}(x|y) = \frac{1}{h_T^4} \sum_{t=2}^{T} \tilde{K} \left( \frac{x_t - x}{h_T} \right) K \left( \frac{y_{t-1} - y}{h_T} \right) \left/ \sum_{t=2}^{T} K \left( \frac{y_{t-1} - y}{h_T} \right) \right.,
\]

where \( \tilde{K} \) and \( K \) are multivariate Gaussian kernel functions, \( h_T \) is the bandwidth, and \( x \) and \( y \) are generic values for the vectors \( X_t \) and \( Y_t \), respectively.\(^7\) The nonparametric kernel estimator \( \hat{\psi} \) of the conditional probability density function of \( Z_{1,t}, \ldots, Z_{4,t} \) given \( Y_t \) is defined similarly.

The two estimators for vector \( \theta_0 \) and the second estimator for \( \phi_0 \) employ the empirical counterparts of the model restrictions given by Equations (5) and (6) and System (7) and depend on the nonparametric kernel estimators for \( \psi_0 \) and \( \phi_0 \). In order to present all these estimators in a compact form, let us first introduce two vectors that collect some model restrictions. These two vectors are functionals of the model parameters \((\theta, \phi, \psi)\) and the notation to denote them highlights the fact that we need to specify the values of the model parameters to compute them. The vector \( U \) is defined as

\[
U(y; \theta, \phi, \psi) := \begin{bmatrix}
E_\psi \left[ \exp \left( \alpha - \sum_{j=1}^{4} \theta_j Z_{j,t} + \beta V_t \right) \right| V_t = v \right] - 1 \\
E_\phi \left[ M_{t+1}(\theta)e^{r_{t+1}} \right| Y_t = y \right] - 1 \\
E_\phi \left[ M_{t+1}(\theta) \left| Y_t = y \right. \right] - 1
\end{bmatrix},
\]

for any value \( y \) of the volatility factor and individual equity return volatility, and when the model parameters are fixed at their values \((\theta, \phi, \psi)\). The components of vector \( U \) are the differences between the LHSs and RHSs of the equation in Assumption 6 and System (7), computed at the generic value \((\theta, \phi, \psi)\) of the model parameters instead of the true value \((\theta_0, \phi_0, \psi_0)\). As mentioned at the very beginning of Section 1.4, when option prices are observed, volatility factor and individual equity return volatility have value \( y^* \). Let us

\(^7\)Since the arguments of function \( \hat{\phi} \) are correlated, in the empirical application multidimensional bandwidth matrices are used (see Appendix B.1). See Bosq (1998), for an introduction on nonparametric time series estimation techniques, and, for applications in financial studies, see, e.g., Ait-Sahalia (1996), Ait-Sahalia and Lo (1998), Pritsker (1998), Chapman and Pearson (2000), Hong and Li (2005), Hong, Tu and Zhou (2007), Li and Zhao (2009) and Ang and Kristensen (2012).
moreover denote by $v^{**}$ the value of the volatility factor on the following day. The vector $L$ is defined as

$$L(\theta, \phi, \psi) := \begin{bmatrix}
E_\psi \left[ \exp \left( \alpha - \sum_{j=1}^{4} \theta_j Z_{j,t} + \beta V_t \right) \middle| V_t = v^{**} \right] - 1 \\
p(h^p_j, k^p_j, y^*; \theta, \phi) - p_j, & \text{for } j = 1, \ldots, M \\
c(h^c_i, k^c_i, y^*; \theta, \phi) - c_i, & \text{for } i = 1, \ldots, N \\
E_\phi [M_{t+1}(\theta)e^{r_{t+1}} | Y_t = y^*] - 1 \\
E_\phi [M_{t+1}(\theta)e^{r_{t+1}} | Y_t = y^*] - 1
\end{bmatrix},$$

when the model parameters are fixed at their values $(\theta, \phi, \psi)$ and where the observed and model-implied option-to-stock price ratios are denoted as in Section 1.4. The components of vector $L$ are the differences between the LHSs and RHSs of the equation in Assumption (6), Equations (5) and (6) and System (7), computed at $(\theta, \phi, \psi)$ instead of $(\theta_0, \phi_0, \psi_0)$. In particular, the first and last two components of vector $L$ are the components of vector $U$ for the values $v^{**}$ and $y^*$ of common volatility risk factor and individual equity return volatility on following days. The remaining $M + N$ components are the differences between model-implied and observed American option-to-stock price ratios.

The restrictions that hold for any value $y$ of the volatility factor and individual equity return volatility are gathered in vector $U$. Differently, the restrictions that hold just for the values $v^{**}$ and $y^*$ of common volatility risk factor and individual equity return volatility on following days are collected in vector $L$. Valuing vectors $U$ and $L$ at the generic value $(\theta, \phi, \psi)$ of the model parameters means that the model-implied American option-to-stock price ratios and the conditional risk-neutral expectations are computed by using this value of the model parameters. From Assumption (6), Equations (5) and (6) and System (7), vectors $U$ and $L$ are null at the true value $(\theta_0, \phi_0, \psi_0)$ of the model parameters. When they are valued at $(\theta, \hat{\phi}, \hat{\psi})$ they collect the empirical restrictions for the value $\theta$ of the SDF parameter vector. In this case, vectors $U$ and $L$ are indeed computed for a generic value $\theta$ of the SDF parameter vector, for the kernel estimator $\hat{\phi}$ as the historical conditional density of vector $(X_t)$ and for the kernel estimator $\hat{\psi}$ of the conditional probability density function of variables $Z_{1,t}, \ldots, Z_{4,t}$ given the $V_t$. For any given $\theta$, the model-implied American option-to-stock price ratios in vector $L(\theta, \hat{\phi}, \hat{\psi})$ can be computed by a dynamic programming approach with kernel regressions. This approach is essentially based on the iterative scheme described by Equations (3)
and (4) in the case of a put option.\textsuperscript{8}

Let us now introduce the two estimation methods for the SDF parameter vector $\theta$. The GMM minimizes a quadratic form in the empirical counterparts of the considered restrictions that hold for any value of the conditioning volatility risk factor and individual equity return volatility. The XMM does similarly considering both the restrictions that hold for any value of the conditioning volatility risk factor and individual equity return volatility and the restrictions that hold just for the value of the same variables at a particular business day. Using the notation of the paper, the GMM minimizes a quadratic form in $U(y; \theta, \hat{\phi}, \hat{\psi})$, for any $y$, and the XMM minimizes a quadratic form in both $U(y; \theta, \hat{\phi}, \hat{\psi})$, for any $y$, and $L(\theta, \hat{\phi}, \hat{\psi})$. In particular, the XMM estimator $\hat{\theta}$ of the SDF parameter vector is defined as

$$\hat{\theta} = \arg\min_{\theta} \left[ h_T^2 L(\theta, \hat{\phi}, \hat{\psi})' L(\theta, \hat{\phi}, \hat{\psi}) + \frac{1}{T} \sum_{t=1}^T U(y_t; \theta, \hat{\phi}, \hat{\psi})' U(y_t; \theta, \hat{\phi}, \hat{\psi}) \right],$$

(9)

where the time series goes up to the day at which options are considered, so that $V_{T+1} = v^{**}$ and $y_T = y^*$. The criterion minimized by estimator $\hat{\theta}$ is a weighted sum of two components. The first component takes all the asset prices actively traded on the considered date into account. The second component exploits the information contained in a time series of equity and T-bill prices. The first component is similar to standard criteria used for the calibration of asset pricing models. It is a scalar product multiplied by the square of the kernel estimator bandwidth to ensure convergence and asymptotic normality (Gagliardini and Ronchetti, 2012). The GMM estimator of the SDF parameter vector minimizes just the second component of the criterion in Equation (9). This component is similar to the minimum distance criterion introduced in Ai and Chen (2003) to estimate conditional moment restrictions models, and used by Nagel and Singleton (2011) in an application to conditional asset pricing models.\textsuperscript{9} In their most general formulation, the GMM and XMM estimators minimize quadratic forms defined by some weighting matrices. The estimation of a particular weighting matrix, for example of the matrix that minimizes the asymptotic variance of the estimator (Hansen, 1982, and Gagliardini and Ronchetti, 2012), could introduce additional statistical errors and lead to large finite sample bias (see, e.g., Altonji and Segal, 1996, and Clark, 1996). For this reason and to reduce the computation burden, identity weighting matrices are used.

