Liquidity crises due to asymmetric information

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This version: January 30, 2013
Job market paper.

The paper contains figures in color, use color printer for best results.

Abstract

This paper considers a model where both market liquidity and funding liquidity depend on the information in the market. Speculators are constrained by margins on their positions, and the margins are set by financiers who have less information than the speculators. The paper shows how this can result in fragile financial markets. The model considers multiple assets, and it is shown that when speculators are specializing, margins may be destabilizing, which does not happen with speculators who only provide liquidity. Margins depend on the initial holdings of the speculator, the initial illiquidity, and the variance of the financier’s information, and may be either increasing or decreasing depending on the trade and the sign of the initial holding of the speculator. Correlation of the returns also has an impact on the margins and stability in the market. The model predicts that uncertainty about future returns is a driver of market liquidity and has several predictions regarding margins and stability of the financial market.

Keywords: Liquidity, Information Quality, Asset Pricing, Correlation Effects, Asymmetric Information.

JEL classification: G01, G12

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Introduction

Why should speculators such as hedge funds diversify? Hedge funds are normally characterized as risk-neutral agents; thus, risk is not taken into consideration in the investments made. Instead, most speculators specialize in small markets and earn thin profits by trading, thereby providing liquidity into the market when liquidity is low. Speculators, thus have an important function in the financial markets. In order to be able to provide liquidity at all times, speculators need funding and often use collateralized trading and high leverage. Hence, speculators need financiers to fund their trading. Speculators are often better informed about the assets — since speculators, as stated above, operate and follow the developments in smaller markets — than their financiers, who finance many speculators and thus operate in many markets.

In normal times, speculators earn positive returns. However, when this form of investments do not have a positive outcome, speculators may suffer great losses instead. Moreover, since speculators are often highly leveraged, this may have severe consequences for the stability of the financial market. We have seen multiple examples of this throughout history, for example in 1998 with the collapse of Long-Term Capital Management (LTCM), a Greenwich, Connecticut hedge fund that, after large losses due to the Russian default, was bailed out by the Federal Reserve.\(^1\) LTCM had suffered large losses on macroeconomic bets and was ultimately rescued in a bailout organized by the Federal Reserve to avoid a further collapse in the financial system.\(^2\) In 2005, convertible hedge funds faced large redemptions of capital gains from investors, leading to binding capital constraints, which again led to massive bond sales. Similarly, in connection with the 1987 stock market crash, many mutual funds specialized and earned profits in merger arbitrage. However, during the crash, many mergers failed, causing great losses to these funds. Thus, speculators also exert potential dangers to the financial system.

In this paper, we combine the information distortions and capital requirements of investors to study how asymmetric information has an impact on the capital restrictions set by financiers and the liquidity of risky assets. Thus, we examine what effects information quality and correlation have on the stability of the financial market when speculators use leveraged trading. Collateral/leveraged trading takes place when an investor buys an asset, uses this asset as collateral, and borrows against it. However, the investor cannot

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\(^1\)See Mitchell et al. (2007).

\(^2\)Similarly, the collapse of Bear Stearns in 2008 was due to great losses in two of its hedge funds and the investment bank was ultimately bought by JP-Morgan Chase with a loan from the Federal Government.
borrow against the entire value of the asset; there is a margin or haircut the against asset, and hence the investor cannot trade against the entire value of the asset. Similarly, when an investor buys an asset short, it requires capital, which the investor has to provide for herself. If the investors initially have losses and are leveraged, they have binding capital restrictions which may lead to assets sales to gain capital. This effect may end up in fire sales and even greater losses for the investors, and the financial market goes in to a losing spiral.

In theory, margins should work as a stabilizing effect, meaning that when prices diverge from fundamentals, the margins should decrease since agents expect that prices converge back to fundamentals. However, financial frictions like asymmetric information may change this effect, and the margins can be destabilizing such that they enhance the capital restrictions of the investor and make the financial market fragile. The margins depend on the expectations of the returns of the assets and the order imbalance or one-sided pressure in the market, which both may only be known to the traders in the market, but not to the financiers. The financier has to use the tradings of the investor to infer expectations about the assets and set the margins accordingly. If the investor is forced to sell assets to uphold his capital constraint, it may result in fire sales and price drops. In addition to the price impact of forced selling, the financier cannot observe whether the kept assets are good liquid assets, which the investor is holding in hope to recover his wealth. The assets could also be bad illiquid assets that could not be sold and which the investor had to keep on his books. The financier will fear the latter, and therefore margins will increase when constrained investors sell other assets. Thus, there may exist two different spirals that make the financial market unstable - the price spiral and the liquidity spiral. Figure 2 shows that margins did increase empirically for the S&P 500 futures during the liquidity crises of 1987, 1990, 1998, and 2007. Compared to the margins of members of the exchange, the margins are a lot higher and we see that to maintain the stability the speculators pay a high fee to participate in the market.

The model can explain how for instance prime brokers set the margin constraints for their customers such as hedge funds. The model shows situations where liquidity can dry up if the broker receives signals indicating unfavorable returns. Markets may be unstable with only one risky asset; however, liquidity spirals will not occur. This is due to the fact that with only one asset the financier does not have to fear that the speculator is only

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3 See Brunnermeier (2009).
4 An illustrative figure is given in Brunnermeier and Pedersen (2009, page 2204).
 Margins at the CME

![Graph showing margins at the CME with key events: Black Monday, US-Iraq war, LTCM, Sub-prime crisis.]

Figure 1: Margins for the speculators on S&P 500 futures on the CME

able to sell the *good* assets. With multiple assets, margins may be destabilizing due to specializing speculators who take his investments into consideration and profit maximize. If we assume that the speculator is passive and is only present in the market to provide liquidity, no liquidity spiral will occur. With multiple risky assets the financier is more affected by the trades of the speculator, since in times when the speculator is restricted to selling assets, the financier will fear that the good and liquid assets are sold first, which reduces the expectations of the kept assets.

The paper is related to several strands of literature: The first deals with fragile financial markets and liquidity crises, as studied in the seminal paper by Brunnermeier and Pedersen (2009), from which the model is also inspired. The authors use the same model setup and show that markets may in some circumstances be fragile, that liquidity dry-ups may occur, and that illiquidity co-move across assets and are related to volatility and are subject to 'flight to quality'. However, the model depends on the fact that the size of the volatility

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5 Other papers regarding fragile markets are Grossman and Miller (1988); Gennette and Leland (1990); Allen and Gale (2004); Wagner (2007). Empirically, this is studied in Gorton and Metrick (2011) from the 2007-2008 sub-prime crisis with focus on the repo market and shows a contagion-effect from credit spreads in securitized bonds, the LIB-OIS spread and repo rates. Mitchell et al. (2007) study three illiquidity crises
depends on shocks to the economy in the previous periods and that investors may not be present in the market in all periods, and the financier does not update whether or not investors are present, although trading happens. Thus, the financier cannot distinguish if the changes in the prices are due to volatility changes or liquidity shocks. In their model both the returns of the assets and the margins are independent of each other. The financier does not view the trading of the speculator as information and the margins do not depend on the investments of the speculator. Hence, the speculator has a simple investment rule: Invest all capital in the asset that provides the largest return per dollar invested, and thus the paper does not use the fact that the trading of speculators also provide information to the market.

The second strand of literature is about liquidity. Pástor and Stambaugh (2003) show the existence of a cross-sectional liquidity premium of stocks. Thus, illiquid stocks have lower prices and higher returns.\(^6\) Acharya and Pedersen (2005) consider a general asset pricing model with an illiquidity cost. They introduce an illiquidity cost, which is correlated with the returns, derive an illiquidity-adjusted CAPM, and show empirical results regarding the theoretical results in Brunnermeier and Pedersen (2009). Comerton-Forde et al. (2010) show that market-maker balance sheet and income statement variables explain time variation in liquidity, suggesting that liquidity-supplier financing constraints matter. They use daily returns of NYSE specialists to show that after specialists lose money on their inventories and/or find themselves holding large positions, effective spreads widen. Lastly, Ng (2011) finds that higher information quality is associated with lower liquidity risk, and the relation is stronger in times with large shocks to liquidity. The last two papers fit well with the results in this paper where the financier may increase margins if the speculator has large losses. Hence, the better informed the financier is, the smaller margins and the higher the equilibrium prices will be. As in Comerton-Forde et al. (2010) the speculator will have a tightening capital constraint if the speculator is subject to initial losses, and hence market liquidity will be lower.

As stated above, liquidity can suddenly dry up and investors may therefore withdraw from the financial markets where agents have asymmetric information. This paradigm is also studied in the third strand of literature in relation to ambiguity aversion.\(^7\) In and show that financial frictions exist such that even a small shock to the capital of investors can have a large effect and that it may take some time before the economy 'bounces' back.

\(^6\)Similar Chordia et al. (2005); Goyenko and Ukhov (2009) show the illiquidity of stocks and bonds are correlated.

\(^7\)See Easley and O’Hara (2009, 2010); Cao et al. (2005); Ozsoylev and Werner (2009)
these papers, investors do not know the correct distribution of the returns. This lack of information will make investors rationally leave the market and hold no assets for an interval of prices, and a bid-ask spread occurs. The result is similar to this paper, however, in this paper the lack of trading is due to the capital restrictions of the speculator, not to rational choices by uninformed investors.

