Optimal Housing, Consumption, and Investment Decisions over the Life-Cycle

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Outline

1. Introduction
2. Model
3. Full flexibility
4. Comparative statics
5. Limited flexibility
Overview and motivation

- **Important features** in life-cycle decisions of individuals:
  - interest rate risk
    - (Sørensen JFQA99; Campbell/Viceira AER01; Munk/Sørensen JBF04)
  - risky labor income
    - (Bodie/Merton/Samuelson JEDC92; Cocco/Gomes/Maenhout RFS05; Munk/Sørensen JFE-forthcoming)
  - housing decisions
    - (Cocco RFS05; Yao/Zhang RFS05; Van Hemert wp08)

- **Existing papers** with income and housing:
  - coarse and computationally intensive numerical solution techniques with unknown precision
  - provide limited understanding of economic forces at play

- **This paper**: explicit, “Excel-ready” solutions
Financial assets

- Available assets: cash, bond(s), stock
- Short-term interest rate (= return on cash):
  \[ dr_t = \kappa (\bar{r} - r_t) \, dt - \sigma_r \, dW_{rt} \]
- Price \( B_t = B(r_t, t) \) of an arbitrary bond:
  \[ dB_t = B_t \left[ (r_t + \lambda_B \sigma_B(r_t, t)) \, dt + \sigma_B(r_t, t) \, dW_{rt} \right] \]
  Zero-coupon bond: \( B_t^u = \exp\{-a(u - t) - B_{\kappa}(u - t)r_t\} \), where \( B_{\kappa}(\tau) = \frac{1}{\kappa}(1 - e^{-\kappa \tau}) \).
- Stock price:
  \[ dS_t = S_t \left[ (r_t + \lambda_S \sigma_S) \, dt + \sigma_S \left( \rho_{SB} \, dW_{rt} + \sqrt{1 - \rho_{SB}^2} \, dW_{St} \right) \right] \]
Housing

“Unit” house price $H_t$:

$$\frac{dH_t}{H_t} = \left( r_t + \lambda_H \sigma_H - r^{imp} \right) dt + \sigma_H \left( \rho_{HB} dW_{rt} + \hat{\rho}_{HS} dW_{St} + \hat{\rho}_H dW_{Ht} \right)$$

Housing positions:

- owning $\varphi_{ot}$ housing units
- renting $\varphi_{rt}$ units at continuous rental rate per unit is $\nu H_t$
- investing in REITs $\varphi_{Rt}$ units, with total return $\frac{dH_t}{H_t} + \nu dt$

Housing consumption: $\varphi_{Ct} = \varphi_{ot} + \varphi_{rt}$

Housing investment: $\varphi_{It} = \varphi_{ot} + \varphi_{Rt}$
Labor income

Income rate $Y_t$ until retirement at $\tilde{T}$:

$$\frac{dY_t}{Y_t} = (\bar{\mu}_Y(t) + br_t) \, dt + \sigma_Y(t) \left( \rho_{YB} \, dW_{rt} + \rho_{YS} \, dW_{St} + \rho_Y \, dW_{Ht} \right)$$

In retirement: $Y_t = \gamma Y_{\tilde{T}}$, $t \in (\tilde{T}, T]$. 

Human wealth/capital:
The human capital is

$$L_t \equiv \mathbb{E}^Q_t \left[ \int_t^T e^{-\int_s^t r_u \, du} Y_s \, ds \right] = \begin{cases} Y_t F(t, r_t), & t < \tilde{T}, \\ Y_{\tilde{T}} F(t, r_t), & t \geq \tilde{T} \end{cases}$$

where

$$F(t, r) = \begin{cases} \int_t^{\tilde{T}} e^{-A(t, s) - (1-b)B_\kappa(s-t)r} \, ds + \gamma \int_{\tilde{T}}^{T} e^{-\tilde{A}(t, s) - (B_\kappa(s-t) - bB_\kappa(\tilde{T}-t))r} \, ds, & t < \tilde{T}, \\ \gamma \int_{\tilde{T}}^{T} e^{-a(s-t) - B_\kappa(s-t)r} \, ds, & t \geq \tilde{T} \end{cases}$$

Here $A(t, s)$ and $\tilde{A}(t, S)$ are deterministic functions stated in Appendix A.
Wealth

Financial/tangible wealth: \( X_t \)

Total wealth: \( X_t + L_t \)

\[
dX_t = \pi_{St} X_t \frac{dS_t}{S_t} + \pi_{Bt} X_t \frac{dB_t}{B_t} + \left[ X_t (1 - \pi_{St} - \pi_{Bt}) - (\varphi_{ot} + \varphi_{Rt}) H_t \right] r_t \, dt \\
+ \varphi_{ot} \, dH_t + \varphi_{Rt} \left( dH_t + \nu H_t \, dt \right) - \varphi_{rt} \nu H_t \, dt - c_t \, dt + Y_t \, dt \\
= \left[ X_t \left( r_t + \pi_{St} \lambda S \sigma_S + \pi_{Bt} \lambda B \sigma_B \right) + \varphi_{lt} \lambda'_H \sigma_H H_t - \varphi_{Ct} \nu H_t - c_t + Y_t \right] \, dt \\
+ \left( \pi_{St} X_t \rho_{SB} \sigma_S + \pi_{Bt} X_t \sigma_B + \varphi_{lt} H_t \rho_{HB} \sigma_H \right) \, dW_{rt} \\
+ \left( \pi_{St} X_t \sigma_S \sqrt{1 - \rho_{SB}^2} + \varphi_{lt} H_t \hat{\rho}_{HS} \sigma_H \right) \, dW_{St} + \varphi_{lt} H_t \hat{\rho}_H \sigma_H \, dW_{Ht},
\]

where

\( \varphi_{Ct} \equiv \varphi_{ot} + \varphi_{rt}, \quad \varphi_{lt} \equiv \varphi_{ot} + \varphi_{Rt}. \)
The individual’s utility maximization problem

\[ J(t, x, r, h, y) = \sup_{(c, \varphi_C, \varphi_I, \pi_B, \pi_S) \in A_t} E_t \left[ \int_t^T e^{-\delta(u-t)} \frac{1}{1-\gamma} \left( c_u \varphi_c^{1-\beta} \right)^{1-\gamma} ds \right. \]

\[ \left. + \varepsilon e^{-\delta(T-t)} \frac{1}{1-\gamma} X_t^{1-\gamma} \right] \]
Parameter values used in illustrations

- **Preferences and wealth:**  $X_0 = \text{USD } 20,000; \delta = 0.03; \gamma = 4; \beta = 0.8; \varepsilon = 0; \bar{T} = 30; T = 50$

- **Interest rate and bond:**  $\kappa = 0.2; r_0 = \bar{r} = 0.02; \sigma_r = 0.015; \lambda_B = 0.1; T_{\text{bond}} = 20 \text{ (running)}; \sigma_B = 0.0736$

- **Stock:**  $\sigma_S = 0.2, \lambda_S = 0.25, \rho_{SB} = 0$

- **Housing:**  $\sigma_H = 0.12; \lambda_H = 0.325; r^{\text{imp}} = \nu = 0.05 \text{ (so } \mu_H = 0.9\%); \rho_{HB} = 0.65; \rho_{HS} = 0.5; H_0 = \text{USD } 250 \text{ per “average standard” sq. foot} \text{ REITS drift } 0.9\%+5\%=5.9\%, \text{ low vol } \sim \text{ attractive investment}$