The second estimator of the transition density of the process $(X_t)$ is a version of the kernel estimator $\hat{\phi}$ adjusted for the no-arbitrage restrictions. This estimator, denoted by $\hat{\phi}^*$, minimizes a statistical divergence

\textsuperscript{8}Each regression function in the computation is estimated by a Nadaraya-Watson estimator. This estimator is asymptotically equivalent to the conditional expectation operator computed by using the transition density for process $(X_t)$ defined in Equation (8).

\textsuperscript{9}Gagliardini and Ronchetti (2012) show that including the no-arbitrage restrictions for the value $y^*$ of the conditioning volatilities in both vectors $L$ and $U(y_T; \ldots)$ increases the asymptotic efficiency of the XMM estimator.
from the estimator $\hat{\phi}$ subject to the no-arbitrage restrictions. This divergence is derived from the Kullback-Leibler divergence $d$ of the transition density from its kernel estimator:

$$d(\phi, \hat{\phi}|y) = \int \log \left( \frac{\phi(x|y)}{\hat{\phi}(x|y)} \right) \phi(x|y) dx,$$

for any value $y$ of the conditioning volatilities. The estimator $\hat{\phi}^*$ of the transition density of the process $(X_t)$ is defined as

$$\hat{\phi}^* = \arg \min_{\phi} \int d(\phi, \hat{\phi}|y)\hat{g}(y) dy, \quad s.t. \quad \left\{ \begin{array}{l} L(\hat{\theta}, \phi) = 0, \\ U(y; \hat{\theta}, \phi) = 0, \text{ for all } y, \end{array} \right.$$ (10)

where $\hat{g}$ is the kernel estimator of the historical marginal joint density of volatility factor and individual equity return volatility:

$$\hat{g}(y) = \frac{1}{Th_T^2} \sum_{t=1}^{T} K\left( \frac{y_t - y}{h_T} \right).$$ (11)

The criterion in the definition of the estimator $\hat{\phi}^*$ is the average Kullback-Leibler divergence, weighted by the kernel density estimator $\hat{g}$, over the entire volatility risk factor and individual equity return volatility space. The constraints for the criterion are the no-arbitrage restrictions evaluated at the XMM estimator $\hat{\theta}$ of the SDF parameter vector defined in Equation (9). In the empirical study reported in this paper, the XMM methodology is characterized by an higher finite sample efficiency than the GMM methodology (see Section 4). For this reason, the constraints for the criterion minimized by estimator $\hat{\phi}^*$ are the no-arbitrage restrictions computed at the XMM estimator and not the GMM estimator of the SDF parameter vector. The minimization of a statistical divergence subject to conditional moment restrictions is used for the estimation of probability densities in model calibrations (Buchen and Kelly, 1996, and Stutzer, 1996) and in information-based approaches to both GMM (Kitamura and Stutzer, 1997, and Kitamura, Tripathi and Ahn, 2004) and XMM (Gagliardini, Gourieroux and Renault, 2011, and Gagliardini and Ronchetti, 2012).

The estimator $\hat{\phi}^*$ defined in Equations (10) is an adaptation of the semi-parametric estimator introduced in Gagliardini and Ronchetti (2012). They consider the first order condition of the associated functional Lagrangian problem and provide an implementable expression for the estimator. Following similar steps, the estimator $\hat{\phi}^*$ satisfies the following implicit equation:

$$\hat{\phi}^*(x|y) = \frac{\hat{\phi}(x|y)T(x, y)}{\int \hat{\phi}(x|y)T(x, y) dx},$$ (12)
for a tilting factor $T$ that depends on vectors $X_t$ and $Y_t$, the estimator $\hat{\theta}$ and the estimator $\hat{\phi}^*$ itself. Hence the representation of the estimator $\hat{\phi}^*$ given in Equation (12) is implicit and yields a fixed point problem. An iterative procedure to solve this problem is implemented, in a similar way as the one suggested by Gagliardini and Ronchetti (2012). This procedure is based on the numerical computation of the estimator $\hat{\phi}^*$ on a grid of points and the choice of a numerical criterion for the convergence of the algorithm.

3 Data

In this section the data are described. Section 3.1 explains the criteria adopted for their selection. Section 3.2 deals with the characteristics of the realizations of risk-free rate, risk factors, equity returns and their volatilities. Section 3.3 illustrates the considered options.

3.1 Data construction

The price of equity shares of each DJIA’s component from January 03, 2006, to August 29, 2008, is taken from the NYSE TAQ database. The firms included in the DJIA during the considered period, with their ICB industry and supersector classifications (http://www.icbenchmark.com/), are listed in Table 1. Extraordinary events affecting both the financial sector and the real economy bring a structural change in the process of formation of asset prices in September 2008. For example, in September 2008 the interbank lending freezes (the TED spread skyrockets to over 450 bps), Lehman Brothers fails, Fannie Mae and Freddie Mac are nationalized, Bank of America acquires Merill Lynch, the FED announces that Goldman Sachs and Morgan Stanley are asked to turn into commercial banks, AIG is rescued by the U.S. Government, the Reserve Primary Fund breaks the buck, JPMorgan Chase acquires Washington Mutual Bank, short selling on many stocks is banned. The general deterioration of the economic and financial general situation leads to a big drop in liquidity in the equity market in the fall of 2008. As an example of the dramatic changes in the market price of an individual equity in the fall of 2008, the time series of the price at close of a share at NYSE of the IBM equity, which is a DJIA’s component, and the bid-ask spread from the highest ask and the lowest bid price for a share on the same day are displayed in the lower panels of Figure 2. As we can see in the left panel, the IBM stock experiences a rally before the fall of 2008 and a plunge in value afterwards. As we can see in the right panel, before this plunge the percentage bid-ask spread computed with the highest ask and the lowest bid price at NYSE on the same day for a single IBM share is almost always lower than
4% and broadens greatly in September 2008.\textsuperscript{10} Therefore, the analysis reported in this paper focuses just on the data generating process for equity, option and T-bill prices during the period from 2006 to the end of August 2008.

Let us denote by $S_t$ the last trading price at NYSE of a share of a DJIA’s component on business day $t$. Let also denote by $D_t$ the dividend per share if $t$ is an ex-dividend day.\textsuperscript{11} The daily cum-dividend geometric return from day $t$ to day $t+1$ on the equity share of the considered DJIA component is

$$ r_{t+1} = \log \left( \frac{S_{t+1} + D_{t+1}}{S_t} \right). $$

Since true volatilities are not observable and the method relies on their empirical realizations, measures of realized volatility (RV) obtained from intraday index levels and equity trades are taken as proxies.\textsuperscript{12} RV measures do not rely on any parametric specification of the stock return dynamics. On business day $t$, the individual equity return RV is defined by intra-day share trading prices $S_{j,t}$ at 5 minutes frequency from 9:30 a.m. to 4:00 p.m. (EST):

$$ RV_t = \sqrt{\frac{1}{T} \sum_j \left( \log \left( \frac{S_{j,t}}{S_{j-1,t}} \right) \right)^2}, $$

When the process of the share price is a square integrable semi-martingale, the realized volatility $RV_t$ converges in probability to the quadratic variation on day $t$ of the process of the log-price $\log (S_t)$, as the number of intra-day trading prices increases (see, e.g., Protter, 2004) For example, this is the case when the share price dynamics is a continuous time stochastic volatility or jump-diffusion process (Andersen, Bollerslev, Diebold and Ebens, 2001, and Andersen, Bollerslev, Diebold and Labys, 2003). The bias induced by the micro-structure effects is small, being the equity shares of DJIA’s components traded at high frequency.

The NYSE tick-size is $0.1$, and return rounding and discreteness effects for the equity shares of DJIA’s components are negligible. For example, for the lowest IBM share closing price in the period ($73.58$), the

\textsuperscript{10}The percentage bid-ask spread for an asset with ask price $ASK_t$ and bid price $BID_t$ is defined as $100 \cdot (ASK_t - BID_t) / [(ASK_t + BID_t) / 2]$. In the period from January 2006 to August 2008 the difference between the highest ask and the lowest bid price of the same day for a single IBM share is always less than $5$.