The model makes predictions about the trading of speculators in times of crisis, and the margins they face in during those times. Hedge funds fit well into this story as speculators who provide liquidity to the financial markets. The trading of hedge funds has been studied extensively since the financial crises. Ben-David et al. (2012) find that hedge funds withdrew capital more extensively than mutual funds. This was mainly due to redemptions and margin calls. Consistent with this paper, the hedge funds sold the liquid stocks first and ended up with the illiquid stocks. Similarly Ang et al. (2011) find that hedge fund leverage is counter-cyclical and that hedge funds use leverage actively to take advantage of mispricing opportunities, which is consistent with this paper. The model has empirical predictions regarding the margins set by prime brokers: 1) The margins are related to the trading of the speculator and asymmetric regarding buys and sales. 2) The margins are related the information the broker has regarding the speculator, e.g. measured by years as a client. 3) Investors with a higher initial share also have smaller margins. 4) Lastly, the numerical examples show that in markets where the assets are positively correlated the initial share the speculator must have for the equilibrium to be fragile is smaller, meaning the the equilibrium is more likely to be fragile.

The paper proceeds as follows: In section 1 we set up the model and specify the knowledge of agents, dynamics of prices, and preferences. The speculator’s demand is derived in section 2. In section 3 we derive the equilibrium for the market, first with only one risky asset, then with two risky assets. Section 4 discusses the empirical prediction. A conclusion is given in section 5. All proofs are provided in appendix A.

1 The Model

The economy has three periods, \( t \in 0, 1, 2, 3 \), in which \( J \) assets are traded. Asset \( j \) has a terminal payoff at time 3 of \( v_j \) and is a random variable on the filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\). The payoff of the set of assets is written as \( v \). We assume that the

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8Teo (2011) finds results of high leverage in hedge funds, and these funds have higher returns. In related papers, Aragon and Strahan (2011) find that stocks held by Lehman-connected hedge funds experienced greater declines in market liquidity following the bankruptcy.
aggregate supply is zero and the risk-free rate is normalized to zero as well, and thus there is no aggregate risk. At time $t$ we write $v_t^j = \mathbb{E}_t[v^j]$ with respect to $\mathcal{F}_t$. The dynamics of the assets is given by

$$v_{t+1} = v_t + D_v \varepsilon_{t+1}, \quad \varepsilon_{t+1} \in \mathcal{F}_{t+1},$$

(1.1)

where $\varepsilon_{t+1} \sim N(0, I)$ and $D_v$ is the Cholesky decomposition of the covariance matrix $\Sigma_v = D_v D_v^\top$. The variable $\varepsilon$ is the shocks in the economy that changes the expected values of the assets, while the variance of the assets are held constant. Thus, we denote $v_t^j$ as the fundamental of asset $j$. For simplicity, we assume that the dimension of $\varepsilon$ is $J$.

The price of asset $j$ at time $t$ is denoted $p_t^j$ and the set of prices are written $p_t$.

There are three sets of agents in the market: customers and speculators trade assets, while the financiers finance the speculators’ positions. The group of customers consist of three risk averse agents. Customer $k$ has an initial endowment of $W^k_0$ in bonds and zero shares, but experiences that he will receive a shock $z^k = (z_1^k, \ldots, z_J^k)^\top$ of shares at time 3. We have that $z^j_k$ is a random variable and assume that $\sum_{k=0}^2 z^j_k = 0$ for all $j \in J$.

These shocks could be due to derivatives trading, which unfold at time 3. To introduce an order imbalance in the market, the customers arrive in the market sequentially such that at time 0 only customer 0 is present, and the rest of the customers enter one in every subsequent period, $t = 1, 2$. Before a customer arrives his demand is $y_t^k = 0$; after he arrives he chooses his security position in each period to maximize his utility at termination. In periods with order imbalance, the risk-neutral speculator is willing to provide liquidity/immediacy. The speculator maximizes profit in each period. However, the speculator faces the constraint that she has to fulfill a capital restriction set by the financier. The speculator has the long (short) shares in asset $j$, of $x_t^{j+}, (x_t^{-})$, and the financier sets the margins, $m_t^{j+}, (m_t^{-})$. The capital restriction is given by:

$$\sum_j \left\{ m_t^{j+} x_t^{j+} + m_t^{-} x_t^{-} \right\} \leq W_t,$$

(1.2)

where $m_t^{j+}, (m_t^{-})$ are the margins for a long (short) position, respectively. Hence, the sum of shares in the speculators inventory times the specific margin cannot be larger than

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9The assets are correlated to examine the effect of correlation across assets, when the financier has to set margins on multiple assets.

10Brunnermeier and Pedersen (2009) use an Arch-model, thus variance depends on former movements.

11We assume that the capital restriction is on each asset and not the entire portfolio. This is for simplicity since the margins are set and would not change the results significantly, although the speculator could invest such that the variance of the portfolio would decrease. We also not view both long and short positions as possible.
the wealth of the speculator. We see that the smaller the margin the smaller is the impact on the capital restriction and the speculator is allowed to buy more. The wealth of the speculator evolves according to

\[ W_{t+1} = W_t + (p_{t+1} - p_t)^\top x_t + \kappa_{t+1}, \quad (1.3) \]

where \( \kappa \) is an independent wealth shock from other activities. We see that if the speculator has gains on his initial trading, the speculator will have a larger wealth to buy assets for. If the speculator looses all her capital, \( W_t \leq 0 \), she can no longer invest due to the capital constraint, (1.3). Thus, she must choose \( x = 0 \). The margins are set, as in Brunnermeier and Pedersen (2009), to limit the financier’s counter party risk. The margins on asset \( j \), \( m_j^+, (m_j^-) \) are set to cover the long (short) positions \( \pi \)-value-at-risk:

\[ \pi = P\left[\left(-\left(p_{t+1}^j - p_t^j\right) > m_j^+ \mid \tilde{F}_t\right]\right], \quad (1.4) \]
\[ \pi = P\left[\left(p_{t+1}^j - p_t^j > m_j^- \mid \tilde{F}_t\right]\right], \quad (1.5) \]

where \( \tilde{F}_t \subset F \) is the information available to the financier at time \( t \). In equation (1.4), the margin on a long position is set such that price drops that exceed the amount of the margin only happens with small probability, \( \pi \). Similarly, in equation (1.5) the margin on a short position is set such that price increases exceed the amount of the margin only with small probability. We have that the margins depend on the financiers information set, \( \tilde{F}_t \).

We assume that the financier does not observe the shocks to fundamentals, \( \varepsilon \), and neither observe the liquidity shocks of the customers, \( z \). However, the financier does observe the prices of the assets, the shares of the speculator and for simplicity also the speculators’ shocks, \( \kappa \). Hence, \( \tilde{F}_t = \sigma \{ v_0, p_0, \ldots, p_t, \kappa_1, \ldots, \kappa_t, x_0, \ldots, x_t, z \} \).

1.1 Margin setting and Liquidity (Time 1)

We assume that the customers have the utility function

\[ U(W_t^k) = -\exp\left[-\gamma W_t^k\right], \quad \text{where} \quad W_t^k \text{ is the wealth of customer } k, \text{ including the endowment shock, } z^k, \text{ and evolves according to} \]

\[ W_{t+1}^k = W_t^k + (p_{t+1} - p_t)^\top (y_t^k + z^k). \quad (1.6) \]

We denote the customers’ value function as \( V \):

\[ V(W_t^k, p_t, v_t) = \max_{y_t^k} -\mathbb{E}_t \left[\exp\left[-\gamma W_t^k\right]\right], \]
\[ = \max_{y_t^k} -\exp\left[-\gamma \left(\mathbb{E}_t \left[W_t^k\right] - \frac{1}{2} \nu_t \left[W_t^k\right]\right)\right], \quad (1.7) \]
and the solution is given by
\[ y_t^k = \frac{1}{\gamma} \Sigma_v^{-1} (v_t - p_t) - z^k. \]  
(1.8)

As explained the financier observes the prices and the investments made by the speculator. This information is used as a signal about the expected return of the assets. We denote \( \Delta x_t = x_t - x_{t-1} \) and \( Z^t = \sum_{k=0}^{t} z^k \). The signed deviation from the fundamental price is denoted, \( \Lambda_t = p_t - v_t \). We have that the shares of the speculator must equal the supply of the customers in the market:
\[
x_0 = -y^0 = -\frac{1}{\gamma} \Sigma_v^{-1} (v_0 - p_0) + Z^0,
\]
\[
x_1 = -(y_0^1 + y_1^1) = -\frac{2}{\gamma} \Sigma_v^{-1} (v_1 - p_1) + Z^1
\]
\[
\Delta x_1 = z^1 - \frac{2}{\gamma} \Sigma_v^{-1} (v_1 - p_1) + \frac{1}{\gamma} \Sigma_v^{-1} (v_0 - p_0).
\]
(1.9)

We assume that the financier has the prior belief that each liquidity shock is independent with mean zero and variance \( \eta^2_j \). We write \( z^1 \sim N(0, \Sigma_z) \). We end up with the signal \( \Psi \):
\[
\Delta x_1 + \frac{1}{\gamma} \Sigma_v^{-1} [\Lambda_0 - 2p_1] = z^1 - \frac{2}{\gamma} \Sigma_v^{-1} v_1 \iff \frac{\gamma}{2} \Sigma_v \Delta x_1 - p_1 + \frac{1}{2} \Lambda_0 = \frac{\gamma}{2} \Sigma_v z^1 - v_1 = \Psi_1.
\]
(1.10)