- **Labor income:**  $Y_0 = \text{USD } 20,000; b = 0.5; \bar{\mu}_Y(t) = 0.01; \sigma_Y(t) = 0.075; \rho_{YB} = -0.3; \rho_{YS} = 0; \rho_{HY} = 0.3509 \text{ (ensures spanning)}$
Solution to the HJB-equation...

\[ J(t, x, r, h, y) = \frac{1}{1-\gamma} g(t, r, h) (x + yF(t, r))^{1-\gamma}, \]

\[ g(t, r, h) = \varepsilon \frac{1}{\gamma} e^{-D\gamma(T-t) - \frac{\gamma-1}{\gamma} B\kappa(T-t) r} + \frac{\eta \nu}{1-\beta} h^k \int_t^T e^{-d_1(u-t) - \beta \frac{\gamma-1}{\gamma} B\kappa(u-t) r} du. \]

Here \( k = (1 - \beta)(1 - 1/\gamma) \), \( \eta = \beta^{1/\gamma} \left( \frac{\beta \nu}{1-\beta} \right)^{k-1} \), and \( D\gamma(\tau) \) and \( d_1(\tau) \) are stated in the appendix.

\[
\hat{\pi}_S \equiv \frac{\pi_S x}{x + yF} = \frac{1}{\gamma} \frac{\xi_S}{\sigma_S} - \frac{\sigma_Y(t) \zeta_S}{\sigma_S} \frac{yF}{x + yF},
\]

\[
\hat{\pi}_B \equiv \frac{\pi_B x}{x + yF} = \frac{1}{\gamma} \frac{\xi_B}{\sigma_B} - \left( \frac{\sigma_Y(t) \zeta_B}{\sigma_B} \frac{yF}{x + yF} - \frac{\sigma_r}{\sigma_B} \frac{yF}{x + yF} \frac{F_r}{F} \right) - \frac{\sigma_r}{\sigma_B} \frac{g_r}{g},
\]

\[
\hat{\pi}_I \equiv \frac{h\varphi_I}{x + yF} = \frac{1}{\gamma} \frac{\xi_I}{\sigma_H} - \frac{\sigma_Y(t) \zeta_I}{\sigma_H} \frac{yF}{x + yF} + \frac{hg_h}{g}
\]

\[
c = \eta \frac{\beta \nu}{1-\beta} h^k \frac{x + yF}{g}, \quad \varphi_c = \eta h^{k-1} \frac{x + yF}{g}.
\]

(when \( t \in [\tilde{T}, T] \): \( \sigma_Y(t) = 0 \) and \( y \) is to be replaced by \( Y_{\tilde{T}} \))
Expected consumption over the life-cycle

\[ C_t = \eta \frac{\beta \nu H_t^k}{1 - \beta} \frac{X_t + Y_t F(t, r_t)}{g(t, r_t, H_t)} \]

\[ \nu H_t \varphi C_t = \frac{1 - \beta}{\beta} C_t \]
Optimal investments – fractions of total wealth

\[ \hat{\pi}_S = \frac{1}{\gamma} \frac{\xi_S}{\sigma_S} - \frac{\sigma_Y \xi_S}{\sigma_S} \frac{yF}{x + yF}, \]
5.5% \quad 0 \leftrightarrow 31%

\[ \hat{\pi}_B = \frac{1}{\gamma} \frac{\xi_B}{\sigma_B} - \left( \frac{\sigma_Y \xi_B}{\sigma_B} - \frac{\sigma_r}{\sigma_B} \frac{F_r}{F} \right) \frac{yF}{x + yF} - \frac{\sigma_r}{\sigma_B} \frac{g_r}{g}, \]
-57% \quad 0 \leftrightarrow 141% \quad 0 \leftrightarrow -43% \quad 49%

\[ \hat{\pi}_I = \frac{1}{\gamma} \frac{\xi_I}{\sigma_H} - \frac{\sigma_Y \xi_I}{\sigma_H} \frac{yF}{x + yF} + \frac{h g_h}{g}, \]
86% \quad 0 \leftrightarrow -104% \quad 15%

speculative \quad adjust \ for \ human \ wealth \quad hedge
Optimal investments and the composition of wealth

Assuming fixed $r$, $h$ and fixed $\frac{F_r}{F}$, $\frac{g_r}{g}$ (vary little over life anyway)
Main life-cycle effect is change in ratio of human-to-total wealth
Expected wealth over the life-cycle
Expected investments over the life-cycle
Housing consumption and over the life-cycle
Robustness of results

- effects of $\gamma$, $\varepsilon$, $H_0$, $\sigma_H$, $Y_0$, $\sigma_Y$, $\gamma$, $\rho_{HY}$: see paper and appendix
Expected income over the life-cycle for three educational groups
Expected wealth over the life-cycle for three educational groups

[Graph showing expected wealth over the life-cycle for three educational groups, differentiated by educational attainment and financial status.]
Expected investments over the life-cycle for three educational groups
Consumption of and investment in housing over the life-cycle for three educational groups
Limited flexibility in housing decisions

Scenarios considered:

1. constant number of units of housing consumed (closed-form solution)
2. infrequent adjustments of housing investment and housing consumption (MC results)

Percentage wealth-equivalent utility loss $L$ due to limited flexibility:

$$J(t, x[1 - L], r, h, y[1 - L]) = J_{\text{limited}}(t, x, r, h, y)$$
Utility loss of constant housing consumption

Note: minimum loss is 0.26% for a constant housing consumption (USD 1,550 out of total wealth USD 596,400).

The individual can almost completely compensate for inflexibility in housing consumption by adjusting perishable consumption and investments.
## Loss due to infrequent housing adjustments

Simulate BM’s and thus wealth, income (10,000 paths, 250 steps/year)
When adjusted, use policies optimal with continuous adjustments

<table>
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<th>Adjustment frequency</th>
<th>Initial income 10,000</th>
<th>Initial income 20,000</th>
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<tr>
<td></td>
<td>2 years</td>
<td>5 years</td>
</tr>
<tr>
<td>Infrequent $\varphi_C$, frequent $\varphi_I$</td>
<td>0.03%</td>
<td>0.08%</td>
</tr>
<tr>
<td>Infrequent $\varphi_I$, frequent $\varphi_C$</td>
<td>0.21%</td>
<td>1.35%</td>
</tr>
<tr>
<td>Infrequent $\varphi_C$ and $\varphi_I$</td>
<td>0.24%</td>
<td>1.52%</td>
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- suggests moderate welfare effects of a well-functioning market for REITs or CSI housing contracts
- suggests moderate effects of housing transactions costs
Portfolio and Consumption Choice with Stochastic Investment Opportunities and Habit Formation in Preferences

(Journal of Economic Dynamics and Control, 2008)

Claus Munk

August 2012
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Motivation

Investment opportunities vary stochastically over time
  ▶ Many papers on optimal portfolio and consumption choice with stochastic investment opportunities
  ▶ Almost all assume utility of terminal wealth, \( E[u(W_T)] \), or time-separable utility of consumption, \( U(c) = E \left[ \int_0^T e^{-\delta t} u(c_t) \, dt \right] \).

Many economists argue that individuals develop habits for consumption.

Preferences with habit formation help explain asset pricing puzzles (Sundaresan 1989, Constantinides 1990, Campbell and Cochrane 1999).