\textsuperscript{11}For example, IBM has a long history of quarterly dividend payments. From January 01, 1998 to August 29, 2008 each period between two subsequent dividend announcements is between 87 and 94 calendar days. IBM’s dividends are usually paid on the ninth or tenth of March, June, September and December. The dividend record date is set about a month earlier and the ex-dividend day is normally two business days before the record date. The ex-dividend days in the considered period are February 08, 2006 ($0.2$ per share); May 08, August 08 and November 08, 2006, and February 07, 2007 ($0.3$ per share); May 08, August 08 and November 07, 2007, and February 06, 2008 ($0.4$ per share); May 09 and August 08, 2008 ($0.5$ per share) (http://www.ibm.com/investor/financials/).

\textsuperscript{12}The RV is a proxy for the historical aggregate volatility. It does not involve the discount for the risk of levels aggregate volatility that are different than expected, as other proxies for the aggregate volatility, such as the VIX index.

\textsuperscript{13}The actual trades that best approximate the 5 minutes frequency are considered.
four possible returns closest to zero are about ±0.12% and ±0.06%.

The market, value/growth, size and momentum risk factors are proxied by the factors provided by Prof. K. R. French in his website (http://mba.tuck.dartmouth.edu/pages/faculty/ken.french). Factor $Z_{1,t}$ is the market factor, i.e., the difference of the value-weight return on all NYSE, AMEX, and NASDAQ stocks and the 1-month T-Bill rate. Factor $Z_{2,t}$ is the size factor, i.e., the Small Minus Big (SMB) FF portfolio. Factor $Z_{3,t}$ is the value/growth factor, i.e., the High Minus Low (HML) FF portfolio. Factor $Z_{4,t}$ is the momentum factor, i.e., the Up Minus Down (UMD) FF portfolio. The realized variance for the S&P 500 and DJIA indexes from data at the 5 minutes frequency provided by the Oxford-Man Institute (http://realized.oxford-man.ox.ac.uk/) is used as a proxy of the common volatility factor (Heber, Lunde, Shephard and Sheppard, 2009). The common volatility factor $V_t$ is proxied by the RV of the return on the S&P 500 index (and also by the return on the DJIA index, used in a robustness check of the analysis).

American call and put options written on DJIA’s components are multiple listed. The option highest ask and the lowest bid price at close across U.S. exchanges are obtained by the Ivy DB OptionMetrics database. These values are used to compute the mid-quotes at the close of the exchanges (4 : 00 p.m. EST). The tick-size is $0.05. Usually, an exchange traded equity option has physical settlement with standard unit of trading (contract lot size): only the trades of lots of options on 100 shares are allowed and the delivery of these shares must take place at the option exercise. U.S. equity put and call options traded in centralized markets expire on the third Saturday of the month and are closed for trading the previous Friday. On any business day, the markets provide the quotes of options for at least four different expiration months. The two earliest expiration months are the current and the next one. The other two months are chosen on the basis of some option issuing cycles. For example, IBM belongs to the January cycle of the U.S. equity options so that the last two expiration months for quoted options are the earliest between January, April, July and October. Then, on any business day in July and August 2008, 1- and 2-months IBM options and IBM options expiring in October and January are quoted.

Option data are filtered on the basis of several criteria. Some options are not considered: options with percentage bid-ask spread at close higher than or equal 100%, options with daily trading activity lower than or equal to 500 contracts, options with time-to-maturity longer than 480 calendar days, options with a moneyness strike less than 0.7 or more than 1.3. The time-to-maturity and percentage bid-ask spread filters imply to retain only options with an ask price at close at most three times the contemporaneous bid price.

\footnote{For example, during July and August, 2008, options on IBM shares are traded at the CBOE, PHLX and AMEX.}

\footnote{The discussion does not apply to Long-term Equity AnticiPation Securities (LEAPs), which are options with time-to-maturity greater than two years when first listed. They expire in January.}
Options with null bid at close are automatically excluded.

The reference daily risk-free rate is obtained from the 1-month Constant Maturity Treasury Rate. The U.S. Department of the Treasury provides publicly this rate, which comes from an interpolation of the daily yield curve, on an annualized basis. T-bills, considered in the period free of default risk, are very liquid and with a broad secondary market. Therefore, in the study investors are assumed to be able to have access to investments with rate or return $r_{f,t}$.

### 3.2 Realizations of the state variables

In the considered period, the U.S equity market is characterized by a bullish and (relatively) stable period, before July 2007, and a successive bearish (relatively) volatile one. The market downturn realizes in a moment of growing widespread concerns about prices and ratings of financial assets, liquidity risk and counterparty risk. For example, in July 2007 the interbank lending sharply declines, liquidity crunch spreads around the shadow banking system, the market for asset-backed commercial papers begins to dry up, the asset-backed securities indices starts a decline, the TED spread starts to fluctuate from around 100 to around 200 bps, the rescue of IKB Deutsche Industriebank opens a series of bailouts in Europe, American Home Mortgage Investment Corporation announces its financial difficulties, the SEC relaxes the uptick rule for stocks traded at NYSE. Overall, a sharp contraction in real economies begins to be largely foreseen. The geometric returns on the S&P 500 and DJIA indexes are about 1.2% and 6.3%, respectively, being the index levels at close about 1268 and 12840 on January 03, 2006, and about 1283 and 11550 on August 29, 2008. Both the indexes reach the highest peak on October 9, 2007: the level at close is about 1565 for the S&P 500 and about 14170 for the DJIA. Before July 2007, the mean and the median RVs are about 0.0058 and 0.0052 for the S&P 500, and about 0.0057 and 0.0053 for the DJIA. After July 2007 the two statistics are almost doubled: they are about 0.0108 and 0.0100 for the S&P 500, and about 0.0105 and 0.0101 for the DJIA. The empirical distribution of the daily individual equity returns is centered at about 0 and it is negatively skewed, displaying the unconditional left tail risk typical of individual equities. The sample kurtosis always exceeds the kurtosis of the standard normal distribution. The unconditional return distributions are then leptokurtic (see also Andersen, Bollerslev, Diebold and Ebens, 2001 for further statistics). As an example of the change in equity price characteristics occurred in July 2007, we can see in the right lower panel of Figure 2 the increase of the percentage bid-ask spread computed with the highest ask and the lowest bid price at NYSE on the same day for a single IBM share. This spread varies indeed approximatively in the range $[1\%, 2\%]$.

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16The Treasury yield curve is estimated daily using a cubic spline proprietary model. Inputs to the model are primarily bid-side yields for on-the-run Treasury securities (http://www.ustreas.gov).
before July 2007 and in the range \([2\%, 4\%]\) afterwards.

In the upper right panel of Figure 2 the time series of the risk-free rate is displayed. The range for the rate in the considered period is \([0.07, 1.46] \times 10^{-4}\), with mean value about \(1.08 \times 10^{-4}\) and median value about \(1.26 \times 10^{-4}\). The value of the risk-free rate is several times smaller than the mean return on an equity. For example, the sample mean and median of the risk-free rate are close to one-sixth and one-fifth of the corresponding statistics of the IBM stock return, respectively. Moreover, the value of this rate oscillates around \(1.35 \times 10^{-4}\) before July 2007 and declines afterwards, reflecting the weakening of financial markets.

Figure 3 reports the values of the first 20 sample Auto-Correlation Function coefficients for the considered proxies of the risk factors. These coefficients are displayed as functions of the lag index. The 95% confidence level bounds are represented by the horizontal lines. At the 95% confidence level, almost every ACF coefficient for the FF proxies of market excess return, size, value and momentum factor is not significant. The few exceptions are just marginally significant. Differently, all the first 20 ACF coefficients for the RVs of both the returns on the S&P 500 and DJIA indexes are statistically significant at the 95% confidence level. Therefore, the FF portfolios are in accordance with the requirements in Assumptions 4 and 5 for the dynamics of the risk factors \(Z_{1,t}, \ldots, Z_{4,t}\), and the S&P 500 and DJIA RVs for the dynamics of the risk factor \(V_t\).