From equation (1.10), we see that to infer the signal, \( \Psi \), the financier combines the price, the change in the shares of the speculator, and the initial illiquidity. Thus, the inferences is similar to the seminal papers by Kyle (1985) and Glosten and Milgrom (1985). Since the financier does not observe both \( v_1 \) and \( z^1 \), she cannot distinguish between liquidity shocks and shocks to fundamentals. Hence, if the price of an asset decreases, the financier cannot tell if the drop is due to decreasing fundamentals or increasing liquidity shocks. Instead, the financier uses the signal \( \Psi_1 \) to form a posterior belief about \( v \). The distribution of the signal is \( \Psi \sim N(0, \Sigma_\Psi) \)

**Lemma 1.** The financier writes the distribution of \( v_1 \) as
\[
v_1 = v_{1|\Psi} + D_{v|\Psi} \epsilon_1
\]
\[
\mathbb{E} [v_1 | \Psi_1] = (I - (H^{-1})^\top) v_0 - (H^{-1})^\top (\frac{1}{2} \Lambda_0 - p_1) - \hat{H} \Delta x_1,
\]
where \( H^{-1} = \Sigma_v \Sigma_{\Psi}^{-1} \) and \( \hat{H} = \Sigma_v \Sigma_{\Psi}^{-1} \Sigma_v \). The margin for a long position at time 1 is
\[
m_1^{+}(x_1^1) = \max \left[ \tau^j \Phi^{-1}(1 - \pi) - v_{1|\Psi} + p_1^j, 0 \right]
\]
(1.12)

and the margin for a short position is written
\[
m_1^{-}(x_1^1) = \left[ \tau^j \Phi^{-1}(1 - \pi) + v_{1|\Psi} - p_1^j, 0 \right],
\]
(1.13)
where \( \tau^j = \sqrt{(\sigma^j_{v})^2 + (\sigma^j_{v|\Psi})^2} \).
From equation (1.11), we see a positive relation between the prices of the assets and the expected value of the assets. Hence, when prices fall, the financier will see this as deterioration of the assets. The relation between the expected value of the assets and the change in inventory is negative which is a little counter intuitive. When the speculator buys more of the assets, the financier will lower the expected value due lower demand from the risk averse customers. The signal is positively related to the liquidity shock of the customers, which lowers the demand of the customers. Therefore, with a larger inventory to the speculator, the financier will expect returns to be lower. We see that, as expected, the margin for a long (short) position is decreasing (increasing) in the expected value of asset $j$ and reverse for the price of the asset. The margins then depend on the speculator’s investments in the assets. From Brunnermeier and Pedersen (2009) liquidity spirals occur, if the margins can be decreasing in the prices for a long position.

We will distinguish between markets with one or two assets, since we need two assets for the financier to fear that the speculator sells off the good assets.\footnote{Alternatively, we could distinguish between independent and correlated assets.}

**Lemma 2.** Margins in a single-asset market. At time 1, the margins on a long position will be

$$m^+_1 = \max \left[ \tau \Phi^{-1} (1 - \pi) - v_0 |_\Psi + p_1, 0 \right],$$  \hspace{1cm} (1.14)

$$= \max \left[ \tau \Phi^{-1} (1 - \pi) - v_0 (1 - \theta) + p_1 (1 - \theta) + \frac{1}{2} \theta \left[ \gamma \sigma^2 \Delta x_1 + \Lambda_0 \right], 0 \right].$$  \hspace{1cm} (1.15)

Similar the margin for the short position is given by

$$m^-_1 = \max \left[ \tau \Phi^{-1} (1 - \pi) + v_0 (1 - \theta) - p_1 (1 - \theta) - \frac{1}{2} \theta \left[ \gamma \sigma^2 \Delta x_1 + \Lambda_0 \right], 0 \right],$$  \hspace{1cm} (1.16)

where $\sigma^2|_\Psi = \sigma^2 - \frac{\sigma^2}{\xi^2 \eta^2 + 1}$, and $\theta = \left[ \frac{\xi^2 \eta^2 + 1}{\xi^2 \eta^2 + 1} \right]^{-1}$.

We see that the margins depend on the current price and the expected value of the asset at time zero. However, since $\theta < 1$, the weight is less than one. The rest of the margin is a weighted sum of the change in inventory, $\Delta x_1$ and the illiquidity at time zero, $\Lambda_0$. Thus, $\theta$ works the weight the financier puts on the different variables. The variance of the liquidity shock, $\eta_2$, has a two-fold impact on the margins:\footnote{We consider a long position, when discussing the changes in the margins.} With a higher variance, $\tau$ is smaller while $\theta$ is smaller as well and the total impact depends on the parameters and the trading of the speculator. The margin is decreasing in $x_0$, meaning that speculators with large shares initially have lower margin requirements. In figure 2, we have the margin
depending on either volatility, $\sigma$, or the variance of the liquidity shock, $\eta$, (the other variables are given in Table 1.) In the left figure we see that margins are monotonically increasing in volatility and the higher the price the higher the margins. On the right we see margins can both be increasing and decreasing in $\eta$: The base-case (the black line) is decreasing. However, if the price is increasing this effect is smaller. In the base case we have an initial selling pressure and the period-1-price is lowered suggesting an even larger selling pressure. Thus, the margin is decreasing in $\eta$ since this allows larger liquidity shocks in period 1. If the speculator is selling, the margin is increasing slowly (the blue line), and if the price is increasing (the green line) the increase is even larger. Intuitively, the fact that the selling pressure is smaller than initially could be explained with a higher expected return, $v_1$. If $\eta$ is increasing, the liquidity shock could destroy this positive relation in $v_1$ and the margin is increasing in $\eta$.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\sigma$</th>
<th>$\eta$</th>
<th>$\Delta x$</th>
<th>$p_0$</th>
<th>$p_1$</th>
<th>$v_0$</th>
<th>$\pi$</th>
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<td>5</td>
<td>10</td>
<td>110</td>
<td>100</td>
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</tr>
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Table 1: The parameters used in Figure 2.

![Figure 2: The margin of a long position dependent on the volatility and the variance and the prior belief of the financier.](image-url)
2 The demand of the speculator at time 1

Here, we derive the optimal strategies for the speculator using dynamic programming starting from time 2. The speculator’s value function is denoted, $F$. At time 2, every customer will be present in the market and there is no illiquidity problem. Thus, the unique equilibrium is $p_2 = v_2$, and the aggregate customer demand, $\sum_k y^k_2$, is zero and the speculator also has a zero demand. The value functions will be:

$$V(W_2^k, p_2, v_2) = -\exp[-\gamma W_2^k] \quad (2.1)$$
$$F(W_2, p_2, v_2, x) = W_2. \quad (2.2)$$

Since the speculator’s capital restriction depends on her trading, she will take this under consideration when choosing the optimal allocation. In the rest of the paper we assume that $Z^{ij} > 0$ for all $j \in J$.\textsuperscript{14} Thus, the speculator will be long in all assets and $v^j \geq p^j$.

We distinguish between an active or a passive speculator. The passive speculator does not consider her own impact on the margins and is only present in the market to provide liquidity, whereas the active speculator profit maximizes and takes the margins into account.\textsuperscript{15} We have that the capital restriction is not linear in the amount bought of each asset. Given a set of prices the speculator has the optimization problem:

$$F(x) = \max_x \mathbb{E} \left[ x^\top (p_2 - p_1) \right] = \max_x x^\top (v_1 - p_1), \quad (2.3)$$

such that

$$\sum_j x^j m^j_1 (x_1) \leq W_1. \quad (2.4)$$

We use the fact at time 1, the expected price of the assets is the current fundamental: $\mathbb{E}_1 [p_2^j] = v_1^j$. The Lagrange is written

$$F(x, \lambda) = x^\top (v - p) - \lambda \left( \sum_j x^j m^j_1 (x_1) - W_1 \right), \quad (2.5)$$

and the first order conditions are given by:

$$\frac{\partial F}{\partial x^j} (x^*, \lambda^*) = v^j - p^j - \lambda^* \left[ m^j(x^*) + x^j \frac{\partial m^j}{\partial x^j}(x_1) + \sum_{i \neq j} x^i \frac{\partial m^i}{\partial x^j}(x_1) \right] = 0, \quad (2.6a)$$

$$\frac{\partial F}{\partial \lambda} (x^*, \lambda^*) = \sum_j x^j m^j_1 (x_1) - W_1 = 0. \quad (2.6b)$$

\textsuperscript{14}The assumption is without loss of generality and Ibbotson (1999) shows that security-brokers and speculators are often net-long.

\textsuperscript{15}Thus, the passive speculator is similar to risk-neutral noise traders.
From equation (2.6a) we have that
\[
\lambda^* = \frac{v^j - p^j}{m^j(x^*) + x^j \frac{\partial m^j(x^*)}{\partial x^j} + \sum_{i \neq j} x^j \frac{\partial m^j(x^1)}{\partial x^j}} \text{ for all } j \in J. \tag{2.7}
\]

The speculator will buy more shares of asset \( j \) until the fraction is constant for all assets. The nominator is the profit pr dollar invested. The denominator is a sum of the margin of the asset, the marginal contribution to the margin with the shares bought, and the marginal contribution to other margins with the share in asset \( j \). Intuitively, if the expected profit is larger, then the denominator has to be larger: Either the margin is larger or the share times the marginal change in the margin or the marginal contribution to the margins of the other assets are larger. Equation (2.6b) simply states that the speculator uses his budget restriction.