Earlier studies of individuals’ optimal portfolio and consumption choice under habit formation assume constant investment opportunities.
Objectives

- Derive optimal portfolio and consumption choice for individuals with habit formation in markets with stochastic investment opportunities.
- Consider both general dynamics of investment opportunities and concrete, special cases.
- Do stochastic investment opportunities affect the decisions of individuals with habit formation differently than individuals with time-additive preferences?
- Are the effects of habit formation in preferences on optimal decisions different in markets with stochastic investment opportunities than in markets with constant investment opportunities?
## Literature review

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<th><strong>time-additive utility</strong></th>
<th><strong>habit formation utility</strong></th>
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<td><strong>general stochastic inv. opp.</strong></td>
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<td>Liu (2007)</td>
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<td>Munk and Sørensen (2004)</td>
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<td><strong>concrete stochastic inv. opp.</strong></td>
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<td>Brennan and Xia (2000)</td>
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The financial market

- Basic uncertainty modeled by $n$-dimensional SBM $z = (z_t)$
- "Bank account": rate of return $r_t$
- $n$ risky assets: $dP_t = \text{diag}(P_t) [(r_t1 + \sigma_t \lambda_t) \, dt + \sigma_t \, dz_t]$
- Assume $\sigma_t$ non-singular
- Complete market – unique state-price deflator:

$$\xi_t = \exp \left\{ - \int_0^t r_s \, ds - \int_0^t \lambda_s^T \, dz_s - \frac{1}{2} \int_0^t \|\lambda_s\|^2 \, ds \right\}$$

(and unique risk-neutral probability measure)
The individual...

... wants to maximize

\[ U(c) = \mathbb{E} \left[ \int_0^T e^{-\delta t} \frac{1}{1-\gamma} (c_t - h_t)^{1-\gamma} \, dt \right], \]

where

\[ h_t = h_0 e^{-\beta t} + \alpha \int_0^t e^{-\beta(t-s)} c_s \, ds. \]

Note: requires \( c_t \geq h_t \) for all \( t \) [not necessarily very restrictive].

... chooses a consumption strategy \( c = (c_t) \) and a portfolio strategy \( \pi = (\pi_t) \) under budget constraint \( \mathbb{E}_t \left[ \int_t^T \xi_s c_s \, ds \right] \leq W_t. \)

... receives no labor income.

Wealth dynamics:

\[ dW_t = \left[ W_t \left( r_t + \pi_t^T \sigma_t \lambda_t \right) - c_t \right] dt + W_t \pi_t^T \sigma_t dz_t. \]

Indirect utility:

\[ J_t = \sup_{(c,\pi) \in \mathcal{A}(t)} \mathbb{E}_t \left[ \int_t^T e^{-\delta(s-t)} \frac{1}{1-\gamma} (c_s - h_s)^{1-\gamma} \, ds \right]. \]
Auxiliary processes

Define the processes \( F = (F_t) \) and \( G = (G_t) \) by

\[
F_t = \mathbb{E}_t \left[ \int_t^T e^{-(\beta-\alpha)(s-t)} \frac{\xi_s}{\xi_t} ds \right] = \int_t^T e^{-(\beta-\alpha)(s-t)} B_s^s \, ds,
\]

\[
G_t = \mathbb{E}_t \left[ \int_t^T e^{-\frac{\delta}{\gamma}(s-t)} \left( \frac{\xi_s}{\xi_t} \right)^{1-\frac{1}{\gamma}} (1 + \alpha F_s)^{1-\frac{1}{\gamma}} ds \right].
\]

Note: \( h_t F_t \) is the cost of ensuring that future consumption equals the habit level (\( c_s = h_s \) for all \( s \geq t \) implies \( h_s = h_t e^{-(\beta-\alpha)(s-t)} \)).

Must assume \( W_0 \geq h_0 F_0 \).

Dynamics:

\[
dF_t = -1 \, dt + F_t \left[ (r_t + \sigma_{Ft}^T \lambda_t) \, dt + \sigma_{Ft}^T \, dz_t \right], \quad \text{where} \quad \sigma_{Ft} \equiv \frac{\int_t^T e^{-(\beta-\alpha)(s-t)} B_t^s \sigma_t^s \, ds}{\int_t^T e^{-(\beta-\alpha)(s-t)} B_t^s \, ds}.
\]

\[
dG_t = G_t \left[ \mu_G \, dt + \sigma_{Gt}^T \, dz_t \right] \quad \text{for some} \quad \mu_G, \sigma_G.
\]

Note: \( \sigma_F \) and \( \sigma_G \) depend on the precise asset price dynamics.
Theorem 1

Under two regularity conditions, the solution is as follows.

Optimal consumption: $$c^*_t = h^*_t + (1 + \alpha F_t)^{-\frac{1}{\gamma}} \frac{W^*_t - h^*_t F_t}{G_t}.$$ 
Indirect utility: $$J_t = \frac{1}{1 - \gamma} G_t^\gamma (W^*_t - h^*_t F_t)^{1 - \gamma}.$$ 
Optimal portfolio:

$$\pi^*_t = \frac{W^*_t - h^*_t F_t}{W^*_t} \left( \frac{1}{\gamma} (\sigma_t^\top)^{-1} \lambda_t + \frac{W^*_t - h^*_t F_t}{W^*_t} (\sigma_t^\top)^{-1} \sigma G_t + \frac{h^*_t F_t}{W^*_t} (\sigma_t^\top)^{-1} \sigma F_t \right).$$

Note: Both the speculative term and the hedge term are dampened due to habit formation. Additional effect on hedge term via $G$ and hence $\sigma_G$.

Jointly allowing for habit and stochastic IO’s has effects beyond their separate effects! Quantitatively important?
Markovian asset prices

Assume \( r_t = r(x_t) \) and \( \lambda_t = \lambda(x_t) \), where

\[
dx_t = m(x_t) \, dt + v(x_t) \, dz_t.
\]

Then \( B^s_t = B^s(x_t, t) \), where \( B^s(x, t) \) solves

\[
\frac{1}{2} \text{tr} \left( \frac{\partial^2 B^s}{\partial x^2} v(x) v(x)^\top \right) + (m(x) - v(x) \lambda(x))^\top \frac{\partial B^s}{\partial x} + \frac{\partial B^s}{\partial t} - r(x) B^s = 0
\]

with \( B^s(x, s) = 1 \), and \( \sigma_{Ft} \) becomes

\[
\sigma_{Ft} = \frac{\int_t^T e^{-(\beta - \alpha)(s-t)} \frac{\partial B^s}{\partial x}(x_t, t) \, ds}{\int_t^T e^{-(\beta - \alpha)(s-t)} B^s(x_t, t) \, ds}.
\]

Moreover, \( G_t = G(x_t, t) \) and \( \sigma_{Gt} = v(x_t)^\top \frac{\partial G}{\partial x}(x_t, t) / G(x_t, t) \). \( G(x, t) \) will satisfy

\[
\frac{\partial G}{\partial t}(x, t) + \left( m(x) - \left( 1 - \frac{1}{\gamma} \right) v(x) \lambda(x) \right)^\top \frac{\partial G}{\partial x}(x, t) + \frac{1}{2} \text{tr} \left( \frac{\partial^2 G}{\partial x^2}(x, t) v(x) v(x)^\top \right) + (1 + \alpha F(x, t))^{1 - \frac{1}{\gamma}} = \left( \frac{\delta}{\gamma} + \left( 1 - \frac{1}{\gamma} \right) r(x) + \frac{\|\lambda(x)\|_2^2}{2\gamma} \left( 1 - \frac{1}{\gamma} \right) \right) G(x, t)
\]

with \( G(x, T) = 0 \).
Constant investment opportunities: solution

If both $r$ and $\lambda$ are constant, then

$$F(t) = \int_t^T e^{-(r+\beta-\alpha)(s-t)} \, ds = \frac{1}{r + \beta - \alpha} \left( 1 - e^{-(r+\beta-\alpha)(T-t)} \right)$$

$$G(t) = \int_t^T e^{-\left( \frac{\delta}{\gamma} + \left[ 1 - \frac{1}{\gamma} \right] r + \frac{1}{2\gamma} \left[ 1 - \frac{1}{\gamma} \right] \| \lambda \|^2 \right) (s-t)} \left( 1 + \alpha F(s) \right)^{1-1/\gamma} \, ds$$

and $\sigma_F = \sigma_G = 0$.