### 3.3 Options

By filtering the option quotes obtained by the Ivy DB OptionMetrics database as described in Section 3.1, the constructed option sample is composed by 7376 put options and 8946 call options. The mean percentage bid-ask spread at close across all the contracts is about 7%. On each day, between 0 and 40 call options and between 0 and 31 put options written on the same stock are considered. The time-to-maturity goes up to 166 business days (corresponding to 243 calendar days) for both put and call options (with the maximum time-to-maturity for JPM and PFE put and JPM call options on July 21, 2008, and INTC and MSFT put and JNJ and MMM call options on August 18, 2008). As an example of the time-varying moneyness strike and the time-to-maturity of the considered options, Figure 4 shows these option characteristics for the IBM options as functions of the date. On each business day, between 4 and 23 IBM options are considered. The moneyness strike varies approximately in the range \([0.71, 1.06]\) for the put options (indicated by crosses) and in the range \([0.85, 1.18]\) for the call options (indicated by circles).

Only 0.3% of the put options and 0.2% of the call options are at-the-money and 32.5% of the put options and 27.5% of the call options are in-the-money. Several authors have shown that the early exercise
premium is not negligible especially for in-the-money options (Shastri and Tandon, 1986, Whaley, 1986, Zivney, 1991, and Dueker and Miller, 2003). Since more than one-fourth of the option dataset is composed by in-the-money options, taking the early exercise possibility into account is important in the study. As often in individual equity option markets, the short-term options have the highest trading volume and lowest percentage bid-ask spread. Closer is the expiration, higher is the rate of change in option value due to time (i.e. higher is the option theta in absolute value) and higher are the potential return and leverage. Closer is the expiration, more incentives for informed investors to enter the option market arise. For example, 82% of the IBM put options and 84% of the IBM call options have time-to-maturity up to 70 business days.

The imaginary investor able to trade at the mid-quote and without incurring in frictions would not find any arbitrage opportunity in the option sample. On any day, the mid-quote $C$ of an American call option written on a individual equity share with price $S$, with strike price $K$ and time-to-maturity $h$ is not greater than $S$ and smaller than $(Se^{-\delta h} - Ke^{-r_fh})^+$, for the dividend yield $\delta$ and risk-free rate $r_f$. Similarly, the mid-quote $P$ of the put option with the same contract characteristics is not greater then $Ke^{-r_fh}$ and smaller than $(Ke^{-r_fh} - Se^{-\delta h})^+$. Then, there would not be any discount arbitrage opportunity. There would not be any bull and bear spread arbitrage opportunity: for a given maturity, the call (put) option mid-quotes are convex decreasing (increasing) in the strike price. There would not be any calendar spread arbitrage opportunity: for any couple of contemporaneous mid-quotes of call (put) options with the same strike price, the mid-quote of the call (put) option with the longer maturity is not valued less than the other.

4 Results

This section discusses the estimates of the SDF and some properties of the historical transition density for process $(X_t)$, obtained by the methods described in Section 2. Section 4.1 reports the results of the estimation of the SDF parameter vector. Section 4.2 shows the differences in the estimation of the historical correlation of an equity return and its volatility and the skewness and kurtosis of the equity return obtained by using the kernel estimator $\hat{\phi}$ and the estimator $\hat{\phi}^*$ that considers the no-arbitrage restrictions.

4.1 SDF

This section contains the description of the GMM and XMM estimates of the SDF parameter vector. The estimates are computed on each business day in July and August 2008 and by using separately the price of equity and individual equity options for each DJIA’s component. At any day and for any firm, the estimation of the SDF parameters is performed by both the GMM and the XMM methodologies. The $43 \times 32 = 1376$
(i.e., number of business days times the number of firms) estimates of the SDF parameter vector provides an empirical distribution of the estimates. In Figure 5 we can see the normalized histograms of the GMM and XMM estimates of each SDF parameter, with the histogram for the GMM estimates in darker color. In the legend, the sample mean and standard deviation (STD) for the two methods are also reported. For each parameter, the mean values are close, but the sample distribution of the XMM estimates has smaller variance. The fact that the GMM and the XMM provide different results supports the idea that the time series of risk factors and equity returns and their volatilities and the cross-section of option mid-quotes carry information on the SDF parameters that are not redundant, and that including option data improves the estimation efficiency. Therefore, the XMM, which allows to consider a time series of stock prices and a cross-section of option prices, is the most efficient econometric methodology to study the SDF parameter vector. In Table 3 some statistics of the samples of the XMM and GMM estimates are reported. For both the GMM and XMM estimates we find in this table the estimated mean, standard deviation, 95% bias-corrected and accelerated bootstrap confidence interval for the mean value and the p-values for the Wilcoxon signed rank test and the sign test. The bootstrap confidence interval is computed by using 9,999 bootstrap samples. The mean of the XMM and GMM estimates, except for parameter $\theta_3$, lies in its 95% bootstrap confidence interval. The null hypothesis for the Wilcoxon signed rank test and the sign test is that data are realizations from a probability distribution with median zero. For the results of both GMM and XMM the estimations, this null hypothesis is always rejected at the 5% significance, with p-values always lower than 0.001, except for parameter $\theta_3$, which is the SDF parameter that measures the discount for the value/growth factor. Then, there is very strong evidence against the null hypothesis in favor of the alternative for any SDF parameter except $\theta_3$. The two tests are considered because both the Kolmogorov-Smirnov and the Lilliefors tests reject at the 5% significance level the hypothesis that the estimates are realizations of a normal random variable. We can see that the normalized histograms in Figure 5 are characterized by asymmetry indeed.

The XMM estimates of parameter $\eta$, which is the SDF parameter that measures the fixed discount related to the rate of time preference, have mean value of about $5 \times 10^{-7}$ and standard deviation of about $10^{-6}$ (i.e., about 0.51 as signal-to-noise ratio). The 73% of these estimates is positive and their standard deviation is about 55% of the standard deviation of the GMM estimates. Including option data improves the efficiency of the estimation of parameter $\eta$. The mean value suggests that the market discounts by $e^{-5 \times 10^{-7}} \approx 1 - 5 \times 10^{-7}$ any payoff over the horizon of a day only because of time preference. This finding is in accordance with the idea that the representative investor for the equity and option markets

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17See Andrews and Buchinsky (2003) for the chosen number of bootstrap samples.
prefers having cash on day $t$ than the same cash adjusted at the risk-free rate on day $t + 1$. However, the magnitude of this discount is very small. The representative investor is patient and, just on the basis of time, weights current and one-day-ahead consumption almost equally. For pricing purpose a payoff one year ahead is indeed decreased by only 1.25 bps net of the risk-free rate.

The XMM estimates of parameter $\theta_1$, which is the SDF parameter that measures the discount for the market factor, have mean value of about 0.44 and standard deviation of about 1.19 (i.e., about 0.37 as signal-to-noise ratio). The 75% of these estimates is positive and their standard deviation is about 93% of the standard deviation of the GMM estimates. The results obtained by the GMM and XMM methodologies are very close. It does not seem that including option data improves dramatically the estimation of parameter $\theta_1$. The estimated positive sign of $\theta_1$ suggests an overall shift of probability towards lower excess return on the market portfolio, when passing from the historical to the risk-neutral conditional density of the state variables. Therefore, the estimation suggests that more probability weight is put on negative values of the excess return on the market portfolio for pricing purpose. This result is consistent with the idea of investors that are risk averse in aggregate and require compensation for taking market risk. In particular, the product between the mean point estimate of parameter $\theta_1$ and the sample fifth quantile of factor $Z_{1,t}$ is about $-0.008$. This value implies that the market discounts by $e^{0.008} \approx 1.008$ any payoff for the situations with market factor value corresponding to its sample fifth quantile. This finding means that, for pricing purpose, the value of any payoff is increased by about 0.8% in case the market value gets its sample fifth quantile as value.

The XMM estimates of parameters $\theta_2$, which is the SDF parameter that measures the discount for the size factor, have mean value of about $-0.46$ and standard deviation of about 2.88 (i.e., about 0.16 as absolute signal-to-noise ratio). The 58% of these estimates is negative and their standard deviation is about 86% of the standard deviation of the GMM estimates. The estimated sign of parameter $\theta_2$ is negative, consistently with the idea that smaller a firm’s capitalization, higher is the expected return of its equity. The products between the mean point estimate of parameter $\theta_2$ and the sample fifth and ninety-fifth quantiles of factor $Z_{2,t}$ are about 0.004 and $-0.004$, respectively (that correspond to discounts by $e^{-0.004} \approx 0.996$ and $e^{0.004} \approx 1.004$). These values indicate that for pricing purpose about 0.4% of the probability weight is taken from the sample fifth quantile and shift to the ninety-fifth quantile.