2.1 Demand with two assets

For simplicity, we use two risky assets. The margins with two assets are written:\(^{16}\)
\[
m^j_1 = \tau^j - (I - (H^{-1})^\top) \pi_0 + \left(\begin{array}{ll} I & (H^{-1})^\top \end{array}\right) p_1 + \frac{\gamma}{2} \hat{H}_{jj} (x_1 - x_0) + \frac{1}{2} (H^{-1})^\top \Lambda_0, \tag{2.8}
\]
\[
= G_j(\hat{\tau}^j, \pi_0, x_0, \Lambda_0) + \left(\begin{array}{ll} I & (H^{-1})^\top \end{array}\right) p_1 + \frac{\gamma}{2} \hat{H}_{jj} x_1. \tag{2.9}
\]

From equation (2.6a), we have
\[
\frac{\partial F}{\partial x^1} = v^1_j - p^1_j - \lambda \left( G_j - \left(\begin{array}{ll} I & (H^{-1})^\top \end{array}\right) \right) p^1_j + \left(\begin{array}{ll} I & (H^{-1})^\top \end{array}\right) p^1_j + \frac{2\gamma}{2} \hat{H}_{jj} x^1_j \tag{2.10}
\]
\[
+ \frac{\gamma}{2} \hat{H}_{ji} + \frac{\gamma}{2} \hat{H}_{ij} x^1_j \]
\[
= 0.
\]

We see that holdings in both assets have a linear contribution to the speculator’s optimal share, and the speculator’s holding in asset \( j \) will have an impact on the margin in asset \( i \). Similarly, from equation (2.6b) we have
\[
\frac{\partial F}{\partial \lambda} = \frac{\gamma}{2} \hat{H}_{11}(x^1_1)^2 + \left( G_1 + \left(\begin{array}{ll} I & (H^{-1})^\top \end{array}\right) \pi_0 \right) x^1_1 + \frac{\gamma}{2} \hat{H}_{22}(x^2_1)^2 \tag{2.11}
\]
\[
+ \left( G_2 + \left(\begin{array}{ll} I & (H^{-1})^\top \end{array}\right) \pi_0 \right) x^2_1 + \frac{\gamma}{2} \left( \hat{H}_{12} + \hat{H}_{21} \right) x^1_1 x^2_1 - W_1 = 0.
\]

We see that in the budget restriction the holdings of the speculator are quadratic since the share also appears in the margins of the speculator and the speculator takes this into account. This feature does not occur in Brunnermeier and Pedersen (2009) where the speculator invests only in the assets where the expected profit pr dollar invested divided

\(^{16}\)Single subscripts of the matrices are the rows as a column vector, and \( \hat{\tau} \equiv \pi \Phi^{-1}(1 - \pi) \).
by the margin is largest. From equation (2.10), we have that

$$\lambda^* = \left( v_j^1 - p_j^1 \right) \left[ \frac{2\gamma}{2} \hat{H}_{jj} x_j^1 + \frac{\gamma}{2} \left( \hat{H}_{12} + \hat{H}_{21} \right) x_j^1 + \left( G_j + (I - (H^{-1})^T) j p_j \right) \right]$$

for all $j \in \{1, 2\}$. (2.12)

We see the relation between the demand of asset one and two:

$$x_2^1 = \frac{\hat{H}_{11}(v_2^1 - p_2^1) - \frac{\sigma_{12}}{\det(\hat{H})} (v_1^1 - p_1^1)}{\hat{H}_{22}(v_1^1 - p_1^1) - \frac{\sigma_{12}}{\det(\hat{H})} (v_2^1 - p_2^1)} x_1^1$$

$$+ \frac{1}{\gamma} \frac{(I - (H^{-1})^T) p_1^1 (v_2^1 - p_2^1) - (G_2 + (I - (H^{-1})^T) p_1^1) (v_1^1 - p_1^1)}{\hat{H}_{22}(v_1^1 - p_1^1) - \frac{\sigma_{12}}{\det(\hat{H})} (v_2^1 - p_2^1)},$$

$$= - \left( \hat{H}_1 \times (v_1 - p_1) \right) x_1^1 + \frac{1}{\gamma} \left( G + (I - (H^{-1})^T) p_1 \right) \times (v_1 - p_1) \frac{1}{\hat{H}_2 \times (v_1 - p_1)}$$

(2.13)

We see that the speculator’s demand for asset 2 is affine in her share of asset 1. This is intuitive since the impact on the margins is linear in the inventory of the other assets. The matrix $\hat{H}$ is the weight the financier puts on the the expected values of the assets when the inventory changes, thus on the margins. Similarly, the matrix, $H^{-1}$ is the weight of the other variables. The second row of $\hat{H}$ is the weight on the expected value of the second asset, and we see that denominator in the relation is the cross-product with respect to the difference of the fundamentals and the prices, $v_1 - p_1$.

**Demand of the speculator**

![Demand of the speculator](image)

Figure 3: The share of $x^2$ as a function of $(\sigma_1, \sigma_2)$, with a correlation of $|\rho| = 0.5$ and $x^1 = 25$. On the left with negative correlation, positive on the right.
Demand of the speculator

![Graph showing demand of the speculator](image)

Figure 4: The share of $x^2$ as a function of the variance, with a correlation of $|\rho| = 0.5$ and $x^1 = 25$. On the left with negative correlation, positive on the right.

In figure 3 we have the share of asset two, $x^2$, and we clearly see the effect of flight to quality. Ceteris paribus, the speculator will invest more in the asset with lowest volatility since this has a lower impact of the capital restriction. In figure 4, we have the share of asset two, $x^2$, as a function of the variance of $z^1$. On the left the returns are negatively correlated, and we see that the change in holdings is very small. We see that the share is increasing in $\eta_2$, but decreasing in $\eta_1$. Thus, the speculator will have a larger share in the asset where the variance of the customers’ demand is largest. Intuitively, if $\eta_i$ is small then the financier is more certain that there has been a change in $v$ instead of an increase in the supply of the assets. The situation is more complex if the returns are positively correlated on the right. If $\eta_1$ is low, we have the same situation as above, but as we increase $\eta_1$ we decrease the share more in the area where $\eta_2$ is also high. The result is that the share is decreasing in $\eta_2$ when $\eta_1$ is high.

3 Equilibrium

Now we examine the equilibrium and consider the margins set by the financier. First, we only consider one risky asset to examine how the trading of a single asset affects the margins and equilibrium.

\footnote{The graph does not change qualitatively with a negative correlation.}
We consider competitive equilibria of the economy:

**Definition 3.** An *equilibrium* is a price process $p_t$ such that

1. $x_t$ maximizes the speculators’ expected profit subject to the capital constraint.
2. Each $y^k_t$ maximizes customer $k$’s expected utility after their arrival at the market and is zero beforehand.
3. The financier correctly update expectations about the assets and sets the margins given his information using Lemma 1.
4. Markets clear: $x_t + \sum_{k=0}^{t} y^k_t = 0$ at each period $t$.

Equilibrium before time 2 depends on the shocks by the customers, $z^k$. We use the demand functions for the customers from above with the equilibrium conditions to find the equilibria. We define equilibrium as *fragile* or *unstable* if excess demand for shares can be non-monotonic in price and prices cannot be chosen continuously in exogenous shocks, $\kappa$ and $\Delta \nu$. Then, if prices increase, the decrease in supply by the customers is not as large as the decreasing demand or possible demand by the speculator. The equilibrium prices do not have explicit formulas but are found with numerical solutions.

### 3.1 Equilibrium at time 1 with a single asset

In a market with only one asset, then we have that the supply of the customers and allowed demand of the speculator must clear and the equilibrium is *fragile* if small changes in the price makes this clearing fail. We use the definition by Brunnermeier and Pedersen (2006):

**Definition 4.** Fragile equilibrium for a single asset. An equilibrium, $p < v$ is fragile if there exists $\delta > 0$ such that for all $\varepsilon \in (-\delta, \delta)$:

$$
\frac{(-\sum_k y^k_t (p + \varepsilon)) - \frac{W_t(p+\varepsilon)}{m(p+\varepsilon)}}{\varepsilon} < 0.
$$

(3.1)

The first term in equation (3.1) is the supply of the customers while the second term is the amount of wealth the speculator has to buy the asset. If $\varepsilon < 0$ and price changes to $p + \varepsilon$, the speculator’s wealth constraint tightens forcing him to sell (the second term in the nominator is lower). If the supply from the customers is not decreased equivalently, however, there is excess supply and the price would have to fall even further, making the equilibrium fragile. Similarly, if $\varepsilon > 0$ and the increase in the wealth of the speculator is larger than the increased supply, the price will converge towards $v_1$. 


Proposition 5. **Fragile equilibrium with only one asset.** There exist \( \bar{x} \) such that for \( |x_0| > \bar{x} \) and of the same sign as \( Z_1 \), the equilibrium is fragile. The speculator has to have an initial position of

\[
x_0 > (v_1 - v_0) \left( \frac{1 - \theta}{1 + \theta} \right) \frac{2}{\gamma \sigma^2} - \frac{Z_1^1 1 - \theta}{1 + \theta} + \frac{2}{\gamma \sigma^2} \frac{1}{1 + \theta} \left( \tau \Phi^{-1}(1 - \pi) + \frac{\theta \Lambda_0}{2} \right)
\]

(3.2)

for this to be possible. Hence, if the inequality is fulfilled in an equilibrium, the equilibrium is fragile.