Optimal consumption: $c_t^* = h_t^* + (1 + \alpha F(t))^{-1/\gamma} \frac{W_t^* - h_t^* F(t)}{G(t)}$.

Optimal portfolio: $\pi_t^* = \frac{1}{\gamma} (\sigma_t^{-1}) \lambda \frac{W_t^* - h_t^* F(t)}{W_t^*}$.
Constant investment opportunities: example

Parameters: $r = 0.03, \sigma = 0.2, \lambda = 0.3; T = 30, \gamma = 2, \delta = 0.02, W = 100, h = 4$.

The thick [thin] curves are for the case with [without] habit formation.

- habit formation dampens risky investment substantially
- optimal stock weight decreases with length of investment horizon (for given $W, h$...): long horizon $\Rightarrow$ put more money aside to cover habit
The financial market

- constant interest rate $r$
- stock index: $dP_t = P_t [(r + \sigma \lambda_t) \, dt + \sigma \, dz_t]$, constant $\sigma$
- market price of risk: $d\lambda_t = \kappa \left[ \bar{\lambda} - \lambda_t \right] \, dt - \sigma \lambda \, dz_t$
- Note: perfect negative correlation – empirically not that unreasonable
- Wachter (2002) derives an explicit solution for time-additive power utility of consumption
- Kim & Omberg (1996) derives an explicit solution for power utility of terminal wealth, allowing for non-perfect correlation (incomplete market)
Theorem 2

Suppose that \( \kappa^2 > \left(1 - \frac{1}{\gamma}\right) \left(\sigma_\lambda^2 \left(1 - \frac{1}{\gamma}\right) + 2\kappa\sigma_\lambda\right) \).

Indirect utility: \( J(W, h, \lambda, t) = \frac{1}{1-\gamma} G(\lambda, t)^\gamma (W - hF(t))^{1-\gamma} \), where

\[
F(t) = \frac{1}{r + \beta - \alpha} \left(1 - e^{-(r+\beta-\alpha)(T-t)}\right)
\]

\[
G(\lambda, t) = \int_t^T (1 + \alpha F(s))^{1-\frac{1}{\gamma}} e^{g_0(s-t)+g_1(s-t)\lambda+\frac{1}{2}g_2(s-t)\lambda^2} \ ds
\]

and \( g_0, g_1, g_2 \) are known in closed-form.

Optimal strategies:

\[
C(W, h, \lambda, t) = h + (1 + \alpha F(t))^{-\frac{1}{\gamma}} \frac{W - hF(t)}{G(\lambda, t)},
\]

\[
\Pi(W, h, \lambda, t) = \frac{\lambda}{\gamma\sigma} \frac{W - hF(t)}{W} - \frac{\sigma_\lambda}{\sigma} D(\lambda, t) \frac{W - hF(t)}{W},
\]

\[
D(\lambda, t) = \int_t^T (g_1(s-t) + g_2(s-t)\lambda) (1 + \alpha F(s))^{1-\frac{1}{\gamma}} e^{g_0(s-t)+g_1(s-t)\lambda+\frac{1}{2}g_2(s-t)\lambda^2} \ ds
\]

\[
\int_t^T (1 + \alpha F(s))^{1-\frac{1}{\gamma}} e^{g_0(s-t)+g_1(s-t)\lambda+\frac{1}{2}g_2(s-t)\lambda^2} \ ds.
\]
Properties of the solution

**Theorem 3:** Suppose $\gamma > 1$ and $\lambda_t > 0$. Then

1. the hedge demand for the stock is positive,
2. both the average and the marginal consumption/wealth ratio increase with $\lambda$,
3. the optimal fraction of free wealth invested in the stock, $\hat{\pi} \equiv \frac{\pi W}{W - hF}$, increases with the horizon $T$.

- Results as in Wachter (with $\hat{\pi} = \pi$ in (iii)...).
- Since $F$ is increasing in $T$, $\pi = \hat{\pi}(1 - hF/W)$ may be decreasing in $T$ – contrary to common advice.
- Typically, middle-aged investors will start to dissave (reduce wealth) and increase consumption growth and habit $\sim hF/W$ will tend to increase and $\pi$ decrease over time. Younger investors tend to build up wealth $\sim \pi$ may increase over time in the beginning of life.
Numerical experiment

Parameters: \( t = 0, \ T = 30, \ \delta = 0.02, \ \gamma = 2, \ W = 100, \ r = 0.03, \ \sigma = 0.2, \ \kappa = 0.3, \ \sigma_\lambda = 0.1, \ \bar{\lambda} = 0.3. \)

State variables: \( \lambda = \bar{\lambda} \) (expected excess return 6%), habit level \( h = 4. \)

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( F(t) )</th>
<th>( D(\lambda, t) )</th>
<th>( G(\lambda, t) )</th>
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<th>( \pi_{\text{hed}} )</th>
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<td>27.4</td>
<td>52.4%</td>
<td>8.8%</td>
<td>16.9%</td>
</tr>
</tbody>
</table>

NOTE: Hedge demand for stocks is dampened slightly more than myopic demand.
NOTE: \( \pi \) is decreasing in the horizon (at least up to \( T \approx 35 \) yrs)
\[ \Rightarrow \text{Horizon-effect due to habit beats horizon-effect due to mean-reversion} \]
The financial market

- CIR model: $dr_t = \kappa (\bar{r} - r_t) \, dt - \sigma_r \sqrt{r_t} \, dz_{1t}$ with $\lambda_{1t} = \lambda_1 \sqrt{r_t}/\sigma_r$.
- Zero-coupon bonds: $B_t^s = B^s(r_t, t) \equiv e^{-a(s-t)-b(s-t)r_t}$
  with $\hat{\kappa} = \kappa - \lambda_1$, $\nu = \sqrt{\hat{\kappa}^2 + 2\sigma_r^2}$, and
  \[
  b(\tau) = \frac{2(e^{\nu\tau} - 1)}{(\nu + \hat{\kappa})(e^{\nu\tau} - 1) + 2\nu},
  \]
  \[
  a(\tau) = -\frac{2\kappa\bar{r}}{\sigma_r^2} \left( \frac{1}{2}(\hat{\kappa} + \nu)\tau + \ln \left( \frac{2\nu}{(\nu + \hat{\kappa})(e^{\nu\tau} - 1) + 2\nu} \right) \right),
  \]
- WLOG assume trade in bank and $T$-zero (weight $\pi^B$) with dynamics:
  \[
  dB_t^T = B_t^T \left[ (r_t + b(T - t)\lambda_1 r_t) \, dt + b(T - t)\sigma_r \sqrt{r_t} \, dz_{1t} \right].
  \]
- Single stock (weight $\pi^S$):
  \[
  dS_t = S_t \left[ (r_t + \sigma_S \psi(r_t)) \, dt + \sigma_S \left\{ \rho \, dz_{1t} + \sqrt{1 - \rho^2} \, dz_{2t} \right\} \right]
  \]
  with $\psi(r) = \rho \frac{\lambda_1}{\sigma_r} \sqrt{r} + \sqrt{1 - \rho^2} \lambda_2$. 