The XMM estimates of parameter $\theta_4$, which is the SDF parameter that measures the discount for the momentum factor, have mean value of about $-0.64$ and standard deviation of about 1.61 (i.e., about 0.40 as absolute signal-to-noise ratio). The 65% of the XMM estimates is negative and their standard deviation
is about 83% of the standard deviation of the GMM estimates. The estimated sign of parameter $\theta_4$ is negative, consistently with the idea that better the performance over the previous 3 to 12 months, higher is the expected return. The products between the mean point estimate of parameter $\theta_4$ and the sample fifth and ninety-fifth quantile of factor $Z_{4,t}$ are about 0.009 and $-0.009$, respectively (that correspond to discounts $e^{-0.009} \approx 0.991$ and $e^{0.009} \approx 1.009$). The interpretation of these values as measuring the shift of probability weight when passing from the historical to the risk-neutral probability distribution is similar as before.

The XMM estimates of parameter $\theta_5$, which is the SDF parameter that measures the discount for the volatility factor, have mean value of about $-1.25$ and standard deviation of about 1.46 (i.e., a relatively high absolute signal-to-noise ratio of about 0.86). The 90% of the XMM estimates is negative and their standard deviation is about 84% the standard deviation of the GMM estimates. The estimated negative sign of $\theta_5$ suggests an overall shift of probability towards higher values of the common volatility factor when passing from the historical to the risk-neutral conditional density of the risk factors and equity returns and their volatilities. This result is consistent with the idea of investors that are risk averse in aggregate and require compensation for taking volatility risk. In particular, this result is consistent with the idea that option traders, who usually holds large portfolios, are not able to eliminate volatility risk, which could cause large losses (Green and Figlewski, 1999), through hedging and diversification. The products between the mean point estimate of parameter $\theta_5$ and the sample fifth quantile, median value and ninety-fifth quantile of factor $Z_{5,t}$ are about $-0.004$, $-0.009$ and $-0.020$, respectively (that correspond to discounts $e^{0.004} \approx 1.004$, $e^{0.009} \approx 1.009$ and $e^{0.020} \approx 1.020$).

With the exception of the volatility factor, the product of any risk factor value included in its sample 5th to 95th inter-quantile range and the estimated related parameter is approximatively included in the interval $[-0.009 : 0.009]$. The risk discount for this range of values of the risk factors is then up to the same order of magnitude $10^{-3}$ as the estimated value 0.005 for parameter $\eta$. The risk-free discount is of lower order, being at most of order $10^{-4}$ (see Table 2). The heaviest discount occurs for situations characterized by high values of the volatility factor. A substantial discount takes place for situations characterized by low and high values of the momentum factor and then for situations characterized by low value of the market factor. More precisely, when discounting a payoff one day ahead, it is inflated of about 2% for the realization of the ninety-fifth quantile of the volatility factor, it gets a discount of about $\pm 90$ bps for the realizations of the fifth and ninety-fifth quantiles of the momentum factor and it is inflated of about 80 bps for the realizations of the fifth quantile of the market factor. This result is in agreement with the finding of the study on the monthly U.S. stock returns in Connor, Hagmann and Linton (2012) that momentum and volatility risk factors are “as
important or more important than size and value in explaining equity return comovements.”

4.2 State variable dynamics

Every individual equity stock has different dynamic features. An improvement in the estimation of its marginal and cross-moment is always obtained by considering the estimator $\hat{\phi}^*$, which accounts for the no-arbitrage restrictions, instead of the nonparametric kernel estimator $\hat{\phi}$. As an example of this improvement, in this section the comparison between the estimates of the historical conditional correlation function between IBM equity return and its volatility and the conditional skewness and kurtosis of the IBM equity return by means of the two estimation methodologies is reported. The conditional expectations involved in the definitions of the estimators are computed by Nadaraya-Watson kernel regression function estimators. All the estimators are asymptotically normal (see Appendix B.2 for the estimators computed by $\hat{\phi}$ and Gagliardini and Ronchetti, 2012 for the estimators computed by $\hat{\phi}^*$). The evolution over time of the mentioned moments is considered in July and August 2008. The full time series of the common volatility factor, the returns on the IBM stock and its volatility from January 03, 2006, to August 29, 2008, is used for the estimation of each moment. The matrix bandwidth for the kernel estimation is proportional to the one chosen by the multivariate generalized Scott’s rule of thumb (Scott, 1992). The proportional constants are 2.5, 2 and 1.25 for the correlation, skewness and kurtosis, respectively. The width of the 95% confidence interval derived from the estimate of the asymptotic variance is smaller than the 10% of the absolute value of the point estimate for most of values of the conditioning RVs. In the three upper panels of Figure 6 the estimates of the three quantities are displayed as functions of the date. They are computed for the contemporaneous value of the conditioning RVs. For each quantity, the dotted line, labeled “Kernel”, indicates the estimate obtained by using $\hat{\phi}$, while the solid line, labeled “Tilted”, indicates the estimate obtained by using $\hat{\phi}^*$. We can see the daily RV of the returns on the S&P 500 index and the IBM stock on the same days in the lower left and center panels. The reported results are obtained by taking the S&P 500 RV as proxy for the common volatility risk factor. Similar results are obtained using the DJIA RV.

Before describing the estimation results, let us consider three potential causes of variation of the estimated quantities over time: the change in the values of the conditioning RVs; the statistical variability in the estimation of the quantities and in the approximation of the true systematic price determinants by means of asset portfolios and realized volatilities; any possible model misspecification. Let us consider separately these three causes. First, the common volatility risk factor and the equity return volatilities vary over time and the considered historical moments of the equity returns and their volatilities are conditional on specific
values of these variables. Therefore, changes in the relevant conditioning set for the density $\phi$ make the moments vary. Second, the estimation of the model parameters is performed on different dates with distinct data samples. This last point has not a big impact on the estimation results. As a matter of fact, the application of the kernel estimator to time series that consist of hundreds of observations and differ only for few of them most likely does not lead to statistically different results. Moreover, the cross-section of option quotes, which includes almost at any day both short- and long-term option quotes (see the right panel of Figure 4 for the cross-sections of IBM option quotes), carry similar information about the data generating process. In addition, the used realizations of risk factors and volatilities are proxies and not true values of the systematic price determinants. Third, the choices of the risk factors and the parametrization of Equation (1) are assumptions. The estimates of the SDF parameter are clearly peaked around some values and support the validity of the asset pricing model. The contribution given by the two last points to the time variation of the estimated quantities is small. A major part of the variation in the time series of the estimated quantities is caused by the changing value of the conditioning volatility risk factor and equity return volatility and only a minor part is due to the statistical variability. Considering that the estimated conditional quantities vary over time and that the estimation is affected by some statistical variability, let us focus on the different insights on the conditional density of the IBM equity return and volatility obtained by the two different estimation methodologies.