The result is similar to Brunnermeier and Pedersen (2009, proposition 3.). We see that the critical share is increasing in \( Z_1 \) and decreasing in the probability \( \pi \). We are mostly interested in understanding how the critical value depends on the volatility of the asset and the variance of the financier’s prior, \( z_1 \).

**The critical value**

![Figure 5: The critical value, \( \bar{x} \) as a function of the volatility, \( \sigma_v \) and the variance of variance of the prior’s belief, \( z_1 \).](image)

From figure 5, we see that the critical value of decreasing in both the volatility of the assets and the variance of \( z_1 \) (the parameters are from Table 1.) Hence, the financial market is more stable the less volatile and the better informed the financier is.

Brunnermeier and Pedersen (2009) defines that the market is in a liquidity-spiral if \( \frac{\partial m^i}{\partial p_1} > 0 \) and a loss-spiral if \( x_0 Z_1 > 0 \). Below we see that in the single asset market no liquidity-spiral will occur.
Corollary 6. The margin for long position in equilibrium is given by

$$m^+_1 = \max \left\{ \Phi^{-1}(1-\theta) \tau - v_0(1-\theta) + p_1 - \theta v_1 + \frac{1}{2} \theta \gamma \sigma^2 (Z_1 - x_0) + \frac{1}{2} \theta \Lambda_0, 0 \right\}, \quad (3.3)$$

and for a short position

$$m^-_1 = \max \left\{ \Phi^{-1}(1-\theta) \tau + v_0(1-\theta) - p_1 + \theta v_1 - \frac{1}{2} \theta \gamma \sigma^2 (Z_1 - x_0) - \frac{1}{2} \theta \Lambda_0, 0 \right\}. \quad (3.4)$$

We have that with only one asset there is no margin spiral since the derivative with respect to $p$ is 1 for the long position. Intuitively, this is because, although the financier does not observe any changes in fundamentals or liquidity shocks, she knows that the changes are only in the one asset. Thus, the financier does not have to fear ending up with the bad assets since these could not be sold by the speculator, and the market is "only" in a loss spiral.

### 3.2 Numerical example

We illustrate the problem of illiquid equilibria. We first show how the supply and demand changes with respect to a given price. Then, we illustrate the equilibrium prices dependent on the initial share, $x_0$, the fundamental value of the asset, $v_1$, and the capital shock of the speculator $\kappa$. The base case of the parameters are given in table 2. In figure 6, we see the supply of the customers and the demand of the speculator—the share she is allowed to buy—dependent on the given price. The demand is clearly decreasing in price and increasing in initial share, $x_0$, and the demand is decreasing slower with a larger initial share. We clearly see that the equilibrium price is increasing in $x_0$.\(^{18}\) This is also consistent with figure 7 where the equilibrium price is found dependent on the initial share of the speculator. We see two distinct equilibria: The solid line is the stable equilibrium and the dashed line is the unstable. The stable equilibrium is the liquid equilibrium, where the speculator is allowed to buy enough until the price is inelastic of shocks to the speculator. The fragile equilibrium occurs when $x_0$ is lower than the liquidity shock at time 1, $Z_1 = 30$. In figure 8 and 9, we have the equilibrium price dependent on the change on fundamental, $v_1 - v_0$, and the capital shock, $\kappa$. In figure 8 we see that for large decreases in $\Delta v_1$, the two stable equilibria have the same price, while with smaller decreases or increases the equilibrium price is larger when the speculator has initial holdings. However, this equilibrium becomes fragile with larger increases in $\Delta v_1$, and the price does not increase as much as with the 'no initial holdings' equilibrium. In the same

\(^{18}\)Note that we hold $W_0$ fixed.

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figure we have an equilibrium that is fragile and the price is lower than for the two other cases. Note that the initial holdings, $x_0$, is larger than the aggregate liquidity shock, $Z^1$, and therefore the speculator will be selling instead of buying, so when we have increases in $\Delta v_1$ the equilibrium may become fragile. In figure 9 we notice that the illiquid and fragile equilibrium does not depend on the capital shock in the range that has been chosen.\footnote{The speculator is almost bankrupt since $p_0 > p_1$.} Although we do not have liquidity spirals with increasing margins, the speculator is still exposed to loosing spirals and the financial market may be fragile.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\sigma$</th>
<th>$\eta$</th>
<th>$v_0$</th>
<th>$p_0$</th>
<th>$v_1$</th>
<th>$Z_1$</th>
<th>$W_0$</th>
<th>$\kappa$</th>
</tr>
</thead>
<tbody>
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<td>10</td>
<td>5</td>
<td>110</td>
<td>110</td>
<td>120</td>
<td>30</td>
<td>1500</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 2: Base case parameters of the example.

3.3 Equilibrium at time 1 with two assets

In this section, we show that with an active speculator, the market may be subjected to liquidity spirals. We first show that this is not the case for the passive speculator. The

Figure 6: The supply and demands of the customers and the speculator. The solid increasing line is the supply of the customers.
Figure 7: The equilibrium price as a function of initial share of the speculator.

reason is that there is no information release when the speculator has to sell and prices fall. This is due to the fact that the speculator does not take price and margin changes into account and just strades the asset at whatever price needed.\footnote{This is similar to the results in Kyle (1985) versus the results in Glosten and Milgrom (1985).}

**Proposition 7.**

1. With a passive speculator no liquidity spirals will occur.

2. In a market, where the speculator profit maximizes, there exist some interval, $I = [x, \bar{x}]$ such that for $x_0 \in I \times I$, liquidity spirals may occur.

With multiple assets in the market, the market may be fragile in one asset due to shocks in another asset:

**Definition 8.** The market is said to be fragile if:

$$\exists \delta > 0 : \forall \varepsilon \in [-\delta, \delta] \exists j \in J :$$

$$- \left( \sum_k y^{ki}(p + \varepsilon 1_j) - \frac{W_i(p + \varepsilon 1_j) - m_i(p + \varepsilon 1_j)(-\sum_k y^{kj}(p + \varepsilon 1_j))}{m_i(p + \varepsilon 1_j)} \right) < 0,$$

(3.5)

where $1_j \in \mathbb{R}^J$ has 1 in the j’th coordinate and zero in the rest.

The intuition is the same as in the single asset case; here the change is that the margin and supply of both assets can make the distortion needed such that the equilibrium is fragile. The general definition of a fragile market has the same intuition, however there may be spillover-effects in the supply-demand equilibrium.
Proposition 9. 1. Given a set of equilibrium prices, \( p \in (v - \frac{\gamma}{2} \Sigma_v Z, v) \), there exist a set of initial holdings \( x_0 \) such that the market is fragile.

2. The set of holdings depend on the correlation: If the correlation is zero, there is a critical minimum for both assets. If the correlation is negative, \( x_0^2 \) is an increasing function of \( x_0^1 \). If the correlation is positive, the set is a half plane.

With a correlation of zero, we have the same result as in proposition 5, and the conditional initial holding of asset 2 for the equilibrium to be fragile with negative correlation is increasing in asset 1. With negative correlation there is less probability that the speculator will have losses on both assets, and the customers have higher demand with negative correlation due to hedging effect. Hence, the supply is less with negative correlation. The result is less intuitive with positive correlation in the returns: If the speculator has a large share in asset 1, she makes the price-1-equilibrium unstable, however it has an stabilizing effect on asset two, see equation (A.52). Hence, if the speculator has a large share in both assets the equilibrium may be stable.\(^{21}\) Although the market in fragile if the supply - demand does not hold for just one asset, we keep distinguishing between in which price direction the market is fragile. We assume that the speculator is active.

\(^{21}\)In the numerical examples this occurs only when the liquid equilibrium is reached.
Figure 9: The equilibrium price as a function of $\kappa_1$

**Numerical example**

We use the same example as with one asset and the same base-case parameters, now for both assets. We include the results with positive and negative correlation, and set $|\rho| = 0.5$. For all of the figures, we have the price of asset 1 on the left and the price of asset 2 on the right.

In figure 13, we have the price depending on $x_0^1$ with negatively correlated returns. In the blue line we have $x_0^2 = 0$ and we see that both prices are increasing in $x_0^1$ since the speculator is allowed to buy more and the price is larger than $p_0^1$. At some point the speculator is allowed to buy of the supply in both assets and the price jumps to the liquid equilibrium. The red line is the equilibrium with $x_0^2 = 40$: We see that the price is higher than with no initial holdings in the low $x_0^1$ region. However, the price-jump to the liquid price happens for a higher $x_0^1$. Before this, the equilibrium has an unstable region. The price is higher for small regions of $x_0^1$ since the speculator is allowed to buy more with initial gains and the margin is not too high. For higher regions of $x_0^1$ and $x_0^2 = 0$, the margins will be low enough for the speculator buy the supplied assets. In figure 10, we have the figure with positive correlation. We see that the equilibrium with $x_0^2 = 40$ is fragile—first in asset 2, then for both assets—only the liquid equilibria are stable. The equilibrium with $x_0^2 = 0$ is stable for low values of $x_0^1$, however for values higher than 30, the prices are fragile until the prices jump to the liquid equilibrium. Changes in the
equilibria happens for qualitatively the same prices for positive and negative correlation. However, with negative correlation the prices jump to the liquid region, whereas with positive correlation the equilibrium becomes unstable. We clearly see that the equilibria are more stable with negative correlation.