CIR model: $dr_t = \kappa (\bar{r} - r_t) \, dt - \sigma_r \sqrt{r_t} \, dz_{1t}$ with $\lambda_{1t} = \lambda_1 \sqrt{r_t}/\sigma_r$. 

Zero-coupon bonds: $B_t^s = B^s(r_t, t) \equiv e^{-a(s-t)-b(s-t)r_t}$ with $\hat{\kappa} = \kappa - \lambda_1$, $\nu = \sqrt{\hat{\kappa}^2 + 2\sigma_r^2}$, and 

\[
 b(\tau) = \frac{2(e^{\nu\tau} - 1)}{(\nu + \hat{\kappa})(e^{\nu\tau} - 1) + 2\nu},
\]

\[
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\]

WLOG assume trade in bank and $T$-zero (weight $\pi^B$) with dynamics:

\[
 dB_t^T = B_t^T \left[ (r_t + b(T - t)\lambda_1 r_t) \, dt + b(T - t)\sigma_r \sqrt{r_t} \, dz_{1t} \right].
\]

Single stock (weight $\pi^S$):

\[
 dS_t = S_t \left[ (r_t + \sigma_S \psi(r_t)) \, dt + \sigma_S \left\{ \rho \, dz_{1t} + \sqrt{1 - \rho^2} \, dz_{2t} \right\} \right]
\]

with $\psi(r) = \rho \frac{\lambda_1}{\sigma_r} \sqrt{r} + \sqrt{1 - \rho^2} \lambda_2$. 
**Theorem 4**

Indirect utility: \( J(W, h, r, t) = \frac{1}{1-\gamma} G(r, t) \gamma (W - hF(r, t))^{1-\gamma} \) with 
\[
F(r, t) = \int_t^T e^{-(\beta-\alpha)(s-t)-a(s-t)-b(s-t)r} \, ds
\]
and \( G(r, t) \) solves the PDE 
\[
\frac{\partial G}{\partial t} + \left( \kappa r - \left[ \kappa - \left( 1 - \frac{1}{\gamma} \right) \lambda_1 \right] r \right) \frac{\partial G}{\partial r} + \frac{1}{2} \sigma^2 r \frac{\partial^2 G}{\partial r^2} 
+ (1 + \alpha F(r, t))^{1-\frac{1}{\gamma}} = \left( \frac{\delta}{\gamma} + \frac{1}{2 \gamma} \left( 1 - \frac{1}{\gamma} \right) \left( \frac{\lambda_1^2}{\sigma_r^2} r + \lambda_2^2 \right) + \left( 1 - \frac{1}{\gamma} \right) r \right) G
\]
with the terminal condition \( G(r, T) = 0 \). Optimal strategies:
\[
C(W, h, r, t) = h + (1 + \alpha F(r, t))^{-\frac{1}{\gamma}} \frac{W - hF(r, t)}{G(r, t)} 
\]
\[
\Pi^B(W, h, r, t) = \frac{W - hF(r, t)}{W} \frac{1}{b(T-t)} \left[ \frac{1}{\gamma \sigma_r} \left( \frac{\lambda_1}{\sigma_r} - \frac{\rho \lambda_2}{\sqrt{1 - \rho^2}} \right) - \frac{\partial G}{\partial r}(r, t) \right] 
+ \frac{h}{W b(T-t)} \int_t^T b(s-t) e^{-(\beta-\alpha)(s-t)-a(s-t)-b(s-t)r} \, ds 
\]
\[
\Pi^S(W, h, r, t) = \frac{W - hF(r, t)}{W} \frac{\lambda_2}{\gamma \sigma_s \sqrt{1 - \rho^2}}.
\]
Properties

- Hedge bond demand positive for $\gamma > 1$.
- Both myopic and hedge bond demand dampened due to habit formation.
- Additional habit effect on hedge demand through $G$.
- “Habit insurance” bond demand is positive – ensure $c \geq h$ by a dynamic portfolio in the bond and the bank account.
- Bond/stock ratio different with habit formation than without.
Explicit solution with time-additive power utility

\[ G(r, t) = \int_t^T e^{-\bar{a}(s-t) - \left(1 - \frac{1}{\gamma}\right) \bar{b}(s-t)r} \, ds, \]
\[ \bar{b}(\tau) = \frac{2 \left(1 + \frac{\lambda_1^2}{2\gamma \sigma_r^2}\right) \left(e^{\bar{\nu} \tau} - 1\right)}{(\bar{\nu} + \bar{\kappa}) \left(e^{\bar{\nu} \tau} - 1\right) + 2\bar{\nu}}, \]
\[ \bar{a}(\tau) = \frac{\delta}{\gamma} \tau + \frac{1}{2\gamma} \left(1 - \frac{1}{\gamma}\right) \lambda_2^2 \tau - \frac{2\kappa \bar{r}}{\sigma_r^2} \left(\frac{1}{2} (\bar{\nu} + \bar{\kappa}) \tau + \ln \frac{2\bar{\nu}}{(\bar{\nu} + \bar{\kappa}) \left(e^{\bar{\nu} \tau} - 1\right) + 2\bar{\nu}}\right), \]

where \( \bar{\kappa} = \kappa + \frac{1 - \gamma}{\gamma} \lambda_1, \bar{\nu} = \sqrt{\bar{\kappa}^2 + 2\sigma_r^2 \left(1 - \frac{1}{\gamma}\right) \left(1 + \frac{\lambda_1^2}{2\gamma \sigma_r^2}\right)}. \)

Optimal investment strategy:

\[ \Pi^B(r, t) = \frac{1}{\gamma \sigma_r b(T - t)} \left(\frac{\lambda_1}{\sigma_r} - \frac{\rho \lambda_2}{\sqrt{1 - \rho^2} \sqrt{r}}\right) \]
\[ + \left(1 - \frac{1}{\gamma}\right) \frac{\int_t^T \bar{b}(s-t) e^{-\bar{a}(s-t) - \left(1 - \frac{1}{\gamma}\right) \bar{b}(s-t)r} \, ds}{b(T - t) \int_t^T e^{-\bar{a}(s-t) - \left(1 - \frac{1}{\gamma}\right) \bar{b}(s-t)r} \, ds}, \]
\[ \Pi^S = \frac{\lambda_2}{\gamma \sigma_S \sqrt{1 - \rho^2}}, \]

Numerical experiment

(Owns a Crank-Nicholson finite difference scheme)

Parameters: \( t = 0, \ T = 30, \ \delta = 0.02, \ \gamma = 2, \ W = 100, \ \sigma = 0.2, \ \kappa = 0.3, \ \bar{r} = 0.05, \ \sigma_r = 0.1, \ \rho = 0.25, \ \lambda_1 = 0.05, \ \lambda_2 = 0.3. \)

State variables: \( r = \bar{r}, \ h = 4. \)

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<th>( \beta )</th>
<th>( F )</th>
<th>( G )</th>
<th>( \pi^S )</th>
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<td>15.3%</td>
<td>33.8%</td>
<td>16.4%</td>
<td>65.5%</td>
</tr>
</tbody>
</table>

- Hedge demand for bond not dampened quite as much as myopic demand.
- Significant “habit insurance” bond demand.
- Bond/stock ratio is very different with habit formation.
The thick [thin] curves are for the case with [without] habit formation.