The range of variation over time of the estimates obtained by $\hat{\phi}^*$ is mainly smaller than the one of the estimates obtained by $\hat{\phi}$. Moreover, the range of variation of the latter estimates has two regimes: it is broader before July 17, when volatility factor and volatility of the IBM return have extreme values, and lower afterwards, when the two variables assume very high but not extreme values. In the first period the IBM equity return varies approximatively in the range $[-0.025 : 0.025]$, the common volatility factor does it in $[0.006 : 0.024]$ and the volatility of the IBM return does it in $[0.01 : 0.025]$. In the second period the return varies approximatively in the range $[-0.02 : 0.015]$, the common volatility factor does it in $[0.007 : 0.013]$ and the volatility of the IBM return does it in $[0.009 : 0.015]$. While we observe two regimes of variability for the estimates obtained by $\hat{\phi}$, no clear separation in different regimes appear when the estimation is performed by $\hat{\phi}^*$. Adopting an arbitrage-free pricing model leads to estimates of some dynamic properties of the state variables that are more stable over time. We see in the upper left panel of Figure 6 the time series of the estimates of the conditional correlation between the state variables. The estimates obtained by $\hat{\phi}^*$ are almost always negative, while the estimates obtained by $\hat{\phi}$ vary approximatively in the range $[-0.22 : 0.15]$ before July 17 and are negative afterwards. While the former estimates support almost always the presence
of contemporaneous leverage and/or volatility feedback effects, the latter does it only after July 17. We
see in the upper center panel the time series of the estimates of the conditional skewness of the returns.
The estimates obtained by \( \hat{\phi} \) vary approximatively in the range \([-0.5 : 0.5]\) before July 17 and are almost
always negative afterwards. Differently, the estimates obtained by \( \hat{\phi}^* \) are almost always negative before
July 17 and positive afterwards. The estimates obtained by \( \hat{\phi}^* \) display a much lower variation over time.
Considering both the time series of the state variables and the option data under an arbitrage-free pricing
model makes the shareholders fear the left tail risk only before July 17 that is the period characterized by
extreme values of volatility factor and IBM equity return volatility. Finally, we see in the upper right panel
the time series of the estimates of the conditional kurtosis of the returns. Before July 17, the estimates
obtained by \( \hat{\phi} \) vary approximatively in the range \([1.5 : 4.2]\), alternating the characterization of platykurtic
and leptokurtic conditional distributions of the IBM equity return. Differently, the estimates obtained by \( \hat{\phi}^* \)
vary approximatively in the range \([3 : 6]\), indicating always a leptokurtic distribution. After July 17, the
distribution of the IBM equity return, estimated in both ways, is leptokurtic.

From the comparison between the estimates of the considered statistical moments and the actual real-
ilizations of the IBM returns, we understand that the observed values are more likely to be generated from
the probability distribution \( \hat{\phi}^* \) than \( \hat{\phi} \), especially on those days with extreme values of volatility risk factor
and IBM return volatility. The IBM equity return time series for the period from January 2006 to August
2008 has unconditional mean value that is close to 0 and unconditional standard deviation close to 0.012. As
we can see in the lower left panel of Figure 6, before July 17 the realizations of the returns are included in
two-and-a-half unconditional standard deviations from the unconditional mean, while afterwards they are no
more than one-and-a-half unconditional standard deviations away from the unconditional mean. Before July
17 the information content in the option quotes suggest to take into account a greater fat tail risk, almost on
any day. The conditional probability distribution of the IBM equity returns estimated by using \( \hat{\phi}^* \) is indeed
often negatively skewed and leptokurtic, while the same distribution estimated by using the kernel estimator
\( \hat{\phi} \) display thinner tails. The extreme returns occurred just before July 17 are then more likely to be generated
from the conditional distribution \( \hat{\phi}^* \) than from \( \hat{\phi} \). After July 17 the conditional probability distribution of
the IBM equity returns estimated by using \( \hat{\phi}^* \) has thinner tails than before and smaller values of the returns
are in fact realized. This fact supports the hypothesis that accounting for an arbitrage-free pricing model and
considering option data improves the characterization of the historical joint probability of the state variables,
even in the absence of parametric assumptions for the dynamics of the state variables.
5 Concluding remarks

This paper reports an investigation of the information content of equity and American option prices. The study does not rely on any parametric specification for the historical dynamics of risk factors, stock returns and their volatilities and takes the option early exercise into account. Three empirical results are obtained. First, considering option prices along equity prices improves the efficiency in the estimation of risk premia. Second, extreme values of the common volatility risk factor brings the heaviest payoff discount, followed in sequence by extreme values of momentum, market and size factors. Third, considering an arbitrage-free pricing model improves the characterization of the historical dynamics of risk factors, equity returns and their volatilities, even in the absence of parametric assumptions. Considering both equity and option prices improves the estimates of risk premia and the conditional probability distributions of risk factors, equity returns and their volatilities.
References


Figure 1: Allowed Granger causality links between risk factors and equity return and volatility at following business days. Each arrow indicates the direction of causality.
Figure 2: Time series of the levels of S&P 500 and DJIA indexes, the 1-month T-Bill rate, the price of an IBM share at NYSE and its bid-ask spread at market close as functions of time. In the upper left panel the time series of the market indexes are reported on two different scales (the left scale refers to the S&P 500 and the right one to the DJIA). As example of the drop in stock market value during the fall of 2008, the time series of the price of an IBM share is reported in the lower left panel. This drop goes along with a broaden in the percentage bid-ask spread from the highest ask and the lowest bid price for a share on the same day, as displayed in the lower right panel.
Figure 3: Sample autocorrelation function for the risk factors in the period from January 03, 2006, to August 29, 2008. The correlations are displayed as functions of the lag index. The 95% confidence interval for the coefficients is \([-0.0773 : 0.0773]\) and its borders are displayed by the straight horizontal lines.
Figure 4: Time-varying characteristics of the considered IBM put and call options. IBM belongs to the January cycle of the U.S. equity options. In the left panel we can see the moneyness strike and in the right panel their times-to-maturity as functions of the date. The time-to-maturity is expressed in business days. Crosses refer to the put options, circles to the call options.
Figure 5: The normalized histograms of the estimates of the components of the SDF parameter vector $\theta$. The estimates are preformed on each business day in July and August 2008 by the Generalized Method of Moments (GMM) and Extended Method of Moments (XMM) methodologies.
Figure 6: Point estimates of some dynamic properties of the IBM equity return and its volatility as functions of the date. We see in the upper panels the estimates of the conditional correlation coefficient between IBM stock returns and its realized volatility, skewness and kurtosis of the IBM stock return. At any day, the moments are for a one-day horizon and conditional on the contemporaneous value of the volatility factor and the volatility of the IBM return. The estimates are obtained by using $\hat{\phi}$ (dotted line) and $\hat{\phi}^*$ (solid line). We see in the lower panels the values of the IBM equity return and realized volatilities of the returns on the S&P 500 index and IBM stock.
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<td>Food Producers</td>
<td>KFT</td>
</tr>
<tr>
<td>Kraft Foods</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Firms included in the DJIA index during the period from January 2006 to August 2008 with their ICB industry and supersector classifications. On February 19, 2008, CVX and BAC replace MO and HON. GM, HON and MO are not currently included in the DJIA. Additionally, KFT and CSCO, current DJIA’s components, are also included in the study.
<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>STD</th>
<th>Minimum</th>
<th>5th quantile</th>
<th>25th quantile</th>
<th>Median</th>
<th>75th quantile</th>
<th>95th quantile</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>rf, t</td>
<td>108 × 10^{-6}</td>
<td>37.7 × 10^{-6}</td>
<td>7.22 × 10^{-6}</td>
<td>37.8 × 10^{-6}</td>
<td>79.2 × 10^{-6}</td>
<td>126 × 10^{-6}</td>
<td>136 × 10^{-6}</td>
<td>145 × 10^{-6}</td>
<td>146 × 10^{-6}</td>
</tr>
<tr>
<td>MKT</td>
<td>0.0000</td>
<td>0.0098</td>
<td>-0.0343</td>
<td>-0.0173</td>
<td>-0.0048</td>
<td>0.0007</td>
<td>0.0053</td>
<td>0.0159</td>
<td>0.0397</td>
</tr>
<tr>
<td>SMB</td>
<td>0.0000</td>
<td>0.0047</td>
<td>-0.0152</td>
<td>-0.0080</td>
<td>-0.0028</td>
<td>-0.0001</td>
<td>0.0029</td>
<td>0.0079</td>
<td>0.0159</td>
</tr>
<tr>
<td>HML</td>
<td>0.0001</td>
<td>0.0043</td>
<td>-0.0242</td>
<td>-0.0053</td>
<td>-0.0018</td>
<td>0.0000</td>
<td>0.0018</td>
<td>0.0060</td>
<td>0.0332</td>
</tr>
<tr>
<td>UMD</td>
<td>0.0004</td>
<td>0.0094</td>
<td>-0.0580</td>
<td>-0.0138</td>
<td>-0.0026</td>
<td>0.0008</td>
<td>0.0045</td>
<td>0.0138</td>
<td>0.0380</td>
</tr>
<tr>
<td>S&amp;P 500 RV</td>
<td>0.0080</td>
<td>0.0043</td>
<td>0.0024</td>
<td>0.0036</td>
<td>0.0049</td>
<td>0.0009</td>
<td>0.0099</td>
<td>0.0160</td>
<td>0.0392</td>
</tr>
<tr>
<td>DJIA RV</td>
<td>0.0078</td>
<td>0.0041</td>
<td>0.0021</td>
<td>0.0036</td>
<td>0.0048</td>
<td>0.0006</td>
<td>0.0098</td>
<td>0.0153</td>
<td>0.0442</td>
</tr>
</tbody>
</table>

Table 2: Unconditional sample properties of the daily 1-month T-bill rate and the systematic determinants of financial asset prices. The sample mean, standard deviation (STD), median, minimum, maximum, 5th, 25th, 75th and 95th quantiles for all the factors are reported. MKT is the market factor $Z_{1,t}$, SMB is the Small Minus Big portfolio $Z_{2,t}$, HML is the High Minus Low portfolio $Z_{3,t}$, UMD is the Up Minus Down portfolio $Z_{4,t}$. The realized variance for both the S&P 500 and DJIA indexes is factor $V_t$ and it is computed from data at the 5 minutes frequency.