In figure 14, we have the equilibrium prices depending on the change in the fundamental for asset 1, $\Delta v_1$, with negative correlation. The equilibrium with $x_0 = 0$ is stable and increasing in $p^1$. However, the price of asset is decreasing although almost constant. The speculator is not allowed to buy the entire supply and the equilibria are illiquid. In the set where $x_0 = 40$, we see that the prices of assets equal the fundamentals for low values of $\Delta v_1$ since the speculator can buy the entire asset. In the region where $\Delta v_1$ is positive, the equilibria becomes fragile since the speculator is selling, but the fundamentals are increasing. In figure 11, we have positive correlation between the returns. We have qualitatively the same figure. However, the prices are lower compared to the negative correlation figure, which is to be expected since margins and the supplies are higher.\footnote{We have that the entries in the matrix $\hat{H}$ is larger with positive correlation and from equation (A.34) we see that the margins are larger with positive correlation.}

The equilibrium depending on the variance of the prior belief of the financier with negatively correlated returns is depicted in figure 15. Recall that the variance of the prior belief is a measure of information quality of the financier. The no-initial-holdings equilibrium has increasing price in asset one, while a slowly decreasing price in asset two, similar to figure 14. Higher variance in $z^1$ help better explain, why the speculator buys in period 1, and she is allowed to buy more. We therefore have a higher price. Due to the negatively correlated returns, the price in asset two is decreasing (we also see that with positive correlation the price is increasing from Figure 12.) When the speculator has initial holdings, the equilibrium is initially liquid. However, for larger values of $\eta_1$ the price drops, is decreasing, and eventually the equilibrium is fragile. When the returns are positively correlated and the speculator has initially no holdings both prices are increasing but still lower than in with negatively correlated returns. When the speculator has initial holdings, the shape of the prices are the same as with negatively correlated returns, but as in Figure 11, the equilibria are fraile. From these figures, we see that when speculators invest in assets that are positively correlated, prices are lower and markets are more likely to be fragile. We have also have that information quality is important in keeping the market stable.
Figure 10: The equilibrium price depending on $x_0^1$ with $x_0^2 = 0$ (blue) and $x_0^2 = 40$ (red) with positive correlation.

4 Empirical predictions

The analysis in this paper has some new testable predictions regarding margins for speculators set by financiers and predictions regarding fragile financial markets. The margin predictions can be tested with data from prime brokers while predictions regarding fragile financial markets can be tested using inventory data from large hedge funds. The first prediction is regarding the margins: Margins are smaller for speculators with large initial holdings. This is tested directly by regressing the margins against the holdings of the speculator in the previous period. Margins also depend on the variance of the prior beliefs of the financier. This can be tested by regressing the margins against characteristics of the speculator such as the period length, the speculator has used the broker. A broker should be more familiar with a client that he has had for longer time. Another possible variable is the number of trades with gains the speculator has done or does in every period of time. A speculator that engages in more trades with positive outcomes has a better knowledge in the market and should be expected to have smaller margins. The margins depend asymmetrically on the trade of the speculator, whether or not the speculator is buying or selling - especially if the sign of the holdings change, which could also be tested by regressing the margins against the specific trades the speculator has done.

From the numerical example we saw that when speculators invest in assets that are
positively correlated, the financial market is more inclined to be fragile. This could be tested using data from the financial crises such as the LTCM crises and the convertible bond crises, and the test is if the hedge funds invested prior to these crises, especially if they invested in a small group of assets with positively correlated returns. Some anecdotal evidence already suggest that this was the case: From Mitchell et al. (2007), we had that hedge funds who had more diversified trading strategies survived and eventually bought the cheap assets.

5 Conclusion

This paper considers how margins are set by financiers for speculators who provide liquidity in the financial market. The financiers do not observe the changes in expected value of the assets and have to rely on the prices and the inventory of the speculator to make the correct inferences regarding the assets. It is shown that the margins depend on the initial holdings of the speculator and the illiquidity in previous periods. The margins can be both increasing or decreasing in the variance of the prior belief of the financier, depending on the specific trade the speculator does and also on the initial holding of the speculator.

We have that the equilibrium prices may become fragile in the sense that small changes in the price creates an imbalance in the supply and demand which causes the price to
Figure 12: The equilibrium price depending on $\eta_1$ with $\eta_2 = 5$ and positive correlation. The blue line has $x_0 = 0$ and the red line has $x_0 = 40$

change even further. We therefor have that prices cannot be chosen to being continuous functions of the shocks in the economy. The speculator may run in a loosing spiral where losses leads to forced sales which leads to even further losses. Similarly, there may be liquidity spirals where the margins are increasing when the prices are decreasing. This happens since the financier fears that the speculator is selling the good liquid assets where the price-decreases are smaller. If this happens, the financier might end up with the bad illiquid assets, which the financier cannot sell. Finally, the paper has empirical predictions regarding the margins set by for instance prime brokers and how financial crises are started by leveraged trading of large speculators that exploit market imbalances.
Equilibrium prices

Figure 13: The equilibrium price depending on $x_0$ with $x_0 = 0$ (blue) and $x_0 = 40$ (red) with negative correlation.

Equilibrium prices

Figure 14: The equilibrium price depending on $\Delta v_1$ with negative correlation. The blue line has $x_0 = 0$ and the red line has $x_0 = 40$. 
Equilibrium prices

Figure 15: The equilibrium price depending on $\eta_1$ with $\eta_2 = 5$ and negative correlation. The blue line has $x_0 = 0$ and the red line has $x_0 = 40$
A Proofs

First we describe the matrices used in the paper: We have that the variance matrices are given by

\[ \Sigma_v = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}, \quad \Sigma_z = \begin{pmatrix} \eta_1^2 & 0 \\ 0 & \eta_2^2 \end{pmatrix}. \]  

(A.1)

From equation (A.8) we get

\[ \Sigma_{\Psi} = \frac{\gamma^2}{4} \Sigma_v \Sigma_z \Sigma_v + \Sigma_v = \Sigma_v H, \]  

(A.2)

\[ \Sigma_{\Psi}^{-1} = H^{-1} \Sigma_v^{-1}, \]  

(A.3)

where

\[ H = \begin{pmatrix} \frac{\gamma^2}{4} \sigma_1^2 \eta_1^2 + 1 & \frac{\gamma^2}{4} \sigma_{12} \eta_1^2 \\ \frac{\gamma^2}{4} \sigma_{12} \eta_1^2 & \frac{\gamma^2}{4} \sigma_2^2 \eta_2^2 + 1 \end{pmatrix}. \]  

(A.4)

From equation (A.12) we need

\[ \Sigma_v \Sigma_{\Psi}^{-1} = -\Sigma_v \Sigma_{\Psi}^{-1} = -(H^{-1})^T. \]  

(A.5)

\[ (H^{-1})^T \Sigma_v = \det(H)^{-1} \begin{pmatrix} \frac{\gamma^2}{4} (\sigma_2^2 \eta_2^2) + 1 & -\frac{\gamma^2}{4} \sigma_{12} \eta_2^2 \\ -\frac{\gamma^2}{4} \sigma_{12} \eta_1^2 & \frac{\gamma^2}{4} \sigma_1^2 \eta_1^2 + 1 \end{pmatrix} \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} \]  

(A.6)

\[ = \det(H)^{-1} \begin{pmatrix} \frac{\gamma^2}{4} \det(\Sigma_v) \eta_2^2 + \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \frac{\gamma^2}{4} \det(\Sigma_v) \eta_1^2 + \sigma_2 \end{pmatrix} \equiv \hat{H}. \]  

(A.7)

Proof of Lemma 1

Proof. The signal \( \Psi_1 \) is normally distributed with mean \( E[\Psi_1] = -v_0 \), variance matrix \( \Sigma_\Psi \), and covariance with \( v_1 \) denoted as \( \Sigma_{v\Psi} \). The conditional variance is denoted \( \Sigma_{v|\Psi} \). We have that

\[ \Sigma_\Psi = \frac{\gamma^2}{4} \Sigma_v \Sigma_z \Sigma_v + \Sigma_v. \]  

(A.8)

The covariance between \( \Psi_1 \) and \( v_1 \) is given by

\[ \Sigma_{v\Psi} = -\Sigma_v. \]  

(A.9)

The conditional variance is given by

\[ \Sigma_{v|\Psi} = \Sigma_v - \Sigma_{v\Psi} \Sigma_{\Psi}^{-1} \Sigma_{v\Psi}. \]  

(A.10)

The financier then has conditional expectation given as

\[ v_{1|\Psi} \equiv E[v_1 \mid \Psi] = E_0 [v_1] + \Sigma_{v\Psi} \Sigma_{\Psi}^{-1} (\Psi + v_0), \]  

(A.11)

\[ = (I + \Sigma_{v\Psi} \Sigma_{\Psi}^{-1}) v_0 + \Sigma_{v\Psi} \Sigma_{\Psi}^{-1} \left[ \frac{\gamma^2}{2} \Sigma_v \Delta x_1 - p_1 + \frac{1}{2} \Lambda_0 \right]. \]  

(A.12)
From equations (A.4), (A.5), and (1.11) we have that the conditional expected value is

$$v_1|\Psi = (I + \Sigma_\psi \Sigma_\psi^{-1}) v_0 + \Sigma_\psi \Sigma_\psi^{-1} \left[ \frac{\gamma}{2} \Sigma_\nu \Delta x_1 - p + \frac{1}{2} \Lambda_0 \right], \quad (A.13)$$

$$= (I - (H^{-1})^T) v_0 - (H^{-1})^T \left[ \frac{\gamma}{2} \Sigma_\nu \Delta x_1 - p_1 + \frac{1}{2} \Lambda_0 \right], \quad (A.14)$$

$$= (I - (H^{-1})^T) v_0 - (H^{-1})^T \left[ \frac{1}{2} \Lambda_0 - p_1 - \frac{\gamma}{2} \hat{H} \Delta x_1 \right]. \quad (A.15)$$