**NOTE:** Total bond demand relatively insensitive to \( T \).
Summary

- Exact characterization of optimal strategies in general setting.
- Detailed analysis in two important special settings (the paper also considers a combination of the two).
- No quantitatively surprising and/or dramatic effects of combining habit formation in preferences and stochastic investment opportunities.
- Main effect of habits: some assets (bonds, cash) are better than others (stocks) at ensuring that future consumption will not fall below the habit level.

Possible extensions:
- Other habit-type utility functions.
- Different habits for different consumption goods.
Dynamic Asset Allocation

Epstein-Zin recursive utility

Claus Munk

Aarhus University

August 2012
Motivation

- Standard time-additive power utility imposes a close relation between the aversion against consumption gambles over states and the aversion against substituting consumption over time.
- No reason to believe in that relation?
- With Epstein-Zin preferences, the risk aversion and the elasticity of intertemporal substitution can be disentangled.
- Discrete-time portfolio application: Campbell & Viceira (QJE 1999), Campbell et al. (EFR 2001)
- Continuous-time portfolio application: Chacko & Viceira (RFS 2005)
- Often used for an infinite time horizon
- Plays an important role in the “long-run risk explanation” of the equity premium puzzle, cf. Bansal & Yaron (JF 2004)
Outline

1. The set-up
2. The case EIS equals 1
3. The case EIS different from 1
4. An example
Epstein-Zin utility in continuous time

The utility index $V_{t}^{c,\pi}$ associated at time $t$ with a given consumption process $c$ and portfolio process $\pi$ over the remaining lifetime $[t, T]$ is recursively given by

$$V_{t}^{c,\pi} = E_t \left[ \int_{t}^{T} f(c_u, V_{u}^{c,\pi}) \; du + \bar{V}_{T}^{c,\pi} \right]. \quad (*)$$

The so-called normalized aggregator $f$ is defined by ($\gamma > 1$)

$$f(c, V) = \begin{cases} \frac{\delta}{1-1/\psi} c^{1-1/\psi} ([1 - \gamma] V)^{1-1/\theta} - \delta \theta V, & \text{for } \psi \neq 1 \\ \delta (1 - \gamma) V \ln c - \delta V \ln ([1 - \gamma] V), & \text{for } \psi = 1 \end{cases}$$

where $\theta = (1 - \gamma)/(1 - 1/\psi)$.

- $\delta$: subjective time preference rate,
- $\gamma$: the degree of relative risk aversion towards atemporal bets,
- $\psi > 0$: the elasticity of intertemporal substitution (EIS) towards deterministic consumption plans.

Note: assume terminal utility $\bar{V}_{T}^{c,\pi} = \frac{\alpha}{1-\gamma} (W_{T}^{c,\pi})^{1-\gamma}$, where $\alpha \geq 0$ and $W_{T}^{c,\pi}$ is the terminal wealth induced by the strategies $c, \pi$. 
Time-additive utility as special case

The special case $\psi = 1/\gamma$ (so that $\theta = 1$) corresponds to the classic time-additive power utility since the recursion (*) is then satisfied by

$$V^c,\pi_t = \delta \left( E_t \left[ \int_t^T e^{-\delta (u-t)} \frac{1}{1-\gamma} c_{u-\gamma}^1 du + \frac{1}{\delta} e^{-\delta (T-t)} \frac{\alpha}{1-\gamma} (W_T^c,\pi)^{1-\gamma} \right] \right),$$

which is a positive multiple of the traditional time-additive power utility specification.

Note that $\alpha = \delta$ corresponds to the case where utility of a terminal wealth of $W$ will count roughly as much as the utility of consuming $W$ over the final year.
Dynamic programming with recursive utility

Indirect utility is defined as \( J_t = \sup_{(c, \pi) \in A_t} V_t^{c, \pi} \).

Duffie and Epstein (1992) have demonstrated the validity of the dynamic programming solution technique with recursive utility.

With a state variable \( x_t \) and no income, wealth dynamics is

\[
dW_t = \left( W_t \left[ r(x_t) + \pi^T \sigma(x_t, t) \lambda(x_t) \right] - c_t \right) dt + W_t \pi^T \sigma(x_t, t) d\bar{z}_t.
\]

Assuming diffusion

\[
dx_t = m(x_t) dt + v(x_t)^T d\zeta_t + \hat{v}(x_t) d\hat{z}_t,
\]

the indirect utility is of the form \( J_t = J(W_t, x_t, t) \), and the HJB equation is

\[
0 = \mathcal{L}^\pi J(W, x, t) + \sup_{c \geq 0} \{ f(c, J(W, x, t)) - cJ_W(W, x, t) \} + \frac{\partial J}{\partial t}(W, x, t) + J_W(W, x, t)Wr(x) + J_x(W, x, t)m(x) + \frac{1}{2}J_{xx}(W, x, t)(v(x)^T v(x) + \hat{v}(x)^2),
\]

where

\[
\mathcal{L}^\pi J = \sup_{\pi \in \mathbb{R}^d} \left\{ J_W W^T \sigma(x, t) \lambda(x) + \frac{1}{2} J_{WW} W^2 \pi^T \sigma(x, t) \sigma(x, t)^T \pi + J_{Wx} W \pi^T \sigma(x, t) v(x) \right\}.
\]

Terminal condition \( J(W, x, T) = \frac{\alpha}{1-\gamma} W^{1-\gamma} \).
Optimal portfolio

The maximization with respect to $\pi$ is exactly as for the case with general time-additive expected utility. The maximizer is

$$
\pi^* = -\frac{J_W(W, x, t)}{WJ_{WW}(W, x, t)} (\sigma(x, t)^\top)^{-1} \lambda(x) - \frac{J_{Wx}(W, x, t)}{WJ_{WW}(W, x, t)} (\sigma(x, t)^\top)^{-1} v(x)
$$

which implies that

$$
\mathcal{L}^\pi J(W, x, t) = -\frac{1}{2} \frac{J_W(W, x, t)^2}{J_{WW}(W, x, t)} \| \lambda(x) \|^2 - \frac{1}{2} \frac{J_{Wx}(W, x, t)^2}{J_{WW}(W, x, t)} \| v(x) \|^2
$$

$$
- \frac{J_W(W, x, t)J_{Wx}(W, x, t)}{J_{WW}(W, x, t)} v(x)^\top \lambda(x).
$$

(Of course, $J$ will be different with Epstein-Zin utility than with time-additive utility.)
Optimal portfolio

We will study $\psi = 1$ and $\psi \neq 1$ separately, but in both cases the indirect utility will be of the form

$$J(W, x, t) = \frac{1}{1 - \gamma} G(x, t)^\gamma W^{1-\gamma}.$$ 

Hence, the optimal portfolio is

$$\pi_t^* = -\frac{J_W(W, x, t)}{W J_{WW}(W, x, t)} \left( \sigma(x, t)^\top \right)^{-1} \lambda(x) - \frac{J_{Wx}(W, x, t)}{W J_{WW}(W, x, t)} \left( \sigma(x, t)^\top \right)^{-1} v(x)$$

$$= \frac{1}{\gamma} \left( \sigma(x_t, t)^\top \right)^{-1} \lambda(x_t) + \frac{G_x(x_t, t)}{G(x_t, t)} \left( \sigma(x_t, t)^\top \right)^{-1} v(x_t).$$