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>STD</th>
<th>95% CI</th>
<th>SRT</th>
<th>ST</th>
<th>Mean</th>
<th>STD</th>
<th>95% CI</th>
<th>SRT</th>
<th>ST</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta$</td>
<td>0.49 × 10^{-6}</td>
<td>[4.39, 5.45] × 10^{-7}</td>
<td>0.97 × 10^{-6}</td>
<td>-18.16</td>
<td>16.32</td>
<td>0.56 × 10^{-6}</td>
<td>[4.89, 6.86] × 10^{-7}</td>
<td>18.7509</td>
<td>-18.16</td>
<td></td>
</tr>
<tr>
<td>$\theta_1$</td>
<td>0.44</td>
<td>1.19</td>
<td>[0.38, 0.51]</td>
<td>-14.86</td>
<td>17.38</td>
<td>0.45</td>
<td>1.27</td>
<td>[0.38, 0.52]</td>
<td>-15.09</td>
<td>14.86</td>
</tr>
<tr>
<td>$\theta_2$</td>
<td>-0.46</td>
<td>2.88</td>
<td>[-0.61, -0.30]</td>
<td>-5.96</td>
<td>-5.36</td>
<td>-0.44</td>
<td>3.33</td>
<td>[-0.61, -0.24]</td>
<td>-6.36</td>
<td>5.96</td>
</tr>
<tr>
<td>$\theta_3$</td>
<td>1.12</td>
<td>4.15</td>
<td>[-0.11, 0.35]</td>
<td>-0.98*</td>
<td>0.53*</td>
<td>0.12</td>
<td>4.51</td>
<td>[-0.12, 0.36]</td>
<td>-1.23*</td>
<td>-0.99*</td>
</tr>
<tr>
<td>$\theta_4$</td>
<td>-0.64</td>
<td>1.61</td>
<td>[-0.73, -0.55]</td>
<td>-12.94</td>
<td>-10.53</td>
<td>-0.64</td>
<td>1.93</td>
<td>[-0.74, -0.53]</td>
<td>-13.67</td>
<td>-12.94</td>
</tr>
<tr>
<td>$\theta_5$</td>
<td>-1.25</td>
<td>1.47</td>
<td>[-1.33, -1.17]</td>
<td>-28.43</td>
<td>-28.50</td>
<td>-1.26</td>
<td>1.75</td>
<td>[-1.34, -1.15]</td>
<td>-29.19</td>
<td>-28.43</td>
</tr>
</tbody>
</table>

Table 3: Marginal statistics of the samples of the XMM and GMM estimates. Mean, standard deviation (STD), 95% bias-corrected and accelerated bootstrap confidence intervals for the mean value (95% CI), z-statistics for the Wilcoxon signed rank test (SRT) and the sign test (ST) are reported, for each SDF parameter estimated by the two methodologies. The bootstrap confidence intervals are computed by using 9,999 bootstrap samples. The null hypothesis for the two tests is that data are realizations of a variable with median zero. The null hypothesis is not rejected at the 5% significance level in the cases with the asterisk *, which is for parameter $\theta_3$ estimated by both the methodologies. All the other cases have p-values lower than 0.001.
A Expectations

Assumptions 2, 4, 5 and 6 and the multiplicative separability of the SDF in Equation (1) w.r.t. risk factors simplify the computation of some risk-neutral expectations considered in the paper. For more clarity in the exposition of this model property, let us introduce a firm index and consider the equity market as composed by equities of $L$ firms. On business day $t$, we now denote the value of the daily return on the traded equity of the $i$-th firm and its volatility by $r_{i,t}$ and $\sigma_{i,t}$, respectively, for any $i = 1, \ldots, L$. Let us denote the conditional expectation operator given these variables as $E \left[ \cdot | F_t \right]$, and consider the risk-neutral expectation of a generic function $\zeta$ of just vector $X_{i,t} := [V_t, r_{i,t}, \sigma_{i,t}]'$. For example, $\zeta$ can be the payoff at exercise of a derivative contract written on an individual equity. From the law of iterated expectations we get

$$E \left[ M_{t+1}(\theta) \zeta(X_{i,t+1}) | F_t \right]$$

$$= E \left[ \exp \left( -r_{f,t+1} - \eta - \sum_{j=1}^{4} \theta_j Z_{j,t+1} - \theta_5 V_{t+1} \right) \zeta(X_{i,t+1}) | F_t \right]$$

$$= E \left[ \exp (-r_{f,t+1} - \eta - \theta_5 V_{t+1}) \zeta(X_{i,t+1}) E_{\psi} \left[ \exp \left( - \sum_{j=1}^{4} \theta_j Z_{j,t+1} \right) \left| X_{i,t+1}, F_t \right. \right] \left| F_t \right. \right].$$

Under Assumptions 2, 4 and 5 the conditioning set in the inner expectation on the last line originates the same information as $V_{t+1}$. The previous expectation can then be written as

$$E \left[ \exp (-r_{f,t+1} - \eta - \theta_5 V_{t+1}) \zeta(X_{i,t+1}) E_{\psi} \left[ \exp \left( - \sum_{j=1}^{4} \theta_j Z_{j,t+1} \right) \left| V_{t+1} \right. \right] \left| F_t \right. \right].$$

Under Assumption 6 the conditioning set for this expectation is simplified, and the expectation becomes

$$E \left[ \exp (-r_{f,t+1} - (\eta - \alpha) - (\theta_5 - \beta) V_{t+1}) \zeta(X_{i,t+1}) | F_t \right].$$

In the last expression the function over which the conditional expectation operator $E \left[ \cdot | F_t \right]$ is applied depends just on $X_{i,t+1}$. Therefore, under Assumptions 2, 4 and 5, the expectation is equal to

$$E_{\phi_i} \left[ \exp (-r_{f,t+1} - (\eta - \alpha) - (\theta_5 - \beta) V_{t+1}) \zeta(X_{i,t+1}) | V_t, \sigma_{i,t} \right],$$

45
for the transition density $\phi_i$ for process $(X_{i,t})$. The last expectation can be written in integral form as

$$\int \exp (-r_{f,t} - (\eta - \alpha) - (\theta_5 - \beta) v) \zeta(x) \phi_i(x|V_t, \sigma_{i,t}) dx,$$

for the integration variable $x = [v \ r \ \sigma]'$. Finally, the first Equation in System (7) is obtained by considering the normalization of the SDF and function $e^{-r_{f,t}+1}$ instead of the generic function $\zeta$.

**B Kernel estimation**

This section describes the kernel estimators. Section B.1 deals with their implementation. Section B.2 with their large sample properties.