From equations (A.4) and (A.5) the margin is:

$$m_j^+ = \hat{\tau}_j - v_{1|\psi} + p_1, \quad (A.16)$$

$$= \hat{\tau}_j + p_1 - \left( (I - (H^{-1})^T) v_0 - (H^{-1})^T \left[ \frac{1}{2} \Lambda_0 - p_1 - \frac{\gamma}{2} \hat{H} \Delta x_1 \right] \right), \quad (A.17)$$

$$= \hat{\tau}_j + ((H^{-1})^T - I) v_0 + (I - (H^{-1})^T) p_1 + \frac{1}{2} (H^{-1})^T \Lambda_0 + \frac{\gamma}{2} \hat{H} \Delta x_1. \quad (A.18)$$

Proof of Proposition 2

By use of the normal distribution we get

$$m_1^+ = \tau \Phi (1 - \pi) - v_{1|\psi} + p_1, \quad (A.19)$$

$$= \tau \Phi (1 - \pi) - \left[ v_0 \left( 1 - \frac{\sigma^2}{\gamma^2 / 4\sigma^4 \eta^2 + \sigma^2} \right) + \frac{\sigma^2}{\gamma^2 / 4\sigma^4 \eta^2 + \sigma^2} \left( -\frac{\sigma^2}{2} \Delta x_1 + p_1 - \frac{1}{2} \Lambda_0 \right) \right] + p_1, \quad (A.20)$$

$$= \tau \Phi (1 - \pi) - v_0 (1 - \theta) + \theta \left( \frac{\gamma}{2} \sigma^2 \Delta x_1 - p_1 + \frac{1}{2} \Lambda_0 \right) + p_1, \quad (A.21)$$

$$= \tau \Phi (1 - \pi) - v_0 (1 - \theta) + p_1 (1 - \theta) + \frac{\theta}{2} \left[ \gamma \sigma^2 \Delta x_1 + \Lambda_0 \right]. \quad (A.22)$$

Proof of Proposition 5.

We assume that $Z_1 > 0$ meaning $v_1 \geq p_1$ and $x_1 \geq 0$. We insert the equilibrium condition $x_1 = -\sum_{k=0}^1 y_k$.

$$m_1^+ \left[ Z_1 - \frac{2}{\gamma \sigma^2} (v_1 - p_1) \right] \leq W_0 - x_0 p_0 + x_0 p_1 + \kappa \Leftrightarrow \quad (A.23)$$

$$m_1^+ \left[ Z_1 - \frac{2}{\gamma \sigma^2} (v_1 - p_1) \right] - c_0 - x_0 p_1 \leq \kappa. \quad (A.24)$$

We leave out that the margin has to be non-negative.
We define the function $G$ as

$$G(p) := m_1 \left[ Z_1 - \frac{2}{\gamma \sigma^2} (v_1 - p_1) \right] - c_0 - x_0 p_1,$$

(A.25)

$$= \left[ \tau \Phi^{-1}(1 - \pi) - v_0(1 - \theta) + p_1 (1 - \theta) + \frac{1}{2} \theta \left[ \gamma \sigma^2 \Delta x_1 + \Lambda_0 \right] \right]$$

$$\times \left[ Z_1 - \frac{2}{\gamma \sigma^2} (v_1 - p_1) \right] - c_0 - x_0 p_1,$$

(A.26)

$$= \tau \Phi^{-1}(1 - \pi) - v_0(1 - \theta) + p_1 + \frac{\theta}{2} \left( \gamma \sigma^2(Z^1 - x_0) + \Lambda_0 \right) \times \left[ Z^1 - \frac{2}{\gamma} (v_1 - p_1) \right]$$

$$- c_0 - x_0 p_1.$$

(A.27)

Along with Brunnermeier and Pedersen (2006) the equilibrium is fragile, if $G$ can be decreasing in $p_1$.\footnote{See Brunnermeier and Pedersen (2006) for details.} Multiplying out, we get an equation on the form:

$$G(p_1) = p_1^2 \frac{2}{\gamma \sigma^2} + p_1 \left( \frac{2}{\gamma \sigma^2} \Phi^{-1}(1 - \pi) \tau - x_0 (\theta + 1) + (\theta + 1) \left( Z_1 - \frac{2}{\gamma \sigma^2} v_1 \right) \right)$$

$$+ \frac{\theta}{\gamma \sigma^2} \Lambda_0 + \frac{2}{\gamma \sigma^2} v_0(1 - \theta) + \left[ Z_1 - \frac{2}{\gamma \sigma^2} v_1 \right]$$

$$\times \left[ \Phi^{-1}(1 - \pi) \tau - \frac{\theta}{2} \left( \gamma \sigma^2 \left( Z_1 - \frac{2}{\gamma \sigma^2} v_1 - x_0 \right) + \Lambda_0 \right) - v_0(1 - \theta) \right] - b_0.$$

(A.28)

The solution depends the position of the vertex: The vertex has to be larger than the minimum price $p_1$, where the speculator is in default. The price is

$$p_1 = v_1 - Z_1 \frac{\gamma \sigma^2}{2}.$$

(A.29)

We have the solution is given by

$$x_0 > \left[ \frac{4}{\gamma \sigma^2} v_1 - 2 Z^1 + \frac{2}{\gamma \sigma^2} \left( \tau \Phi^{-1}(1 - \pi) - v_0(1 - \theta) + \frac{\theta}{2} \Lambda_0 \right) + (\theta + 1) \left( Z^1 - \frac{2}{\gamma \sigma^2} v_1 \right) \right]$$

$$\times (1 + \theta)^{-1}.$$

(A.30)

**Proof of proposition 7**

First we prove the first statement: The margin for asset on in given by:

$$m^1 = G_1 + \left( I - (H^{-1})^\top \right)_1 \mathbf{p} + \frac{\gamma}{2} \hat{H}_{11} x^1 + \frac{\gamma}{2} \hat{H}_{12} x^2,$$

(A.31)

$$= G_1 + (1 - H_{11}^{-1}) \mathbf{p}^1 - (H^{-1})_{12} p^2 + \frac{\gamma}{2} \hat{H}_{11} \left[ Z^1 - \frac{2}{\gamma} \Sigma^{-1}_v (v - \mathbf{p}) \right] + \frac{\gamma}{2} \hat{H}_{12} \left[ Z^2 - \frac{2}{\gamma} \Sigma^{-1}_v (v - \mathbf{p}) \right],$$

(A.32)

$$= p^1 \left( 1 - H_{11}^{-1} + \hat{H}_{11} \Sigma^{-1}_v + \hat{H}_{12} \Sigma^{-1}_v \right) + p^2 \left( (H^{-1})_{12} + \hat{H}_{11} \Sigma^{-1}_v + \hat{H}_{12} \Sigma^{-1}_v \right) + G_1$$

$$+ \hat{H}_{11} \left[ \frac{\gamma}{2} Z^1 - \Sigma^{-1}_v v \right] + \hat{H}_{12} \left[ \frac{\gamma}{2} Z^2 - \Sigma^{-1}_v v \right],$$

(A.33)

$$= p^1 + G_1 + \hat{H}_{11} \left[ \frac{\gamma}{2} Z^1 - \Sigma^{-1}_v v \right] + \hat{H}_{12} \left[ \frac{\gamma}{2} Z^2 - \Sigma^{-1}_v v \right].$$

(A.34)
We now prove the second statement: We insert the condition from the speculator’s demand in the margin for asset 1:

\[ m^1 = G_1 + (I - (H^{-1})^\top) p + \frac{\gamma}{2} v^1 \left( \hat{H}_1 + \frac{1}{\hat{H}_2} \frac{\hat{H}_1 \times (v - p)}{(v - p) \times \hat{H}_2} \right) \]

\[ \quad + \frac{1}{2} \hat{H}_{12} \frac{G + (I - (H^{-1})^\top) p \times (v - p)}{(v - p) \times \hat{H}_2}. \]  

(A.35)

The derivative with respect to \( p^1 \) is given by

\[ \frac{\partial m^1}{\partial p^1} = -1 - H_{11}^{-1} + \frac{1}{2} \hat{H}_{12} \left( \frac{G_2}{(v - p) \times \hat{H}_2} + \hat{H}_{22} \frac{G \times (v - p)}{(v - p) \times \hat{H}_2} + \frac{\hat{H}_1 \times (v - p)}{(v - p) \times \hat{H}_2} \right)^2 \]

\[ + \frac{\gamma}{2} Z \hat{H}_{12} \left[ \hat{H}_{12} \left( \frac{\hat{H}_1 \times (v - p)}{(v - p) \times \hat{H}_2} \right)^2 + \hat{H}_{22} \frac{\hat{H}_1 \times (v - p)}{(v - p) \times \hat{H}_2} \right] + \hat{H}_{11} \Sigma_{\nu}^{-1} \]

\[ + \frac{\hat{H}_{12}}{(v - p) \times \hat{H}_2} \left[ \Sigma_{\nu}^{-1} \hat{H}_1 \times (v - p) - \Sigma_{\nu}^{-1} (v - p) \right] - \Sigma_{\nu}^{-1} (v - p) \hat{H}_{22} \frac{\hat{H}_1 \times (v - p)}{(v - p) \times \hat{H}_2} \]  