- speculative component the same as for time-additive power utility
- hedge component may be different
- we have to determine $G$...
HJB with $\psi = 1$

Recall that for $\psi = 1$,

$$f(c, J) = \delta(1 - \gamma)J \ln c - \delta J \ln ([1 - \gamma]J) .$$

Therefore HJB equation becomes

$$0 = \mathcal{L}^\pi J(W, x, t) + \mathcal{L}^c J(W, x, t) - \delta J(W, x, t) \ln ([1 - \gamma]J(W, x, t)) + \frac{\partial J}{\partial t}(W, x, t)$$

$$+ J_W(W, x, t)Wr(x) + J_x(W, x, t)m(x) + \frac{1}{2} J_{xx}(W, x, t)(v(x)^\top v(x) + \hat{v}(x)^2),$$

where

$$\mathcal{L}^c J = \sup_{c \geq 0} \{ \delta(1 - \gamma)J \ln c - cJ_W \} .$$

The first-order condition for the consumption choice is

$$\delta(1 - \gamma)J \frac{1}{c} = J_W \iff c = \delta(1 - \gamma)JJ_W^{-1},$$

which implies that

$$\mathcal{L}^c J = \delta(1 - \gamma)J (\ln \delta + \ln ([1 - \gamma]J) - \ln J_W) - \delta(1 - \gamma)J$$

$$= \delta(1 - \gamma)J \{ \ln \delta + \ln ([1 - \gamma]J) - \ln J_W - 1 \} .$$
Solving the HJB

Substituting $\mathcal{L}^\pi J$ and $\mathcal{L}^c J$ into HJB:

$$0 = -\frac{1}{2} \frac{J_W^2}{J_{WW}} \| \lambda(x) \|^2 - \frac{1}{2} \frac{J_{Wx}^2}{J_{WW}} \| \nu(x) \|^2 - \frac{J_W J_{Wx}}{J_{WW}} \nu(x)^T \lambda(x)
+ \delta (1 - \gamma) J \{ \ln \delta + \ln ([1 - \gamma] J) - \ln J_W - 1 \}
- \delta J \ln ([1 - \gamma] J) + \frac{\partial J}{\partial t}
+ J_W W r(x) + J_x m(x) + \frac{1}{2} J_{xx} (\nu(x)^T \nu(x) + \hat{v}(x)^2),$$

Conjecture $J(W, x, t) = \frac{1}{1 - \gamma} G(x, t)^{\gamma} W^{1-\gamma}.$

Then optimal consumption is $c^*_t = \delta W_t.$

We need $G(x, T) = \alpha^{1/\gamma}$ and

$$0 = \frac{1}{2} \left( \| \nu(x) \|^2 + \hat{v}(x)^2 \right) G_{xx} + \left( m(x) - \frac{\gamma - 1}{\gamma} \lambda(x)^T \nu(x) \right) G_x + \frac{\gamma - 1}{2} \hat{v}(x)^2 \frac{G_x^2}{G}
+ \frac{\partial G}{\partial t} - \left( \delta \ln G + \frac{\gamma - 1}{\gamma} \delta [\ln \delta - 1] + \frac{\gamma - 1}{\gamma} r(x) + \frac{\gamma - 1}{2\gamma^2} \| \lambda(x) \|^2 \right) G.$$
Solving for $G$

If $\| \mathbf{v}(x) \|^2$, $\hat{\mathbf{v}}(x)^2$, $m(x)$, $\lambda(x)^\top \mathbf{v}(x)$, $r(x)$, and $\| \lambda(x) \|^2$ are all affine functions of $x$, the PDE for $G$ has the solution

$$G(x, t) = \alpha^{1/\gamma} e^{-\frac{\gamma-1}{\gamma} A_0(T-t) - \frac{\gamma-1}{\gamma} A_1(T-t) x},$$

where $A_0$ and $A_1$ solve ODE’s with $A_0(0) = A_1(0) = 0$.

$$A'_1(\tau) = r_1 + \frac{\Lambda_1}{2\gamma} + \left( m_1 - \frac{\gamma-1}{\gamma} K_1 + \delta \right) A_1(\tau) - \frac{\gamma-1}{2\gamma} (V_1 + \gamma \hat{\mathbf{v}}_1) A_1(\tau)^2.$$

Note: closed-form solution with utility of intermediate consumption and incomplete markets – contrasts the results for time-additive power utility.

Then $G_x/G = -\frac{\gamma-1}{\gamma} A_1(T-t)$ so the optimal portfolio becomes

$$\pi_t^* = \frac{1}{\gamma} \left( \sigma(x_t, t)^\top \right)^{-1} \lambda(x_t) - \frac{\gamma-1}{\gamma} \left( \sigma(x_t, t)^\top \right)^{-1} \mathbf{v}(x_t) A_1(T-t).$$

Only difference to time-additive power utility is the $\delta$-term in the ODE!
HJB with $\psi \neq 1$

Recall that for $\psi \neq 1$,

$$f(c, J) = \frac{\delta}{1 - 1/\psi} c^{1-1/\psi} ( [1 - \gamma] J )^{1-1/\theta} - \delta \theta J.$$ 

Therefore HJB equation becomes

$$0 = \mathcal{L}^\pi J(W, x, t) + \mathcal{L}^c J(W, x, t) - \delta \theta J(W, x, t) + \frac{\partial J}{\partial t} (W, x, t)$$

$$+ J_W(W, x, t) Wr(x) + J_x(W, x, t) m(x) + \frac{1}{2} J_{xx}(W, x, t) (v(x)^T v(x) + \hat{v}(x)^2),$$

where

$$\mathcal{L}^c J = \sup_{c \geq 0} \left\{ \frac{\delta}{1 - 1/\psi} c^{1-1/\psi} ( [1 - \gamma] J )^{1-1/\theta} - c J_W \right\}.$$ 

The first-order condition for the consumption choice is

$$c = \delta^\psi J_W^{-\psi} ( [1 - \gamma] J )^{\psi (1 - 1/\theta)},$$

which implies that

$$\mathcal{L}^c J = \frac{1}{\psi - 1} \delta^\psi J_W^{1-\psi} ( [1 - \gamma] J )^{\psi (1 - 1/\theta)}.$$
Solving the HJB

With conjecture \( J(W, x, t) = \frac{1}{1-\gamma} G(x, t)^\gamma W^{1-\gamma} \) we get the optimal consumption

\[
c_t^* = \delta^\psi G(x_t, t)^{-\psi \gamma / \theta} W_t
\]

and \( G \) must solve

\[
0 = \frac{1}{2} \left( \|v(x)\|^2 + \hat{v}(x)^2 \right) G_{xx} + \left( m(x) - \frac{\gamma - 1}{\gamma} \lambda(x)^\top v(x) \right) G_x + \frac{\gamma - 1}{2} \hat{v}(x)^2 \frac{G_x^2}{G}
\]

\[
+ \frac{\partial G}{\partial t} + \frac{\theta}{\gamma \psi} \delta^\psi G^{\gamma \psi - 1} - \left( \frac{\delta \theta}{\gamma} + \frac{\gamma - 1}{\gamma} r(x) + \frac{\gamma - 1}{2 \gamma^2} \|\lambda(x)\|^2 \right) G
\]

with terminal condition \( G(x, T) = \alpha^{1/\gamma} \).