**B.1 Implementation**

The estimates considered in this paper are obtained through the estimation of several regression functions. The conditional expectation of vector $X_t$, evaluated for the kernel estimator $\hat{\phi}$, is replaced by a Nadaraya-Watson estimator:

$$E_{\hat{\phi}}[X_{t+1}|Y_t = y] \approx \frac{\sum_{t=1}^{T-1} X_{t+1} K \left(H^{-1} (y_t - y)\right)}{\sum_{t=1}^{T-1} K \left(H^{-1} (y_t - y)\right)}.$$

Following the generalized Scott’s rule of thumb, the used bandwidth matrix to compute the Nadaraya-Watson estimators and the multivariate kernel density estimators defined in Equations (8) and (11) are proportional to $T^{-\frac{1}{2}} V_{1/2}$, where $V$ is the sample unconditional variance-covariance matrix of the realized aggregate and individual volatilities and $T$ is the sample size. The estimation of conditional and unconditional densities with this bandwidth matrix is equivalent to a three steps estimation procedure: standardizing the data, applying a linear transformation to make them uncorrelated and finally transforming the density back to the original scale. The bandwidth matrix minimizes the integrated mean squared error of the density estimate in case of normally distributed data (Scott, 1992). For any given $\theta$, the American put option-to-stock price ratio is estimated in an iterative way, using Equations (3) and (4) evaluated at the kernel estimator $\hat{\phi}$ of the conditional density. The estimation of the ratio at time-to-maturity $h > 0$ requires the estimation of the discounted conditional expectation of the ratio at time-to-maturity $h - 1$. In particular, this estimation needs the value of the ratio at time-to-maturity $h - 1$, for any value of volatility factor, individual equity return RV.

---

18 The first two steps are known as Mahalanobis transformation.
and moneyness strike. Therefore, on any business day the value of the American put option mid-quote-to-stock price ratio is computed recursively on a grid on the realized aggregate and individual volatilities and moneyness strike domain. The American call option mid-quote-to-stock price ratio is computed in a similar way. For the computation of any American option-to-stock price ratio, when the ratio on a point outside the grid is necessary, the nearest grid point is selected. The lowest and highest returns on the grid are 1.5 times the most negative and positive return on the return time series. The extremes for the grid for realized aggregate and individual volatilities are the first and ninety-nine percentiles of the RVs. The extremes of the moneyness strike grid are 0.5 and 1.5. Any grid for moneyness strike and logarithmic volatility are divided in 15 and 30 equally spaced points, respectively. The option mid-quote-to-share price ratio when the considered moneyness strike is higher than 1.5, for a put option, or lower than 0.5, for a call option, is obtained by a linear extrapolation procedure. When the considered moneyness strike is lower than 0.5, for a put option, or greater than 1.5, for a call option, the option-to-stock price ratio is set to 0.

B.2 Large sample properties

This appendix provides a derivation of the large sample properties of the kernel estimators of the historical conditional correlation function between equity return and volatility and the conditional skewness and kurtosis of the equity return. The properties are first obtained for the estimator of a generic functional of the transition density for process \( (X_t) \) and then adapted to the estimators of interest. Let us indicate by \( Q_{\phi_0} \) the true value of this generic functional and let us consider a real scalar function \( \zeta \) and a real stochastic vector \( \xi_t \) such that this generic functional can be written in the form

\[
Q_{\phi_0}(y) = \zeta \left( E_{\phi_0} [\xi_{t+1} | Y_t = y] \right).
\]

Each statistical moments can be written in the form of function \( Q_{\phi_0} \) for an appropriate choice of \( \zeta \) and \( Z_t \). The kernel estimator \( \hat{Q}_\phi \) of this functional is defined by considering the kernel regression estimator \( E_{\phi_\hat{\phi}} [\cdot | Y_t = y] \) in place of the true conditional expectations. Let us consider the risk factors and individual equity return and its volatility as rescaled, such that a common bandwidth \( h_T \) can be used for the nonparametric estimation. From the theory of kernel estimators, for any value \( y \) of volatility factor and individual equity return volatility, the kernel regression estimator is pointwise asymptotically normal with \( \sqrt{T h_T^2} \)-rate of convergence:

\[
\sqrt{T h_T^2} \left( E_{\phi_\hat{\phi}} [\xi_{t+1} | Y_t = y] - E_{\phi_0} [\xi_{t+1} | Y_t = y] \right) \overset{D}{\to} \mathcal{N}(0, V(y)),
\]
where \( V(y) \) is defined as

\[
V(y) = V_{\phi_0} [\xi_{t+1} | Y_t = y] \left( \int K^2(x)dx / g(y) \right),
\]

for the conditional variance operator \( V_{\phi_0} [\cdot | Y_t = y] \) given \( Y_t = y \), and the true historical marginal joint density \( g \) of volatility factor and individual equity return volatility (See, e.g., Bosq, 1998). The asymptotic distribution of estimator \( \hat{Q}_{\phi} (y) \) can be derived by the delta method:

\[
\sqrt{T} \hat{Q}_{\phi} (y) - Q_{\phi_0} (y) \xrightarrow{D} N \left( 0, \gamma(y)' V(y) \gamma(y) \right),
\]

where the vector \( \gamma(y) \) is defined as

\[
\gamma(y) = \frac{\partial \zeta}{\partial b} (E_{\phi_0} [\xi_{t+1} | Y_t = y]).
\]

for the real vector \( b \) with the same dimension of \( \xi_t \). Let us adapt the expressions to the estimator of the three considered statistical moments.

i) Conditional correlation between an individual equity return and its volatility

Let us take the vector \( \xi_t = [r_t \ \sigma_t \ \sigma_t^2 \ \sigma_t^3 \ \sigma_t^4]' \) and the scalar function \( \zeta \) that depends on the real 5-dimensional vector \( b = [b_1 \ b_2 \ b_3 \ b_4 \ b_5]' \):

\[
\zeta(b) = (b_5 - b_1 b_2) (b_3 - b_1^2)^{-0.5} (b_4 - b_2^2)^{-0.5}.
\]

The derivative of \( \zeta \) w.r.t. \( b \) at \( a = [a_1 \ a_2 \ a_3 \ a_4 \ a_5]' \) is

\[
\frac{\partial \zeta}{\partial b} (a) = \zeta(a) \begin{bmatrix}
    a_1 (a_3 - a_1^2)^{-1} - a_2 (a_5 - a_1 a_2)^{-1} \\
    a_2 (a_4 - a_2^2)^{-1} - a_1 (a_5 - a_1 a_2)^{-1} \\
    -0.5 (a_3 - a_1^2)^{-1} \\
    -0.5 (a_4 - a_2^2)^{-1} \\
    (a_5 - a_1 a_2)^{-1}
\end{bmatrix}.
\]

ii) Conditional skewness of an individual equity return

Let us take the vector \( \xi_t = [r_t \ r_t^2 \ r_t^3]' \) and the real scalar function \( \zeta \) that depends on the real 3-dimensional
real vector $b = [b_1 \ b_2 \ b_3]'$:

$$\zeta(b) = (b_3 - 3b_1b_2 + 2b_1^3) (b_2 - b_1^2)^{-1.5}.$$ 

The derivative of $\zeta$ w.r.t. $b$ at $a = [a_1 \ a_2 \ a_3]'$ is

$$\frac{\partial \zeta}{\partial b}(a) = \begin{bmatrix}
    (-3a_2 + 6a_1^2 + 3a_1 (a_2 - a_1^2)^{0.5} g(a)) (a_2 - a_1^2)^{-1.5} \\
    (-3a_1 - 1.5 (a_2 - a_1^2)^{0.5} g(a)) (a_2 - a_1^2)^{-1.5} \\
    (a_2 - a_1^2)^{-1.5}
\end{bmatrix}.$$

iii) Conditional kurtosis of an individual equity return

Let us take the vector $\xi_t = [r_t \ r_t^2 \ r_t^3 \ r_t^4]'$ and the real scalar function $\zeta$ that depends on the real 4-dimensional real vector $b = [b_1 \ b_2 \ b_3 \ b_4]'$:

$$\zeta(b) = (b_4 - 4b_1b_3 + 6b_1^2b_2 - 3b_1^4) (b_2 - b_1^2)^{-2}$$

The derivative of $\zeta$ w.r.t. $b$ at $a = [a_1 \ a_2 \ a_3 \ a_4]'$ is

$$\frac{\partial \zeta}{\partial b}(a) = \begin{bmatrix}
    (-4a_3 + 12a_1a_2 - 12a_1^3 + 4a_1 (a_2 - a_1^2) g(a)) (a_2 - a_1^2)^{-2} \\
    (6a_1^2 - 2 (a_2 - a_1^2) g(a)) (a_2 - a_1^2)^{-2} \\
    -4a_1 (a_2 - a_1^2)^{-2} \\
    (a_2 - a_1^2)^{-2}
\end{bmatrix}.$$