(A.36)

Given a value for \( y = (v - p) \times \hat{H}_2 \) we solve as a function of \( y \):

\[ 0 = \frac{1}{2} \hat{H}_{12} G_2 + \frac{1}{2} \hat{H}_{12} \left( \gamma (1 - H_{11}^{-1}) + v^1 (H^{-1})_{21} - 2p^1 (H^{-1})_{21} + p^2 \left( H_{11}^{-1} - H_{22}^{-1} \right) \right) \]

\[ \quad + \frac{\gamma}{2} Z (\hat{H}_{12})^2 + \hat{H}_{12} \left( \Sigma_{\nu}^{-1} \hat{H}_1 \times (v - p) - \Sigma_{\nu}^{-1} (v - p) \right) \]

\[ \quad + y^2 \left[ 1 - \hat{H}_{11} + \hat{H}_{11} \Sigma_{\nu}^{-1} \right] \]

\[ \quad + \frac{1}{2} \hat{H}_{12} \hat{H}_{22} \cdot G + \hat{H}_{22} (I - (H^{-1})^\top) p - \Sigma_{\nu}^{-1} (v - p) \hat{H}_{22} \cdot \hat{H}_1 \right] \times (v - p), \]  

(A.38)

\[ \equiv A y^2 + B y + C. \]  

(A.39)

Set \( D = B^2 - 4AC \) and the condition is fulfilled if

\[ y < \frac{-B \pm \sqrt{D}}{2A}. \]  

(A.40)

The term \( D \) is increasing in \( x_0 \) due to the square of \( B \) (recall that \( x_0 \) only appears in \( B \) from \( G \).)

For the term regarding \( x_0 \) in \( B \), we total multiply by

\[ \check{B} = -\frac{\gamma}{2} \hat{H}_{12} \hat{H}_2 = \begin{cases} -\frac{\gamma}{2} \frac{\sigma_{12}}{\det(H)} \sigma_{12}^2 \det(H) & \text{for asset 1} \\ -\frac{\gamma}{2} \frac{\sigma_{12}}{\det(H)} \frac{\sigma_{12}^2 + \sigma_2^2}{\det(H)} & \text{for asset 2} \end{cases} \]  

(A.41)

Hence from equation (A.40), the solution is increasing in \( x_0 \) if the term in (A.41) is negative. This is clearly fulfilled for asset 1. Regarding asset 2 the solution is fulfilled for

\[ \sigma_{12} > -\frac{\gamma^2 \det \Sigma_{\nu}^2}{4 \sigma_2^2}. \]  

(A.42)
Proof of Proposition 9

From the proof of Proposition 7 we have that in equilibrium we have

\[ m_1 = p_1 + G + H_{11} \left[ \frac{\gamma}{2} Z^1 - \Sigma_{e1}^{-1} v \right] + H_{12} \left[ \frac{\gamma}{2} Z^2 - \Sigma_{e2}^{-1} v \right]. \]  

(A.43)

Similar we have for asset 2:

\[ m_2 = p_2 + G + H_{21} \left[ \frac{\gamma}{2} Z^1 - \Sigma_{e1}^{-1} v \right] + H_{22} \left[ \frac{\gamma}{2} Z^2 - \Sigma_{e2}^{-1} v \right]. \]  

(A.44)

where

\[ G_j = \tilde{\tau}_j - (I - (H^{-1})^\top) v_0 + \frac{1}{2} (H^{-1})^\top \Lambda_0 - \frac{\gamma}{2} H_j x_0. \]  

(A.45)

As for the single asset case, we define

\[ G(p_1, p_2) = x_1^\top m_1 + x_2^\top m_2 - W_1, \]  

(A.46)

and similar from Proposition 5 the equilibrium is unstable if \( G \) can be decreasing. Denote

\[ F^1 = G_1 + H_{11} \left[ \frac{\gamma}{2} Z^1 - \Sigma_{e1}^{-1} v \right] + H_{12} \left[ \frac{\gamma}{2} Z^2 - \Sigma_{e2}^{-1} v \right], \]  

(A.47a)

\[ F^2 = G_2 + H_{21} \left[ \frac{\gamma}{2} Z^1 - \Sigma_{e1}^{-1} v \right] + H_{22} \left[ \frac{\gamma}{2} Z^2 - \Sigma_{e2}^{-1} v \right]. \]  

(A.47b)

By repeated insertion of the equilibrium condition we get:

\[ G(p_1, p_2) = \left[ Z^1 - \frac{2}{\gamma} \Sigma_{e1}^{-1} (v - p) \right] [F^1 + p^1] - W_0 - x_0^\top (p_1 - p_0) \]

\[ + \left[ Z^2 - \frac{2}{\gamma} \Sigma_{e2}^{-1} (v - p) \right] [F^2 + p^2], \]  

(A.48)

\[ = \frac{2}{\gamma} (\Sigma_{e1}^{-1} p_1^1)^2 + p_1^1 p_2^1 \left( \frac{2}{\gamma} \Sigma_{e12}^{-1} + \frac{2}{\gamma} \Sigma_{e21}^{-1} \right) + \frac{2}{\gamma} \Sigma_{e22}^{-1} (p_2^2)^2 \]

\[ + p_1^1 \left( \frac{2}{\gamma} (\Sigma_{e11}^{-1} F^1 + \Sigma_{e21}^{-1} F^2) - x_0^1 + Z^1 - \frac{2}{\gamma} \Sigma_{e1}^{-1} v \right) \]

\[ + p_2^2 \left( \frac{2}{\gamma} (\Sigma_{e12}^{-1} F^1 + \Sigma_{e22}^{-1} F^2) - x_0^2 + Z^2 - \frac{2}{\gamma} \Sigma_{e2}^{-1} v \right) - W_0 + x_0^\top p_0 \]

\[ + F^1 \left[ Z^1 - \frac{2}{\gamma} \Sigma_{e1}^{-1} v \right] + F^2 \left[ Z^2 - \frac{2}{\gamma} \Sigma_{e2}^{-1} v \right]. \]  

(A.49)

We need that \( G \) is decreasing in both prices:

\[ \frac{\partial G}{\partial p_1} = \frac{4}{\gamma} (\Sigma_{e11}^{-1} p_1^1 + \Sigma_{e12}^{-1} p_2^1) + \frac{2}{\gamma} (\Sigma_{e1}^{-1} K_1^1 + \Sigma_{e21}^{-1} K_2^2) + Z^1 - \frac{2}{\gamma} \Sigma_{e1}^{-1} v \]

\[ - x_0^1 (1 + H_{11}^{-1}) - x_0^2 H_{12}^{-1} < 0, \]  

(A.50)

\[ \frac{\partial G}{\partial p_2} = \frac{4}{\gamma} (\Sigma_{e22}^{-1} p_2^2 + \Sigma_{e21}^{-1} p_1^1) + \frac{2}{\gamma} (\Sigma_{e12}^{-1} K_1^1 + \Sigma_{e22}^{-1} K_2^2) + Z^2 - \frac{2}{\gamma} \Sigma_{e2}^{-1} v \]

\[ - x_0^2 (1 + H_{22}^{-1}) - x_1 H_{21}^{-1} < 0, \]  

(A.51)

where \( K^j = F^j + \frac{\gamma}{2} H_j x_0 \). Hence

\[ x_0^1 (1 + H_{11}^{-1}) + x_0^2 H_{12}^{-1} > C_1, \]

\[ x_0 H_{21}^{-1} + x_0 (1 + H_{22}^{-1}) > C_2, \]  

(A.52)
where

\[ C_1 = \frac{4}{\gamma} \left( \Sigma^{-1} v_1 p_1 + \Sigma^{-1} v_2 p_2 \right) + \frac{2}{\gamma} \left( \Sigma^{-1} K_1 + \Sigma^{-1} K_2 \right) + Z^1 - \frac{2}{\gamma} \Sigma^{-1} v, \]

\[ C_2 = \frac{4}{\gamma} \left( \Sigma^{-1} v_2 p_2 + \Sigma^{-1} v_2 p_1 \right) + \frac{2}{\gamma} \left( \Sigma^{-1} K_1 + \Sigma^{-1} K_2 \right) + Z^2 - \frac{2}{\gamma} \Sigma^{-1} v. \]

(A.53)

We have three cases depending on the sign of the correlation. Assume that the correlation is zero:

Then the condition is that

\[ x_0^1 > \frac{C_1}{1 + H^{-1}_{11}}, \]

\[ x_0^2 > \frac{C_2}{1 + H^{-1}_{22}}. \]

(A.54)

Assume the correlation is positive: We have the conditions

\[ x_0^2 < \frac{C_1 - x_0^1 (1 + H^{-1}_{11})}{H^{-1}_{12}}, \]

\[ x_0^2 > \frac{C_2 - x_0^1 H^{-1}_{21}}{1 + H^{-1}_{22}}. \]

(A.55)

(A.56)

Finally with negative correlation, we have that given an \( x_0^1 \), we need

\[ x_0^2 > \max \left[ \frac{C_1 - x_0^1 (1 + H^{-1}_{11})}{H^{-1}_{12}}, \frac{C_2 - x_0^1 H^{-1}_{21}}{1 + H^{-1}_{22}} \right]. \]

(A.57)

To see the second statement in the proposition, note that the sign of \( H^{-1}_{12} \) and \( H^{-1}_{21} \) depend only on the covariance, \( \sigma_{12} \).
References