Term with \( G^{\gamma \psi - 1} \) prevents explicit solution – unless \( \psi = 1 / \gamma \), then time-additive CRRA utility as in earlier chapters.

Approximate closed-form solution (next!) or numerical solution (maybe set up the problem in discrete time from the beginning)
Approximate closed-form solution

With constant investment opportunities:

\[ 0 = G'(t) + \frac{\theta}{\gamma \psi} \delta \psi G(t) \frac{\gamma \psi - 1}{\gamma - 1} - AG(t), \quad A = \frac{\delta \theta}{\gamma} + \frac{\gamma - 1}{\gamma} r + \frac{\gamma - 1}{2 \gamma^2} \| \lambda \|^2. \]

A Taylor approximation of \( z \mapsto e^z \) around \( \hat{z} \) gives \( e^z \approx e^{\hat{z}} (1 + z - \hat{z}) \), so

\[ G(t)^{\frac{\gamma \psi - 1}{\gamma - 1}} = G(t) G(t)^{\frac{\gamma(\psi - 1)}{\gamma - 1}} = G(t) e^{\frac{\gamma(\psi - 1)}{\gamma - 1} \ln G(t)} \]

\[ \approx G(t) e^{\frac{\gamma(\psi - 1)}{\gamma - 1} \ln \hat{G}(t)} \left( 1 + \frac{\gamma(\psi - 1)}{\gamma - 1} [\ln G(t) - \ln \hat{G}(t)] \right) \]

\[ = G(t) \hat{G}(t)^{\frac{\gamma(\psi - 1)}{\gamma - 1}} \left( 1 + \frac{\gamma(\psi - 1)}{\gamma - 1} [\ln G(t) - \ln \hat{G}(t)] \right). \]

Using that approximation in the ODE, we get

\[ 0 = G'(t) - a(t) G(t) - b(t) G(t) \ln G(t), \]

\[ a(t) = A - \delta \psi \hat{G}(t)^{\frac{\gamma(\psi - 1)}{\gamma - 1}} \left( \frac{\theta}{\gamma \psi} + \ln \hat{G}(t) \right), \quad b(t) = \delta \psi \hat{G}(t)^{\frac{\gamma(\psi - 1)}{\gamma - 1}}. \]

Solution with \( G(T) = \alpha^{1/\gamma} \) is

\[ G(t) = \alpha^{1/\gamma} e^{-D(t)}, \quad D(t) = \int_t^T e^{-\int_s^t b(u) \, du} \left( a(s) + b(s) \frac{1}{\gamma} \ln \alpha \right) ds. \]
Approximate closed-form solution, cont’d

Consumption: \( c_t^* = \delta^\psi G(x_t, t)^{-\psi \gamma / \theta} \text{W}_t = \delta^\psi \alpha^{-\psi / \theta} e^{\psi \gamma / \theta} D(t) \text{W}_t \)

How to choose \( \hat{G}(t) \)?

One possibility: presume that the optimal consumption/wealth ratio \( c_t^*/\text{W}_t = \delta^\psi G(x_t, t)^{-\psi \gamma / \theta} \) is close to \( \delta \) which is the optimal consumption/wealth ratio for \( \psi = 1 \):

\[
\delta^\psi G(t)^{-\psi \gamma / \theta} \approx \delta \quad \Rightarrow \quad G(t) \approx \delta^{-\frac{\gamma - 1}{\gamma}} \equiv \hat{G}(t).
\]

In that case, the functions \( a \) and \( b \) are simply constants,

\[
b = \delta, \quad a = A - \delta \left( \frac{\theta}{\gamma^\psi} - \frac{\gamma - 1}{\gamma} \ln \delta \right),
\]

so that \( D(t) \) reduces to

\[
D(t) = \left( \frac{A}{\delta} - \frac{\theta}{\gamma^\psi} + \frac{\gamma - 1}{\gamma} \ln \delta + \frac{1}{\gamma} \ln \alpha \right) \left( 1 - e^{-\delta(T-t)} \right).
\]

Precision of the approximation is not clear!
Approximate closed-form solution, cont’d

The approximation can be generalized to **stochastic investment opportunities**, but will then involve recursive procedure.

Not clear how the hedge portfolio with Epstein-Zin utility differs from the hedge portfolio with time-additive power utility – study on a case-by-case basis?
The model


- Constant interest rate \( r \)
- Single risky asset with price dynamics

\[
\frac{dP_t}{P_t} = \mu \, dt + \sqrt{\frac{1}{y_t}} \, dz_{St},
\]

with

\[
dy_t = \kappa [\bar{y} - y_t] \, dt + \sigma \sqrt{y_t} \left[ \rho \, dz_{St} + \sqrt{1 - \rho^2} \, dz_{yt} \right].
\]

Note:

\[
\mu = r + \sigma_t \lambda_t = r + \sqrt{\frac{1}{y_t}} \lambda_t \implies \lambda_t = (\mu - r) \sqrt{y_t}
\]

High \( y \sim \) low volatility \( \sim \) good state (in contrast to Liu-Pan model)

- \( \rho > 0 \) implies that high \( y_t \sim \) low volatility AND high stock price
- Epstein-Zin utility with infinite time horizon
Exact solution for $\psi = 1$

\[
J(W, y) = \frac{1}{1 - \gamma} e^{Ay + B W^{1-\gamma}},
\]

\[
c_t^* = \delta W_t,
\]

\[
\pi_t^* = \frac{1}{\gamma} (\mu - r) y_t + \frac{\gamma - 1}{\gamma} (-\rho) \sigma \hat{A} y_t,
\]

\[
\frac{\lambda_t}{\sigma_t} = \lambda_t \sqrt{y_t}
\]

negative hedge for $\rho > 0$

where $A, B$ are constants, and $\hat{A} = A/(1 - \gamma) > 0$. 
The set-up

The case EIS equals 1

The case EIS different from 1

An example

Approximate solution for \( \psi \neq 1 \)

\[
J(W, y) = \frac{1}{1 - \gamma} e^{-\gamma (A_1 y + B_1) W^{1-\gamma}},
\]

\[
c_t^* = \delta e^{-A_1 y_t - B_1 W_t},
\]

\[
\pi_t^* = \frac{1}{\gamma} (\mu - r) y_t + \frac{\gamma - 1}{\gamma} (-\rho) \sigma \hat{A}_1 y_t,
\]

again negative for \( \rho > 0 \)

where \( A_1, B_1 \) are constants, and \( \hat{A}_1 = A_1/(1 - \gamma) > 0 \).
Some quantitative results: portfolios

E.I.S.

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A. Mean optimal allocation to stocks (%): $E[\pi_t(y_t)] = \pi(\theta) \times 100$

B. Ratio of hedging demand over myopic demand (%)

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Table 2 from Chacko and Viceira (2005).

Little action across EIS.
Some quantitative results: consumption

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A. Consumption–wealth ratio (%) \( C_t/X_t = \exp\{E[c_t-x_t]\} \times 100

B. Long-term expected return on wealth (%) \( (\pi(\theta)(\mu-r) + r) \times 100

Table 4 from Chacko and Viceira (2005).

Lots of action across EIS.